F. Bagemihl
Nagoya Math. J.
Vol. 43 (1971), 157–159

A PROPERTY OF THE PRINCIPAL CLUSTER SETS OF A CLASS OF HOLOMORPHIC FUNCTIONS*

F. BAGEMIHL

Let D be the open unit disk and Γ be the unit circle in the complex plane, and denote by Ω the Riemann sphere. If $f(z)$ is a meromorphic function in D, and if $\zeta \in \Gamma$, then the principal cluster set of f at ζ is the set

$$\Pi(f, \zeta) = \bigcap_{\Lambda} C_{\Lambda}(f, \zeta),$$

where Λ ranges over all arcs at ζ, and the chordal principal cluster set of f at ζ is the set

$$\Pi_{\chi}(f, \zeta) = \bigcap_{X} C_{X}(f, \zeta),$$

where X ranges over all chords at ζ; it is evident that $\Pi(f, \zeta) \subseteq \Pi_{\chi}(f, \zeta)$.

In [1] we studied the relation between $\Pi(f, \zeta)$ and $\Pi_{\chi}(f, \zeta)$, and we proved, among other results, the following [1, Theorem 9, Corollary 1, Corollary 2, Corollary 3]:

(I) *There exists a nonconstant holomorphic function $f(z)$ in D such that $\Pi(f, \zeta) = \pi_{\chi}(f, \zeta) = \{\infty\}$ for every $\zeta \in \Gamma$.*

(II) *If $\omega \in \Omega$, then there exists a nonconstant meromorphic function $f(z)$ in D such that $\Pi(f, \zeta) = \Pi_{\chi}(f, \zeta) = \{\omega\}$ for every $\zeta \in \Gamma$.*

(III) *If $\omega \in \Omega$, then there exists a nonconstant holomorphic function $f(z)$ in D such that $\Pi_{\chi}(f, \zeta) = \{\omega\}$ for every $\zeta \in \Gamma$.*

(IV) *If ω is a finite complex number, then there exists a nonconstant holomorphic function $f(z)$ in D such that, for every $\zeta \in \Gamma$ with at most enumerably many exceptions, $\Pi(f, \zeta) = \Pi_{\chi}(f, \zeta) = \{\omega\}$.*

A comparison of these results makes it natural to inquire [1, Remark 6]

Received August 22, 1970.
* Supported by the U.S. Army Research Office-Durham.

whether (IV) remains valid if the phrase "with at most enumerably many exceptions" is deleted. The purpose of this note is to answer this in the negative by showing that an everywhere dense exceptional subset of Γ is always present.

THEOREM. *Let $f(z)$ be a nonconstant holomorphic function in D, and suppose the finite complex number ω to be such that $\Pi_\chi(f, \zeta) = \{\omega\}$ for every $\zeta \in \Gamma$. If E denotes the set of points $\zeta \in \Gamma$ with the property that $\Pi(f, \zeta) = \phi$, then E is everywhere dense on Γ.*

Proof. Consider an arbitrary open subarc Γ_0 of Γ. It clearly suffices to show that $E \cap \Gamma_0 \neq \phi$.

According to Plessner's theorem [3, p. 217], almost every point of Γ_0 is either a Fatou point or a Plessner point of f. If a point $\zeta \in \Gamma$ is a Fatou point of f, then f has the angular limit ω at ζ, since by hypothesis $\Pi_\chi(f, \zeta) = \{\omega\}$. Hence, by Priwalow's uniqueness theorem [3, p. 210], since f is not identically constant, almost every point of Γ_0 is a Plessner point of f. Let $\zeta_0 \in \Gamma_0$ be a Plessner point of f. Then, in particular, $C(f, \zeta_0) = \Omega$, where $C(f, \zeta_0)$ denotes the cluster set of f at ζ_0 relative to D. Since $f(z)$ is holomorphic in D, the value ∞ is omitted by f, so that ∞ does not belong to the set $R(f, \zeta_0)$, the range of f at ζ_0; in symbols, we have $\infty \in \Omega - R(f, \zeta_0)$. It follows from this and from a consequence [2, p. 131] of Collingwood-Cartwright's theorem in the small, that either $\infty \in \Phi(f, \zeta_0)$ or $\infty \in \chi^*(f, \zeta_0)$.

Now the relation $\infty \in \Phi(f, \zeta_0)$ implies (cf. [2, p. 96]) simply that $f(z) \to \infty$ along a so-called Koebe sequence of arcs in D that converges to a closed subarc Γ_1 of Γ; but this is impossible, because it would follow that if ζ is an interior point of Γ_1, then $\infty \in \Pi_\chi(f, \zeta)$, so that $\omega = \infty$, which contradicts the hypothesis that ω is finite. Therefore we must have $\infty \in \chi^*(f, \zeta_0)$. This implies (cf. [2, p. 123]) in particular the existence of a value $\lambda \in \Omega$, where either $\lambda = \infty$ or $|\omega| < |\lambda| < \infty$, such that $f(z)$ converges to λ on an asymptotic path Λ whose end is contained in the arc Γ_0. The end of Λ is either a closed subarc Γ_2 of Γ_0 or a point $\zeta_1 \in \Gamma_0$. The first case is impossible, however, because it would imply that if ζ is an interior point of Γ_2, then $\lambda \in \Pi_\chi(f, \zeta)$, so that $\omega = \lambda$, which is absurd. In the second case, if $\Pi(f, \zeta_1) \neq \phi$, we must have $\lambda \in \Pi(f, \zeta_1)$, which contradicts the fact that $\Pi(f, \zeta_1) \subseteq \Pi_\chi(f, \zeta_1) = \{\omega\}$. Thus $\zeta_1 \in E \cap \Gamma_0$, and the proof is complete.

The proof has shown actually that under the hypothesis of the theorem, every open subarc Γ_0 of Γ contains a point ζ_1 at which the function f has an asymptotic value $\lambda \neq \omega$ on an arc Λ at ζ_1. This implies that, since $\Pi_\chi(f, \zeta_1) = \{\omega\}$, there exists a chord X at ζ_1 such that $\lambda \notin C_X(f, \zeta_1)$. Thus $C_\Lambda(f, \zeta_1) \cap C_X(f, \zeta_1) = \phi$, so that ζ_1 is an ambiguous point of f, and we have the following

COROLLARY. *If $f(z)$ is a nonconstant holomorphic function in D, and if ω is a finite complex number such that $\Pi_\chi(f, \zeta) = \{\omega\}$ for every $\zeta \in \Gamma$, then the set of ambiguous points of f is everywhere dense on Γ.*

REFERENCES

[1] F. Bagemihl, Some results and problems concerning chordal principal cluster sets, Nagoya Math. J. **29** (1967), 7–18.
[2] E.F. Collingwood and M.L. Cartwright, Boundary theorems for a function meromorphic in the unit circle, Acta Math. **87** (1952), 83–146.
[3] I.I. Priwalow, Randeigenschften analytischer Funktionen, Berlin, 1956.

University of Wisconsin-Milwaukee

https://doi.org/10.1017/S0027763000014434 Published online by Cambridge University Press