

ON THE FIRST PASSAGE TIME FOR BROWNIAN MOTION SUBORDINATED BY A LÉVY PROCESS

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Abstract

In this paper we consider the class of Lévy processes that can be written as a Brownian motion time changed by an independent Lévy subordinator. Examples in this class include the variance-gamma (VG) model, the normal-inverse Gaussian model, and other processes popular in financial modeling. The question addressed is the precise relation between the standard first passage time and an alternative notion, which we call the first passage of the second kind, as suggested by Hurd (2007) and others. We are able to prove that the standard first passage time is the almost-sure limit of iterations of the first passage of the second kind. Many different problems arising in financial mathematics are posed as first passage problems, and motivated by this fact, we are led to consider the implications of the approximation scheme for fast numerical methods for computing first passage. We find that the generic form of the iteration can be competitive with other numerical techniques. In the particular case of the VG model, the scheme can be further refined to give very fast algorithms.

Keywords: Brownian motion; first passage; time change; Lévy subordinator; stopping time; financial models

2000 Mathematics Subject Classification: Primary 60J75
Secondary 60G51; 91B28

1. Introduction

First passage problems are a classic aspect of stochastic processes that arise in many areas of application. In mathematical finance, for example, first passage problems lie at the heart of such issues as credit risk modeling, pricing barrier options, and the optimal exercise of American options. If X_t is any process with initial value $X_0 = x_0$, the first passage time to a lower level b is defined to be the stopping time

$$t_b^*(x_0) = \inf\{t \geq 0 \mid X_t \leq b\}.$$

The distributional properties of t^* can be easily obtained when the underlying process X is a diffusion (see [7, pp. 25–27]), but, when X has jumps, the situation is much more challenging.

Results on Wiener–Hopf-type factorizations (see [3, p. 336], [6, pp. 159–166], [15, pp. 145–158], and [16]) have proved to be very useful for studying first passage time problems for Lévy processes. Probably the best known result of this type is the following identity (see [6, p. 165])

Received 30 May 2008; revision received 5 December 2008.

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and [15, p. 147]):

$$\frac{q}{q + \psi(\lambda)} = \Phi_q^+(\lambda)\Phi_q^-(\lambda). \quad (1)$$

Here $\psi(\lambda)$ is the characteristic exponent of X_t ; $\Phi_q^+(\lambda)$ and $\Phi_q^-(\lambda)$ are characteristic functions of infinitely divisible random variables $S_{\tau(q)}$ and $X_{\tau(q)} - S_{\tau(q)}$, where $S_t = \sup\{X_s : s \leq t\}$ is the supremum process and $\tau(q)$ is an exponential random variable with parameter q , independent of X_t .

We can efficiently recover functions Φ^\pm using (1) when $\psi(\lambda)$ is a rational function. A well-studied class of processes for which this approach works well consists of Lévy processes with *phase-type distributed jumps* (see [1], [3], [4, p. 81], and [14]). Phase-type distributions are defined as the first passage time for a continuous-time, finite-state Markov chain, they form a dense class in the set of all distributions on \mathbb{R}^+ , and, most importantly, if a Lévy process has phase-type jumps, its characteristic exponent is a rational function (though the converse is not true; see [19]). However, if the jumps of process X_t are not of phase-type, we would need to approximate the jump measure of X_t with a sequence of phase-type measures. The first problem with this approach is that there do not exist any efficient algorithms on how to achieve this. The second problem is that the degree of the polynomial equation $q + \psi(\lambda) = 0$ would necessarily grow to ∞ , which will make solving this equation very complicated. Also, see [3] and [19] for an interesting example of a distribution with rational transform which would require an infinite-degree phase-type representation.

A second general approach to first passage is to solve the Fokker–Planck equation for the probability density of X_t conditioned on the set $\{t^* > t\}$. For Lévy processes, this amounts to solving a certain linear partial integral differential equation (PIDE) with nonlocal Dirichlet conditions (see [9, Proposition 12.6, Section 12.2] and [10]). In the case of Lévy processes the PIDE approach has the advantage that we can utilize the fast Fourier transform (FFT) to perform efficient computation of the convolutions involved; however, the method also involves the truncation of the state space (the real line in our case) and discretization in the x and t variables, and the resulting errors are not easy to control.

Our purpose here is to present a new approach to first passage problems applicable whenever the underlying Lévy process can be realized as a Lévy subordinated Brownian motion (LSBM), that is, whenever X can be constructed as $\tilde{W} \circ T$, where \tilde{W} is a standard drifting Brownian motion and T is a nondecreasing Lévy process independent of \tilde{W} . The class of Lévy processes that are realizable as LSBMs is identified in [9, Theorem 4.3], and is broad enough to include many of the Lévy processes that have so far been used in finance, such as a four-parameter subclass of the Kou–Wang model, the variance gamma (VG) model, the normal-inverse Gaussian (NIG) model, and a four-parameter subclass of the generalized tempered stable process.

The basis for our approach is that, for processes that are realizable as time-changed Brownian motions, there is an alternative notion that is also relevant, namely, the first time the time change exceeds the first passage time of the Brownian motion. This notion, called the *first passage of the second kind* in [12], shares some characteristics with the usual first passage time and can be applied in a similar way. The usefulness of this new concept is that it can be computed efficiently in many cases where the usual first passage time cannot.

In the present paper we study the first passage for LSBMs and show how the first passage of the second kind is the first of a sequence of stopping times that converges almost surely to the first passage time. Expressed differently, first passage can be viewed as a stochastic sum of first passage times of the second kind. This sequence leads to a convergent and computable expansion for the first passage probability distribution function p^* in terms of a similar function

p_1^* that describes the first passage distribution of the second kind. The outline of the paper is as follows. In Section 2 we define the objects needed to understand the first passage time, and we prove the expansion formula for first passage. In Section 3 we demonstrate the usefulness of this expansion by proving several explicit two-dimensional integral formulae for p_1^* , the first passage distribution of the second kind. In Section 4 we provide two proofs of the convergence of the expansion. The first proof is a proof of convergence in distribution and the second is a proof of convergence in the pathwise (almost-sure) sense. In Section 5 we focus on the special case of the VG model. In this important example, the formula for p_1^* is reduced to a one-dimensional integral (involving the exponential integral function). In Section 6, the expansion of the function p^* is studied numerically, and found to be numerically stable and efficient.

