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RANGE INCLUSION FOR MULTILINEAR MAPPINGS: APPLICATIONS

BY

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Dedicated to the memory of Robert Arnold Smith

ABSTRACT. The result of S. Grabiner [5] on range inclusion is applied for establishing the following two theorems: 1. For $A, B \in L(H)$, two operators on the Hilbert space H, we have $D_B C_0(H) \subseteq D_A L(H)$ if and only if $D_B C_1(H) \subseteq D_A L(H)$, where D_A is the inner derivation which sends $S \in L(H)$ to $AS - SA$, $C_1(H)$ is the ideal of trace class operators and $C_0(H)$ is the ideal of finite rank operators. 2. (Due to Fialkow [3]) For $A, B \in L(H)$, we write $T(A, B)$ for the map on $L(H)$ sending S to $AS - SB$. Then the range of $T(A, B)$ is the whole $L(H)$ if it includes all finite rank operators in $L(H)$.

1. **The Result**. The following two theorems concerning range inclusion for bounded linear mappings between Banach spaces were proved by M. Embry [2]. Subsequently by B. E. Johnson and J. P. Williams [6].

THEOREM A. *If $F \in L(X, W)$, $G \in L(Y, W)$ and if $FX \subseteq GY$, then there exists a positive number M such that $\|F'f\| \leqq M\|G'f\|$ for all $f \in W'$. (Here, X, Y, and W are real or complex Banach spaces, $L(X, W)$ is the space of all bounded linear mappings from X into W, W' is the dual of W and F': $W' \to X'$ is the dual map of F.)*

THEOREM B. *For $F \in L(X, Y)$ and $G \in L(X, W)$, we have $F'Y' \subseteq G'W'$ if and only if there exists $M > 0$ such that $\|Fx\| \leq M\|Gx\|$ for all x in X.*

S. Grabiner ([5], Lemma 2.1) generalized Theorem A to bounded multilinear mappings. Recall that a multilinear mapping F: $X_1 \times X_2 \times \ldots \times X_n \to Y$ (where $X_1, \ldots, X_n$, Y are Banach spaces) is bounded if

$$\|F\| \equiv \sup\{\|F(x_1, \ldots, x_n)\|: x_j \in X_j, \quad \|x_j\| \leq 1\}$$

is finite. Now we state Grabiner's result as follows.

THEOREM C. *If F is a bounded multilinear map from $X_1 \times \ldots \times X_n$ into W, if G is a bounded multilinear map from $Y_1 \times \ldots \times Y_m$ into W and if*

$$F(X_1 \times \ldots \times X_n) \subseteq \text{linear span of } G(Y_1 \times \ldots \times Y_m),$$

then there exists $M > 0$ such that $\|f \circ F\| \leq M\|f \circ G\|$ for all f in W'. (Note that $f \circ F$

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is a multilinear functional on $X_1 \times \ldots \times X_n$).

This result can be easily deduced from the multilinear version of uniform boundedness principle; for details, see [5]. The purpose of the present note is to illustrate that applying this theorem can give some rather surprising results.

2. **Application to Inner Derivations**. Let H be a (complex) Hilbert space and $L(H)$ be the algebra of all (bounded, linear) operators on H. For $A \in L(H)$, we write D_A for the inner derivation on $L(H)$ induced by A: $D_AX = AX - XA$ for $X \in L(H)$. We will write $C_0(H)$ for the algebra of all finite rank operators on H. We will also write $C_1(H)$ and $C_2(H)$ for the trace class and the Hilbert-Schmidt class respectively. In [4], we showed that, for A, $B \in L(H)$ where A is a normal operator, then the following conditions are equivalent:

(1) $D_BC_1(H) \subseteq D_AL(H)$.

(2) $D_BC_2(H) \subseteq D_AC_2(H)$.

(3) $B = f(A)$ for some Lipschitz continuous function on $\sigma(A)$. (Here $\sigma(A)$ stands for the spectrum of A.) Note that condition (1) is obviously weaker than (2). In the present section we show that condition (1) can be replaced by a still weaker one:

(0) $D_BC_0(H) \subseteq D_AL(H)$.