2. First passage for LSBMs

Let X_t be a general Lévy process with initial value $X_0 = x_0$ and characteristics $(b, c, \nu)_h$ with respect to a truncation function $h(x)$ (see [2] or [13]). This means that X is an infinitely divisible process with identical independent increments and càdlàg paths (those that are continuous from the right with left limits) almost surely. Here $b, c \geq 0$ are real numbers and ν is a sigma-finite measure on $\mathbb{R} \setminus 0$ that integrates the function $1 \wedge x^2$. By the Lévy–Khintchine formula, the log-characteristic function of X_1 is

$$\log E[\exp(iuX_1)] = iub - \frac{cu^2}{2} + \int_{\mathbb{R} \setminus 0} (e^{iux} - 1 - xh(x))\nu(dx).$$

In what follows we will find it convenient to focus on the Laplace exponent of X :

$$\psi_X(u) := -\log E[\exp(-uX_1)].$$

For simplicity of exposition, we specialize slightly by assuming that ν is continuous with respect to the Lebesgue measure $\nu(dx) = \nu(x) dx$, and integrates $1 \wedge |x|$, allowing us to take $h(x) = 0$. In this setting, the Markov generator of the process X_t applied to any sufficiently smooth function $f(x)$ is

$$[\mathcal{L}f](x) = b\partial_x f + \frac{c}{2}\partial_{xx}^2 f + \int_{\mathbb{R} \setminus 0} (f(x + y) - f(x))\nu(y) dy.$$

Definition 1. For any $b \in \mathbb{R}$, the random variable $t_b^* = t_b^*(x_0) := \inf\{t \mid X_t \leq b\}$ is called *the first passage time for level b*. When $b = 0$, we drop the subscript and $t^* := \inf\{t \mid X_t \leq 0\}$ is simply called *the first passage time of X*.

Remarks. 1. Since distributions of the increments of X are invariant under time and state space shifts, we can reduce computations of $t_b^*(x_0)$ to computations of $t^*(x_0 - b)$.

2. A general Lévy process is a mixture of a continuous Brownian motion with drift and a pure-jump process. We say that ‘downward creeping’ occurs if $X_{t^*} = 0$ and does not occur if $X_{t^*} < 0$. Under the assumption that ν integrates $1 \wedge |x|$, Corollaries 3 and 4 of [21] prove that there is almost surely no downward creeping if and only if the diffusive part is 0 (i.e. $c = 0$) and the drift $b \geq 0$. In what follows we will exclude the possibility of downward creeping. In this case, $X_{t^*} - X_{t^*-} \neq 0$, so X jumps across 0, and we can define the *overshoot* to be X_{t^*} .

The central object of study in this paper is the joint distribution of t^* and the overshoot X_{t^*} , in particular the joint probability density function

$$p^*(x_0; s, x_1) = E_{x_0}[\delta(t^* - s)\delta(X_{t^*} - x_1)],$$

where we invoke the Dirac delta function $\delta(\cdot)$. The marginal density of t^* is

$$p^*(x_0; s) = \int_{-\infty}^0 p^*(x_0; s, x_1) dx_1.$$

In the introduction we noted that results on the first passage for general Lévy processes, in particular results on the functions p^* , are difficult to obtain. For this reason, we now focus on a special class of Lévy processes that can be expressed as a drifting Brownian motion subjected to a time change by an independent Lévy subordinator. Such LSBMs have been studied in [9, Section 4.4] and [12]. The general LSBM is constructed as follows.

1. For an initial value $x_0 > 0$ and drift β , let $\tilde{W}_T = x_0 + W_T + \beta T$ be a drifting Brownian motion.
2. For a Lévy characteristic triple $(b, 0, \mu)$ with $b \geq 0$ and $\text{supp}(\mu) \subset \mathbb{R}^+$, let the time-change process T_t be the associated nondecreasing Lévy process (a subordinator), taken to be independent of W .
3. The time-changed process $X_t = \tilde{W}_{T_t}$ is defined to be an LSBM.

So constructed, it is known that X_t is itself a Lévy process. The process X_t will allow creeping if and only if $b > 0$: we henceforth assume for simplicity that $b = 0$. Theorem 4.3 of [9] provides a characterization for Lévy processes that are LSBMs. While, unlike the class of phase-type Lévy processes, the class of LSBMs is not dense in the class of Lévy processes, their analytic properties make them a useful and flexible class. It was observed in [12] that, for any LSBM X_t , we can define an alternative notion of the first passage time, which we denote here by \tilde{t} .

Definition 2. For any LSBM $X_t = \tilde{W}_{T_t}$, we define the *first passage time of \tilde{W}* to be $T^* = T^*(x_0) = \inf\{T : x_0 + W_T + \beta T \leq 0\}$. Note that $T^*(x_0) = 0$ when $x_0 \leq 0$. The *first passage time of the second kind* of X_t is defined as $\tilde{t} = \tilde{t}(x_0) = \inf\{t : T_t \geq T^*(x_0)\}$.

This definition of \tilde{t} , and its relation to t^* , is illustrated in Figure 1.

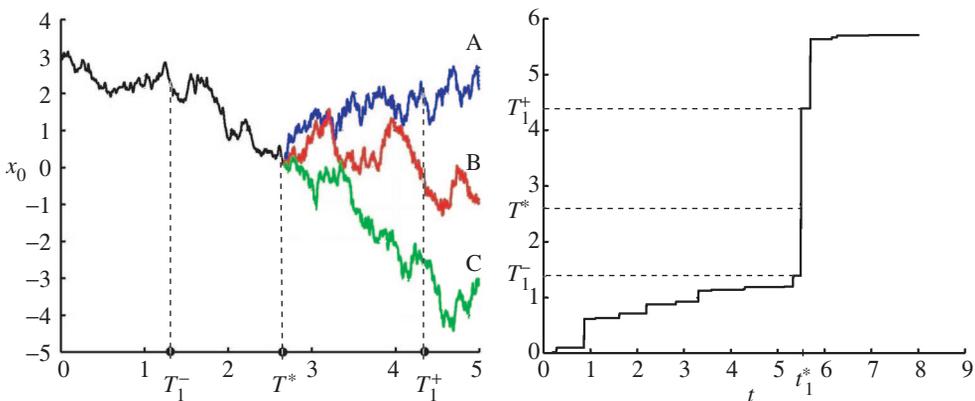


FIGURE 1: Three trajectories of the Brownian motion X_t with the same T^* and the sample path of the time change T_t that illustrate, in general, $\tilde{t} \leq t^*$. On the paths B and C, $X_{t_1^*} = \tilde{W}_{T_1^+} \leq 0$, and on these paths, $t_1^* = \tilde{t} = t^*$. On path A, $X_{t_1^*} = \tilde{W}_{T_1^+} > 0$ and so $t_1^* = \tilde{t} < t^*$.

We now show that $\tilde{t}(x_0) := t_1^*(x_0)$ is the first of a sequence of approximations $\{t_i^*(x_0)\}_{i=1,2,\dots}$ to the stopping time t^* . In Figure 1 we illustrate the first excursion overjump, t_1^* , which may be either t^* or not. The construction of $t_i^*(x_0, \pi)$ is pathwise. We introduce the second argument $\pi \in \Omega$, which denotes a *sample path*, that is, a pair (ω, τ) , where ω is a continuous drifting Brownian path \tilde{W} and τ is a càdlàg sample path of the time change T . Thus, $\pi : (S, s) \rightarrow (\omega(S), \tau(s))_{S,s \geq 0}$. The natural ‘big filtration’ $(\mathcal{F}_t)_{t \geq 0}$ for time-changed Brownian motion has

$$\mathcal{F}_t = \sigma\{\omega(S), \tau(s), S \leq \tau(t), s \leq t\}.$$

For any $t \geq 0$, there is a natural ‘time translation’ operation on paths $\rho_t : (\omega, \tau) \rightarrow (\omega', \tau')$, where $\omega'(S) = \omega(S + \tau(t))$ and $\tau'(s) = \tau(s + t) - \tau(t)$.

The construction of $\{t_i^*(x_0, \pi)\}_{i=1,2,\dots}$ for a given sample path π is as follows. Inductively, for $i \geq 2$, we define the *time of the i th excursion overjump* by

$$t_i^*(x_0, \pi) = \inf\{t \geq t_{i-1}^*(x_0, \pi) : T_t - T_{t_{i-1}^*} \geq T^*(X_{t_{i-1}^*}, \pi')\},$$

where $\pi' = \rho_{t_{i-1}^*}(\pi)$ denotes a time-shifted sample path. Note that $t_i^*(x_0, \pi) = t_{i-1}^*(x_0, \pi)$ if and only if $X_{t_{i-1}^*}(x_0, \pi) \leq 0$ or $t_{i-1}^*(x_0, \pi) = \infty$. At any excursion overjump event t_i^* , the time interval which covers the event has left and right endpoints $T_i^- = T_{t_i^*}^-$ and $T_i^+ = T_{t_i^*}$. Let

$$p_i^*(x_0; s, x) = E_{x_0}[\delta(t_i^* - s)\delta(X_{t_i^*} - x)]$$

denote the joint distribution of $t_i^*(x_0)$ and $X_{t_i^*}(x_0)$.

The definition of this sequence of stopping times is summarized by the pathwise equation

$$t_i^*(x_0, \pi) = t_{i-1}^*(x_0, \pi) \mathbf{1}_{\{X_{t_{i-1}^*} \leq 0\}} + (t_1^*(X_{t_{i-1}^*}, \pi') + t_{i-1}^*(x_0, \pi)) \mathbf{1}_{\{X_{t_{i-1}^*} > 0\}}, \quad i \geq 2,$$

where $\pi' = \rho_{t_{i-1}^*}(\pi)$. The identical increments property of the LSBM implies that the joint probability densities satisfy the recursive relation

$$p_i^*(x_0; s, x) = p_1^*(x_0; s, x) \mathbf{1}_{\{x \leq 0\}} + \int_0^\infty dy \int_0^s du p_1^*(x_0; u, y) p_{i-1}^*(y; s - u, x), \quad i \geq 2. \tag{2}$$

Similarly, the probability density function (PDF) of the first passage time t^* satisfies the relation

$$p_i^*(x_0; s) = \int_{-\infty}^0 p_1^*(x_0; s, x) dx + \int_0^\infty dy \int_0^s du p_1^*(x_0; u, y) p_{i-1}^*(y; s - u), \quad i \geq 2. \tag{3}$$

We note in passing that when downward creeping is included, the probability function includes an atom at $x = 0$, and that when interpreted in that light, (2) and (3) are still correct.

2.1. Examples of LSBMs

We note here three classes of Lévy processes that can be written as LSBMs and have been used extensively in financial modeling.

1. The exponential model with parameters (a, b, c) arises by taking T_t to be the increasing process with drift $b \geq 0$ and jump measure $\mu(z) = ce^{-az}$, $c, a > 0$, on $(0, \infty)$. The Laplace exponent of T is

$$\psi_T(u) := -\log E[\exp(-uT_1)] = bu + \frac{uc}{a + u}.$$

We can show using [9, Equation 4.14] that the resulting time-changed process $X_t := \tilde{W}_{T_t}$ has triple $(\beta b, b, \rho)$ with

$$\rho(y) = \frac{c}{\sqrt{\beta^2 + 2a}} \exp(-(\sqrt{\beta^2 + 2a} - \beta)(y)^+ - (\sqrt{\beta^2 + 2a} + \beta)(y)^-),$$

where $(y)^+ = \max(0, y)$ and $(y)^- = (-y)^+$. This forms a four-dimensional subclass of the six-dimensional family of exponential jump diffusions studied in [14].

2. The VG model [17] arises by taking T_t to be a gamma process with drift defined by the characteristic triple $(b, 0, \mu)$ with $b \geq 0$ (usually, b is taken to be 0) and jump measure $\mu(z) = (\nu z)^{-1} e^{-z/\nu}$, $\nu > 0$, on $(0, \infty)$. The Laplace exponent of T_t , $t = 1$, is

$$\psi_T(u) := -\log E[\exp(-uT_1)] = bu + \frac{1}{\nu} \log(1 + \nu u).$$

The resulting time-changed process has triple $(\beta b, b, \rho)$ with

$$\rho(y) = \frac{1}{\nu|y|} \exp\left(\beta x - \sqrt{\frac{2}{\nu} + \beta^2|x|}\right).$$

3. The NIG model with parameters $\tilde{\beta}$ and $\tilde{\gamma}$ [5] arises when T_t is the first passage time for a second independent Brownian motion with drift $\tilde{\beta} > 0$ to exceed the level γt . Then

$$\psi_T(u) = \tilde{\gamma}(\tilde{\beta} + \sqrt{\tilde{\beta}^2 + 2u}),$$

and the resulting time-changed process has Laplace exponent

$$\psi_X(u) = x\mu + \tilde{\gamma}(\tilde{\beta} + \sqrt{\tilde{\beta}^2 - u^2 + 2\tilde{\beta}u}).$$

3. Computing first passage of the second kind

We have just seen that the first passage for LSBMs admits an expansion as a sum of first passage times of the second kind. In this section we show that this expansion can be useful, by proving several equivalent integral formulae for computing the structure function $p_1^*(x_0; s, x_1)$ for general LSBMs. While the equivalence of these formulae can be demonstrated analytically, their numerical implementations will perform differently: which formula will be superior in practice is not a priori clear, but will likely depend on the range of parameters involved. For a complete picture, we provide independent proofs of the two given formulae.