In fact, we have:

PROPOSITION 1. *For* $A, B \in L(H)$, *the following conditions are equivalent:*

(a) $D_BC_0(H) \subseteq D_AL(H)$.

(b) $D_BC_1(H) \subseteq D_AL(H)$.

(c) *there exists* $M > 0$ *such that, for all* $T \in C_1(H)$,

$$\|D_BT\| \leq M\|D_AT\|_1 .$$

(Here, $\|\cdot\|_1$ stands for the trace norm.)

PROOF. That (b) implies (a) is obvious. To show that (c) implies (b), we note that $L(H)$ can be regarded as the dual space of $C_1(H)$; (see, e.g. [7], IV.1). In fact, if f is a bounded linear functional on $C_1(H)$, then there exists a unique $S \in L(H)$ such that $f(T) = \operatorname{tr}(ST)$ for all T in $C_1(H)$, where $\operatorname{tr}(\cdot)$ stands for the trace function. Also, $C_1(H)$ can be regarded as a subspace of the dual of $L(H)$, since the latter is the bidual of $C_1(H)$. Let G be the restriction of D_A to $C_1(H)$. Then G is a bounded linear map from $C_1(H)$ into itself. Regarding $L(H)$ as the dual of $C_1(H)$, the dual of G can be identified with $-D_A$ on $L(H)$. Let F: $C_1(H) \to L(H)$ be the restriction of D_B to $C_1(H)$. Then, by (c), we have $\|F(T)\| \leq M\|G(T)\|_1$ for $T \in C_1(H)$. It follows from Theorem B that $F'(C_1(H)) \subseteq F'(L(H)') \subseteq G'(L(H))$ and hence (b) follows. It remains to show that (a) $\Rightarrow$ (c) and we need Theorem C to accomplish this.

For x, y in H, we write $x \otimes y$ for the rank one operator (or zero) given by $(x \otimes y)z = (z, y)x$. Consider the map F: $H \times H \to L(H)$ defined by $F(x, y) = D_B(x \otimes y) = Bx \otimes y - x \otimes B^*y$, where B^* is the adjoint of B. Then (a) means $F(H \times H) \subseteq D_AL(H)$. Obviously F is a bounded real bilinear map. Hence, by Theorem C, there exists $M > 0$ such that $\|f \cdot F\| \leq M\|f \circ D_A\|$ for all f in $L(H)'$. For $T \in C_1(H)$,

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define $f_T \in L(H)'$ by putting $f_T(S) = \mathrm{tr}(ST)$. (Thus $T \to f_T$ is the canonical embedding of $C_1(H)$ into $L(H)'$ described in the previous paragraph.) Then, for $x, y \in H$, $T \in C_1(H)$, we have

$$f_T \circ F(x, y) = \mathrm{tr}(T(Bx \otimes y - x \otimes B^*y)) = -((D_B T)x, y).$$

Hence $\|f_T \circ F\| = \|D_B T\|$. On the other hand, for $S \in L(H)$, $T \in C_1(H)$,

$$f_T \circ D_A(S) = \mathrm{tr}(T(AS - SA)) = \mathrm{tr}((D_A T)S).$$

and hence $\|f_T \circ D_A\| = \|D_A T\|_1$. Now (c) follows from the inequality $\|f_T \circ F\| \leq M\|f_T \circ D_A\|$. The proof is complete.