Theorem 1. *Let the time change T_t have $b = 0$ and Laplace exponent $\psi(u)$, and let \tilde{W} have drift $\beta \neq 0$. Then*

$$p_1^*(x_0; s, x_1) = \frac{\exp(\beta(x_1 - x_0))}{4\pi^2} \iint_{\mathbb{R}^2} \frac{\psi(iz_1) - \psi(iz_2)}{i(z_1 - z_2)} \frac{\exp(-s\psi(iz_1))}{\sqrt{\beta^2 - 2iz_2}} \times \exp(-x_0\sqrt{\beta^2 - 2iz_1} - |x_1|\sqrt{\beta^2 - 2iz_2}) dz_1 dz_2. \quad (4)$$

Provided that the time change is not a compound Poisson process, then

$$\begin{aligned}
 p_1^*(x_0; s, x_1) = & -\frac{2 \exp(\beta(x_1 - x_0))}{\pi^2} \text{PV} \iint_{(\mathbb{R}^+)^2} dk_1 dk_2 \frac{k_2 \cos |x_1| k_1 \sin x_0 k_2}{k_1^2 - k_2^2} \\
 & \times \exp\left(-s\psi\left(\frac{k_1^2 + \beta^2}{2}\right)\right) \psi\left(\frac{k_2^2 + \beta^2}{2}\right) \\
 & - \frac{\exp(\beta(x_1 - x_0))}{\pi} \int_{\mathbb{R}^+} \sin |x_1| k \sin x_0 k \\
 & \times \exp\left(-s\psi\left(\frac{k^2 + \beta^2}{2}\right)\right) \psi\left(\frac{k^2 + \beta^2}{2}\right) dk.
 \end{aligned}$$

Here PV denotes that the principal value contour is taken.

Remark. The equivalence of these two formulae can be demonstrated directly by performing the change of variables $k_j = i(\beta^2 - 2iz_j^2)^{1/2}$, $j = 1, 2$, followed by a deformation of the contours. Justification of the contour deformation (from the branch of a left–right symmetric hyperbola in the upper half k_j -plane to the real axis) depends on the decay of the integrand and the computation of certain residues.

3.1. First proof of Theorem 1

For a fixed level $h > 0$, the first passage time and the overshoot of the process T_t above the level h are defined to be $\tilde{t}(h) = \inf\{t > 0 \mid T_t > h\}$ and $\tilde{\delta}(h) = T_{\tilde{t}(h)} - h$. The Pecherskii–Rogozin identity [20] applied to the nondecreasing process T says that

$$\int_0^\infty \exp(-z_1 h) E[\exp(-z_2 \tilde{\delta}(h) - z_3 \tilde{t}(h))] dh = \frac{\psi(z_1) - \psi(z_2)}{z_1 - z_2} (z_3 + \psi(z_1))^{-1}.$$

Inversion of the Laplace transform in the above equation then leads to

$$E[\exp(-z_2 \tilde{\delta}(h) - z_3 \tilde{t}(h))] = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\psi(iz_1) - \psi(z_2)}{iz_1 - z_2} (z_3 + \psi(iz_1))^{-1} \exp(iz_1 h) dz_1.$$

The first passage time of the Brownian motion with drift is defined as $T^* = T^*(x_0) = \inf\{T > 0 \mid x_0 + W_T + \beta T < 0\}$. Next, we need to find the joint Laplace transform of $t_1^* = \inf\{t \mid T_t > T^*\} = \tilde{t}(T^*)$ and the overshoot $\delta^* = \tilde{\delta}(T^*)$. Since T_t is independent of W_T , we find that

$$\begin{aligned}
 E[\exp(-z_2 \delta^* - z_3 t_1^*)] &= E[E[\exp(-z_2 \tilde{\delta}(T^*) - z_3 \tilde{t}(T^*)) \mid T^*]] \\
 &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\psi(iz_1) - \psi(z_2)}{iz_1 - z_2} (z_3 + \psi(iz_1))^{-1} E[\exp(iz_1 T^*)] dz_1 \\
 &= \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\psi(iz_1) - \psi(z_2)}{iz_1 - z_2} (z_3 + \psi(iz_1))^{-1} \exp(-x_0(\beta + \sqrt{\beta^2 - 2iz_1})) dz_1,
 \end{aligned}$$

where in the last equality we have used the following well-known result for the characteristic function of the first passage time of Brownian motion with drift:

$$E[\exp(iz_1 T^*(x_0))] = \exp(-x_0(\beta + \sqrt{\beta^2 - 2iz_1})).$$

Next we use the Fourier transform of the PDF of the Brownian motion with drift to obtain

$$\begin{aligned} E[\delta(\tilde{W}_t - x_1)] &= \frac{\exp(-(x_1 - \beta t)^2/2t)}{\sqrt{2\pi t}} \\ &= \frac{\exp(\beta x_1)}{2\pi} \int_{\mathbb{R}} \exp(-iz_2 t) \frac{\exp(-|x_1|\sqrt{\beta^2 - 2iz_2})}{\sqrt{\beta^2 - 2iz_2}} dz_2. \end{aligned}$$

Thus, using the fact that \tilde{W} is independent of t_1^* and δ^* , we obtain

$$\begin{aligned} E[\exp(-z_3 t_1^*) \delta(\tilde{W}_{\delta^*} - x_1)] &= E[E[\exp(-z_3 t_1^*) \delta(\tilde{W}_{\delta^*} - x_1) \mid \delta^*]] \\ &= \frac{\exp(\beta x_1)}{2\pi} \int_{\mathbb{R}} E[\exp(-z_3 t_1^* - iz_2 \delta^*)] \frac{\exp(-|x_1|\sqrt{\beta^2 - 2iz_2})}{\sqrt{\beta^2 - 2iz_2}} dz_2 \\ &= \frac{\exp(\beta(x_1 - x_0))}{4\pi^2} \iint_{\mathbb{R}^2} \frac{\psi(iz_1) - \psi(iz_2)}{i(z_1 - z_2)} \frac{(z_3 + \psi(iz_1))^{-1}}{\sqrt{\beta^2 - 2iz_2}} \\ &\quad \times \exp(-x_0\sqrt{\beta^2 - 2iz_1} - |x_1|\sqrt{\beta^2 - 2iz_2}) dz_1 dz_2. \end{aligned}$$

Now, the statement of the theorem follows after one additional Fourier inversion:

$$\begin{aligned} p_1^*(x_0; s, x_1) &= E[\delta(t_1^* - s)(\tilde{W}_{\delta^*} - x_1)] \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \exp(iz_3 s) E[\exp(-iz_3 t_1^*) \delta(\tilde{W}_{\delta^*} - x_1)] dz_3 \\ &= \frac{\exp(\beta(x_1 - x_0))}{4\pi^2} \iint_{\mathbb{R}^2} \frac{\psi(iz_1) - \psi(iz_2)}{i(z_1 - z_2)} \frac{\exp(-s\psi(iz_1))}{\sqrt{\beta^2 - 2iz_2}} \\ &\quad \times \exp(-x_0\sqrt{\beta^2 - 2iz_1} - |x_1|\sqrt{\beta^2 - 2iz_2}) dz_1 dz_2, \end{aligned}$$

where we have also used the Fourier integral

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{\exp(iz_3 s)}{iz_3 + \psi(iz_1)} dz_3 = \exp(-s\psi(iz_1)).$$

3.2. Second proof of Theorem 1

The strategy of the proof is to compute

$$I(u) = E_{0,x_0}[\mathbf{1}_{\{s < t_1^* \leq s+u\}} \delta(X_{s+u} - x_1)]$$

and then take the limit of $I(u)/u$ as $u \rightarrow 0+$. The key idea is to note that $X_{s+u} = X_{s-} + \tilde{W}'_{T'_u}$, where \tilde{W}' and T' are copies of \tilde{W} and T , independent of the filtration \mathcal{F}_{s-} . We can then perform the above expectation via an intermediate conditioning on \mathcal{F}_{s-} :

$$\begin{aligned} E[\delta(X_{s+u} - x_1) \mathbf{1}_{\{s < t_1^* \leq s+u\}} \mid \mathcal{F}_{s-}] &= \mathbf{1}_{\{s < t_1^*\}} E[\delta(\ell + \tilde{W}'_{T'_u} - x_1) \mathbf{1}_{\{u \geq t_1^*\}} \mid X_{s-} = \ell]. \end{aligned}$$

To evaluate the expectations that arise, we will need Lemma 1(ii) and (iii), below, that were stated and proved in [12].

Lemma 1. ([12].) (i) For any $s > 0$,

$$E_{0,x}[\mathbf{1}_{\{s < t_1^*\}} \delta(X_s - y)] = \mathbf{1}_{\{y > 0\}} \frac{e^{\beta(y-x)}}{2\pi} \int_{\mathbb{R}} (e^{iz(x-y)} - e^{iz(x+y)}) \times \exp\left(-s\psi\left(\frac{z^2 + \beta^2}{2}\right)\right) dz.$$

(ii) For any $s > 0$ and $\varepsilon \in \mathbb{R}$,

$$E_{0,x}[\mathbf{1}_{\{s \geq t_1^*\}} \delta(X_s - y)] = \frac{e^{\beta(y-x)}}{2\pi} \int_{\mathbb{R}+i\varepsilon} e^{iz(x+|y|)} \exp\left(-s\psi\left(\frac{z^2 + \beta^2}{2}\right)\right) dz. \tag{5}$$

(iii) For any k in the upper half-plane,

$$E_{0,x}[\mathbf{1}_{\{s < t_1^*\}} \exp(-\beta X_s + ik X_s)] = \frac{e^{-\beta x}}{2\pi} \int_{\mathbb{R}} \left(\frac{i}{k-z} - \frac{i}{k+z}\right) e^{izx} \exp\left(-s\psi\left(\frac{z^2 + \beta^2}{2}\right)\right) dz. \tag{6}$$

First, using (5), we find that

$$E[\delta(X_{s+u} - x_1) \mathbf{1}_{\{s < t_1^* \leq s+u\}} \mid X_{s-} = \ell] = \mathbf{1}_{\{s < t_1^*\}} \frac{\exp(\beta(x_1 - \ell))}{2\pi} \int_{\mathbb{R}+i\varepsilon} dk \exp(ik(\ell + |x_1|)) \exp\left(-u\psi\left(\frac{k^2 + \beta^2}{2}\right)\right).$$

When we paste this expression into the final expectation over X_{s-} , we can use Fubini to interchange the expectation and integral, providing that we choose $\varepsilon > 0$. Then we find that

$$I = \frac{\exp(\beta x_1)}{2\pi} \int_{\mathbb{R}+i\varepsilon} dk \exp(ik|x_1|) \exp\left(-u\psi\left(\frac{k^2 + \beta^2}{2}\right)\right) E_{0,x_0}[\exp(ik X_s - \beta X_s) \mathbf{1}_{\{s < t_1^*\}}].$$

We can now use (6) to obtain

$$I = \frac{\exp(\beta(x_1 - x_0))}{(2\pi)^2} \iint_{(\mathbb{R}+i\varepsilon) \times \mathbb{R}} \exp(ik|x_1| + izx_0) \left(\frac{i}{k-z} - \frac{i}{k+z}\right) \times \exp\left(-u\psi\left(\frac{k^2 + \beta^2}{2}\right) - s\psi\left(\frac{z^2 + \beta^2}{2}\right)\right) dz dk.$$

Noting that $I(0) = 0$ and taking $\lim_{u \rightarrow 0} I(u)/u$ now gives

$$p_1^*(x_0; s, x_1) = \frac{\exp(\beta(x_1 - x_0))}{2\pi^2} \iint_{(\mathbb{R}+i\varepsilon) \times \mathbb{R}} dk dz \frac{iz}{k^2 - z^2} \exp(ik|x_1| + izx_0) \times \exp\left(-s\psi\left(\frac{z^2 + \beta^2}{2}\right)\right) \psi\left(\frac{k^2 + \beta^2}{2}\right). \tag{7}$$

Here the arbitrary parameter $\varepsilon > 0$ can be seen to ensure the correct prescription for dealing with the pole at $k^2 = z^2$.

Finally, the complex integration in (7) can be expressed in the following manifestly real form:

$$\begin{aligned}
 p_1^*(x_0; s, x_1) = & -\frac{2 \exp(\beta(x_1 - x_0))}{\pi^2} \text{PV} \iint_{(\mathbb{R}^+)^2} dk dz \frac{z \cos |x_1| k \sin x_0 z}{k^2 - z^2} \\
 & \times \exp\left(-s \psi\left(\frac{k^2 + \beta^2}{2}\right)\right) \psi\left(\frac{z^2 + \beta^2}{2}\right) \\
 & - \frac{\exp(\beta(x_1 - x_0))}{\pi} \int_{\mathbb{R}^+} \sin |x_1| z \sin x_0 z \\
 & \times \exp\left(-s \psi\left(\frac{z^2 + \beta^2}{2}\right)\right) \psi\left(\frac{z^2 + \beta^2}{2}\right) dz,
 \end{aligned}$$

involving a principal value integral plus explicit half residue terms for the poles $k = \pm z$.