3. **Application to the Transform $S \to AS - SB$**. For operators A and B on a Banach space X, we write $T(A, B)$ for the operator on $L(X)$ defined by $T(A,B)S = AS - SB$ where $S \in L(X)$. Fialkow ([3], Theorem 2.1) showed that, in the case that X is a Hilbert space, the range of $T(A, B)$ includes all operators in $L(X)$ if it includes all finite-rank operators in $L(X)$. In view of a result due to C. Davis and P. Rosenthal ([1], Theorem 5) (stating that, for Hilbert space operators A and B, $T(A, B)$ is surjective if and only if $\sigma_d(A) \cap \sigma_a(B) = \emptyset$,) Fialkow reduced the proof of his theorem to the verification of the following statement: if $\sigma_d(A) \cap \sigma_a(B) \neq \emptyset$, then there are finite rank operators not in the range of $T(A, B)$. (Here, $\sigma_a(B)$ is the approximate point spectrum of B and $\sigma_d(A)$ is the approximate defect spectrum of B.) By using Theorem C, we find a short proof of this fact in the Banach space setting.

PROPOSITION 2 *for* $A, B \in L(X)$, *if* $\sigma_d(A) \cap \sigma_a(B) \neq \emptyset$, *then* $T(A, B)L(X) \not\supseteq C_0(x)$.

(Again, $C_0(X)$ stands for the set of all finite rank operators in $L(X)$).

PROOF. Assume the contrary that $T(A, B)L(X) \supseteq C_0(X)$. Define the bilinear map F: $X \times X' \to L(X)$ by $F(x, f) = x \otimes f$, where $x \otimes f$ is the operator in $L(X)$ sending every $y \in X$ to $f(y)x$. Then $F(X \times X') \subseteq T(A,B)L(X)$. By Theorem C, there exists $M > 0$ such that $\|G \circ F\| \leq M\|G \circ T(A,B)\|$ for $G \in L(X)'$. For $y \in X$ and $g \in X'$, we define $G_{y,g} \in L(X)'$ by putting $G_{y,g}(S) = g(Sy)$. Then

$$\begin{aligned}\|G_{y,g} \circ F\| &= \sup\{|G_{y,g}(x \otimes f)| : \|x\| \leq 1, \|f\| \leq 1\}\\ &= \sup\{|g(x)f(y)| : \|x\| \leq 1, \|f\| \leq 1\} = \|g\|\|y\|\end{aligned}$$

Suppose $c \in \sigma_d(A) \cap \sigma_a(B)$. Then there are sequences $\{y_n\}$ in X and $\{g_n\}$ in X' such that $\|y_n\| = \|g_n\| = 1$, $\|A'g_n - cg_n\| \to 0$ and $\|By_n - cy_n\| \to 0$. Let $G_n = G_{y_n, g_n}$. Then we have

$$\begin{aligned}1 = \|g_n\|\|y_n\| = \|G_n \circ F\| &\leq M\|G_n \circ T(A,B)\|\\ &= M \sup\{|g_n((AS - SB)y_n)| : \|S\| \leq 1\}\\ &= M \sup\{|(A'g_n - cg_n)(Sy_n) + g_n(S(By_n - cy_n))| : \|S\| \leq 1\}\\ &\leqslant M\|A'g_n - cg_n\| + M\|By_n - cy_n\|.\end{aligned}$$

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But the last expression tends to zero as $n \to \infty$ and thus we have arrived at a contradiction. The proof is complete.

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REFERENCES

1. C. Davis and P. Rosenthal, *Solving linear operator equations*, Canad. Math. J., **26** (1974), pp. 1384–1389.

2. M. R. Embry, *Factorizations of operators on Banach space*, Proc. Amer. Math. Soc., **38** (1973), pp. 587–590.

3. L. Fialkow, *A note on normed ideals and the operator $X - AX - XB$*, Israel J. Math., **32** (1973), pp. 331–348.

4. C. K. Fong, *Range inclusion for normal derivations*, Glasgow J. Math., **25** (1984), pp. 255–262.

5. S. Grabiner, *Operator ranges and invariant subspaces*, Indiana Univ. Math. J., **28** (1979), pp. 845–857.

6. B. E. Johnson and J. P. Williams, *The range of a normal derivation*, Pacific J. Math. **58** (1975), pp. 105–122.

7. R. Schatten, *Normed ideals of completely continuous operators*, Springer, Berlin (1960).

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