4. The iteration scheme and its convergence

The next theorem shows that (2) can be used to compute $p^*(x_0; s, x)$. We define a suitable L^∞ norm for functions $f(x_0; u, x)$:

$$\|f\|_\infty = \sup_{x_0 \geq 0} \left[\int_0^\infty \int_0^\infty |f(x_0; u, x)| du dx \right].$$

Theorem 2. *The sequence $(p_n^*)_{n \geq 1}$ converges exponentially in the L^∞ norm.*

Proof. First we find from (2) that

$$\begin{aligned}
 & p_{n+1}^*(x_0; s, x_1) - p_n^*(x_0; s, x_1) \\
 & = \int_0^\infty dy \int_0^s du p_1^*(x_0; s - u, y) [p_n^*(y; u, x_1) - p_{n-1}^*(y; u, x_1)];
 \end{aligned}$$

thus,

$$\|p_{n+1}^* - p_n^*\|_\infty \leq C \|p_n^* - p_{n-1}^*\|_\infty,$$

where

$$C = \sup_{x_0 \geq 0} \left[\int_0^\infty \int_0^\infty p_1^*(x_0; u, x) du dx \right].$$

The proof is based on the probabilistic interpretation of the constant C . By definition, $p_1^*(x_0; u, x)$ is the joint density of t_1^* and $X_{t_1^*}$; thus, we obtain

$$C = \sup_{x_0 \geq 0} P(t_1^* < +\infty, X_{t_1^*} > 0 \mid X_0 = x_0).$$

Next, using the fact that $\tilde{W}_{T^*} = 0$ (T^* is the first passage time of X_T and X is a continuous process) and the strong Markov property of the Brownian motion, we find that

$$\begin{aligned}
 C & = P(t_1^* < +\infty, \tilde{W}_{T_1^*} - \tilde{W}_{T^*} > 0 \mid \tilde{W}_0 = x_0) \\
 & = P(t_1^* < +\infty, W_{\delta^*} + \beta \delta^* > 0 \mid W_0 = 0),
 \end{aligned}$$

where the Brownian motion W_t is independent of T_t and $\delta^* = \delta^*(x_0) = T_{t_1^*} - T^*$ is the overshoot of the time change above T^* . Thus, we need to prove that

$$C = \sup_{x_0 \geq 0} P(t_1^* < +\infty, W_{\delta^*} + \beta\delta^* > 0 \mid W_0 = 0) < 1,$$

where $t_1^* = t_1^*(x_0)$, $\delta^* = \delta^*(x_0)$, and the Brownian motion W is independent of t_1^* and δ^* .

First we will consider the case where $\beta < 0$. In this case we obtain

$$\begin{aligned} &P(t_1^* < +\infty, W_{\Delta^*} + \beta\delta^* > 0 \mid W_0 = 0) \\ &\leq P(W_{\delta^*} + \beta\delta^* > 0 \mid W_0 = 0) \\ &= \int_0^\infty P(W_t + \beta t > 0 \mid W_0 = 0) P(\delta^* \in dt) \\ &< \int_0^\infty \frac{1}{2} P(\delta^* \in dt) \\ &= \frac{1}{2}, \end{aligned}$$

where we have used the facts that W_t is independent of the overshoot δ^* and $P(W_t + \beta t > 0 \mid W_0 = 0) < \frac{1}{2}$ for any t and any $\beta < 0$. Thus, in the case where the drift β is negative we obtain an estimate $C < \frac{1}{2}$.

The case where the drift β is positive is more complicated. We cannot use the same techniques as before, since the bound $P(W_t + \beta t > 0 \mid W_0 = 0) < \frac{1}{2}$ is no longer true: in fact, $P(W_t + \beta t > 0 \mid W_0 = 0)$ monotonically increases to 1 as $t \rightarrow \infty$.

First we will consider the case where x_0 is bounded away from 0: $x_0 \geq c > 0$. Then $x_0 + W_t + \beta t$ has a positive probability of escaping to $+\infty$ and never crossing the barrier at 0; thus,

$$\begin{aligned} P(t_1^* < +\infty, W_{\delta^*} + \beta\delta^* > 0 \mid W_0 = 0) &\leq P(t_1^*(x_0) < +\infty) \\ &< P(t_1^*(c) < +\infty) \\ &= 1 - \varepsilon_1(c). \end{aligned}$$

Now we need to consider the case where $x_0 \rightarrow 0^+$. The proof in this case is based on the following sequence of inequalities:

$$\begin{aligned} &P(t_1^* < +\infty, W_{\delta^*} + \beta\delta^* > 0 \mid W_0 = 0) \\ &\leq P(W_{\delta^*} + \beta\delta^* > 0 \mid W_0 = 0) \\ &= 1 - P(W_{\delta^*} + \beta\delta^* < 0 \mid W_0 = 0) \\ &= 1 - \int_0^\infty P(W_t + \beta t < 0 \mid W_0 = 0) P(\delta^* \in dt) \\ &< 1 - \int_0^\tau P(W_t + \beta t < 0 \mid W_0 = 0) P(\delta^* \in dt) \\ &< 1 - \int_0^\tau P(W_\tau + \beta\tau < 0 \mid W_0 = 0) P(\delta^* \in dt) \\ &= 1 - P(W_\tau + \beta\tau < 0 \mid W_0 = 0) P(\delta^* < \tau), \end{aligned} \tag{8}$$

where τ is any positive number and the last inequality is true since $P(W_t + \beta t < 0 \mid W_0 = 0)$ is a decreasing function of t .

Since $x_0 \rightarrow 0^+$, we also have $T^*(x_0) \rightarrow 0^+$ with probability 1. Since δ^* is the overshoot of T^* , and $T^* \rightarrow 0^+$ as $x_0 \rightarrow 0^+$, we see that the distribution of the overshoot $\delta^*(x_0)$ converges either to the distribution of the jumps of T_t if the time-change process T_t is a compound Poisson process or to the Dirac delta distribution at 0 if T_t has infinite activity of jumps. Therefore, in the case where T_t is a compound Poisson process with the jump measure $\nu(dx)$ we choose τ such that $\nu([0, \tau]) > 0$, and if T_t has infinite activity of jumps, we can take any $\tau > 0$. Then we obtain $\lim_{x_0 \rightarrow 0^+} P(\delta^*(x_0) < \tau) = \xi$, where $\xi = \nu([0, \tau])$ in the case of the compound Poisson process and $\xi = 1$ in the case of the process with infinite activity of jumps. Using (8), we find that, as $x_0 \rightarrow 0^+$,

$$P(t_1^*(x_0) < +\infty, W_{\delta^*} + \beta\delta^* > 0 \mid W_0 = 0) < 1 - P(W_\tau + \beta\tau < 0 \mid W_0 = 0)\xi < 1 - \varepsilon_2.$$

To summarize, we have proved that the function

$$P(x_0) = P(t_1^*(x_0) < +\infty, W_{\delta^*} + \beta\delta^* > 0 \mid W_0 = 0)$$

satisfies the following properties:

- for any $c > 0$, there exists $\varepsilon_1 = \varepsilon_1(c) > 0$ such that $P(x_0) < 1 - \varepsilon_1(c)$ for all $x_0 > c$,
- there exists $\varepsilon_2 > 0$ such that $\lim_{x_0 \rightarrow 0^+} P(x_0) < 1 - \varepsilon_2$.

Therefore, we conclude that there exists $\varepsilon > 0$ such that $P(x_0) < 1 - \varepsilon$ for all $x_0 \geq 0$; thus, $C < 1 - \varepsilon$. This completes the proof in the case where $\beta > 0$.

For a complementary point of view, the next result shows that the sequence $(t_i^*(x_0))_{i \geq 1}$ converges pathwise.

Theorem 3. *For any time-changed Brownian motion with Lévy subordinator T_t and Brownian motion with drift β , the sequence of stopping times $(t_i^*(x_0))_{i \geq 1}$ converges almost surely to t^* .*

Proof. If $t^* = \infty$ then certainly $t_i^* \rightarrow \infty$, so we suppose that $t^* < \infty$. In this case, if $t_i^* = t_{i+1}^*$ for some i , the sequence converges, and, thus, the only interesting case to analyze is if $t^* \neq t_i^*$ for all $i < \infty$. Then we have $t_1^* < t_2^* < \dots < t_i^* < \dots$. Correspondingly, we have an infinite sequence of excursion overjump intervals which do not overlap: let their endpoints be $T_{i-}^* := T_{t_i^*-} < T_{i+}^* := T_{t_i^*}$. The following observations lead to the conclusion.

1. By monotonicity and boundedness of the sequences (T_{i-}^*) and (T_{i+}^*) , $\lim_{i \rightarrow \infty} T_{i-}^* = \lim_{n \rightarrow \infty} T_{i+}^* = T_\infty$ exists.
2. $x_0 + W_{T_\infty} + \beta T_\infty = 0$ by the continuity of Brownian motion.
3. $\lim_{i \rightarrow \infty} t_i^* = t_\infty$ exists and $t_\infty \leq t^*$.
4. Jump times are totally inaccessible, so there is no time jump at time t_∞ almost surely. Hence, $T_{t_\infty} = T_\infty$.
5. $X_{t_\infty} = 0$ and so $t_\infty \geq t^*$; hence, $t_\infty = t^*$.

5. The VG model

The VG process described in Section 2 is the LSBM where the time-change process T_t is the Lévy process with jump measure $\mu(z) = (\nu z)^{-1}e^{-z/\nu}$ on $(0, \infty)$ and Laplace exponent $\psi_T(u) = (1/\nu) \log(1 + \nu u)$. In this section we take $b = 0$. This model has been widely used for option pricing, where it has been found to provide a better fit to market data than the Black–Scholes model, while preserving a degree of analytical tractability. The main result in this section reduces the two-dimensional integral representation for $p_1^*(x_0; s, x_1)$ given in (4) to a one-dimensional integral and leads to greatly simplified numerical computations.

Theorem 4. Define $\alpha = \sqrt{2/\nu + \beta^2}$. Then

$$\begin{aligned}
 p_1^*(x_0; s, x_1) = & \frac{\exp(\beta(x_1 - x_0))}{2\pi\nu} \int_{\mathbb{R}} \frac{(1 + i\nu z)^{-s/\nu}}{\sqrt{\beta^2 - 2iz}} \exp(-x_0\sqrt{\beta^2 - 2iz}) \\
 & \times (\exp(|x_1|\sqrt{\beta^2 - 2iz})Ei(-|x_1|(\alpha + \sqrt{\beta^2 - 2iz})) \\
 & - \exp(-|x_1|\sqrt{\beta^2 - 2iz})Ei(-|x_1|(\alpha - \sqrt{\beta^2 - 2iz}))) dz, \tag{9}
 \end{aligned}$$

where $Ei(x)$ is the exponential integral function (see [11, p. 883]).

Proof. Consider the function $I(z_1)$ which represents the outer integral in (4):

$$I(z_1) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\log(1 + i\nu z_1) - \log(1 + i\nu z_2)}{i(z_1 - z_2)} \frac{\exp(-|x_1|\sqrt{\beta^2 - 2iz_2})}{\sqrt{\beta^2 - 2iz_2}} dz_2.$$

First we perform the change of variables $u = i\sqrt{\beta^2 - 2iz_2}$ and obtain

$$I(z_1) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\log(1 + (\nu/2)(u^2 + \beta^2)) - \log(1 + i\nu z_1)}{u^2 + \beta^2 - 2iz_1} \exp(i|x_1|u) du,$$

where the contour L obtained from \mathbb{R} under the map $z_2 \rightarrow u = i\sqrt{\beta^2 - 2iz_2}$ is transformed into the contour \mathbb{R} (this is justified since the integrand is an analytic function in this region for any z_1). To complete the proof, we separate the logarithms, i.e.

$$\log\left(1 + \frac{\nu}{2}(u^2 + \beta^2)\right) - \log(1 + i\nu z_1) = \log(u + i\alpha) + \log(u - i\alpha) - \log\left(\frac{2}{\nu} + 2iz_1\right),$$

and use the partial fractions decomposition

$$\frac{1}{u^2 + \beta^2 - 2iz_1} = \frac{1}{2i\sqrt{\beta^2 - 2iz_1}} \left(\frac{1}{u - i\sqrt{\beta^2 - 2iz_1}} - \frac{1}{u + i\sqrt{\beta^2 - 2iz_1}} \right)$$

to obtain six integrals, which can be computed by shifting the contours of integration and using the following Fourier transform formulae (see [11, Formulae 6.232, p. 639]):

$$\begin{aligned}
 \int_{i\epsilon + \mathbb{R}} \log\left(1 + \frac{iy}{b}\right) e^{ixy} \frac{dy}{y} &= -2\pi i Ei(-bx), & b > 0, \\
 \int_{i(b+\epsilon) + \mathbb{R}} \log\left(\frac{iy}{b} - 1\right) e^{ixy} \frac{dy}{y} &= -2\pi i Ei(bx), & b > 0.
 \end{aligned}$$

Remark. Using the change of variables $u = i\sqrt{\beta^2 - 2iz}$ and simplifying the expression, we can obtain a simpler formula for $p_1^*(x_0; s, x_1)$:

$$p_1^*(x_0; s, x_1) = \exp(\beta(x_1 - x_0)) \frac{(v/2)^{-s/v-1}}{2\pi i} \int_{\mathbb{R}} (\alpha^2 + y^2)^{-s/v} \sin(x_0 y) \exp(i|x_1|y) \times Ei(-|x_1|(\alpha + iy)) dy.$$

Applying the Plancherel formula to the above expression gives us the following representation for p_1^* :

$$p_1^*(x_0; s, x_1) = \exp(\beta(x_1 - x_0) - \alpha(x_0 + |x_1|)) \frac{(v\alpha^2/2)^{-s/v}}{\sqrt{\pi} v(x_0 + |x_1|)} \frac{\Gamma(s/v + 1/2)}{\Gamma(s/v + 1)} + \sqrt{\frac{2\alpha^3}{\pi}} \exp(\beta(x_1 - x_0) - \alpha|x_1|) \times \frac{(\alpha v)^{-s/v-1}}{\Gamma(s/v)} \int_0^\infty u^{s/v-1/2} K_{s/v-1/2}(\alpha u) f(x_0, x_1; u) du,$$

where

$$f(x_0, x_1; u) = \frac{\exp(-\alpha(u + x_0))}{u + x_0 + |x_1|} - \operatorname{sgn}(u - x_0) \frac{\exp(-\alpha|u - x_0|)}{|u - x_0| + |x_1|} - 2 \frac{\exp(-\alpha(u + x_0))}{x_0 + |x_1|}.$$

The above expression is useful for computations when s is small. In particular, when $s = 0$, we find that

$$p_1^*(x_0; 0, x_1) = \frac{\exp(\beta(x_1 - x_0) - \alpha(x_0 + |x_1|))}{v(x_0 + |x_1|)}. \tag{10}$$

6. Numerical implementation for the VG model

The algorithm for computing the functions $p^*(x_0; s, x)$ and $p^*(x_0; s)$ can be summarized as follows.

1. Choose the discretization step sizes δ_x and δ_t and the discretization intervals $[-X, X]$ and $[0, T]$. The grid points are $t_i = i\delta_t$, $1 \leq i \leq N_s$, and $x_j = (j + \frac{1}{2})\delta_x$, $-N_x \leq j \leq N_x$.
2. Compute the three-dimensional array $p_1^*(x_i; t_j, x_k)$. For $j > 0$, use (9) and, for $j = 0$, use explicit formula (10).
3. Iterate (2) or (3). This step can be considerably accelerated if the convolution in the u -variable is done using FFT methods. We used the midpoint rule for integration in the y - and u -variables.

Theorem 2 implies that step 3 in the above algorithm has to be repeated only a few times. In practice, we found that 3–4 iterations were usually enough. An important empirical fact is that the above algorithm works quite well with just a few discretization points in the x -variable. We found that if we used a nonlinear grid (which places more points x_i near $x = 0$) then the above algorithm produced reasonable results with values of N_x as small as 10 or 20.

We compared our algorithm for the PDF $p^*(x_0; s)$ to a finite-difference method that was implemented as follows. First we approximated the first passage time by its discrete counterpart:

$$\hat{t}^* = \hat{t}^*(x) = \min\{t_i : X_{t_i} < 0 \mid X_0 = x\},$$

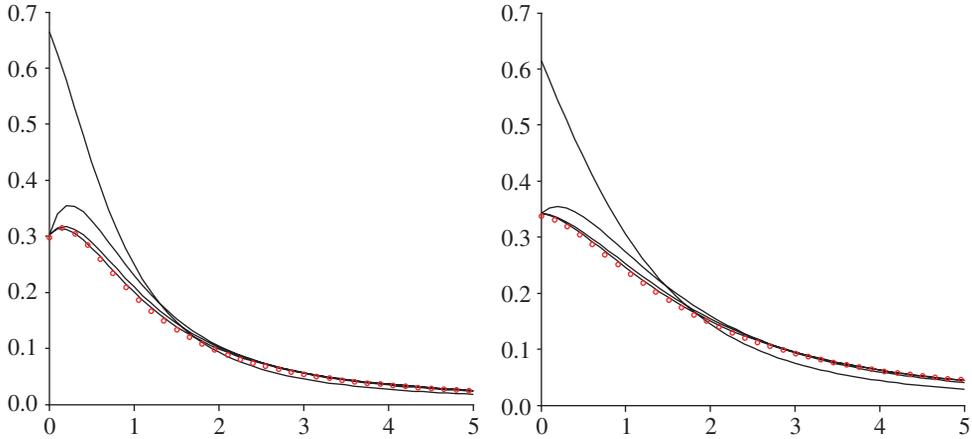


FIGURE 2: The density of the first passage time for the two sets of parameters. The circles show the ‘exact’ result and the three solid lines show the first three approximations.

where $t_i = i\delta_t$, $0 \leq i \leq n_t$, is the discretization of the interval $[0, T]$. The probabilities $f_i(x) = P(\hat{t}^* > t_i \mid X_0 = x)$ satisfy the iteration

$$f_{i+1}(x) = \mathbf{1}_{\{x>0\}} \int_{\mathbb{R}} p(\delta_t, x - y) f_i(y) dy, \quad i \geq 1, \tag{11}$$

with $f_0(x) = \mathbf{1}_{\{x>0\}}$, and can be computed numerically with the following steps.

1. Discretize the space variables $x = i\delta_x$ and $y = j\delta_x$, $0 < i, j < n_x$.
2. Compute the array of transitional probabilities $\hat{p}_i = p(\delta_t, x_i)$, and normalize \hat{p}_0 so that $\sum_i \hat{p}_i = 1$.
3. Use the convolution (based on the FFT) to iterate (11) n_t times.
4. Compute the approximation of the first passage time density

$$\hat{p}^*\left(x, t_i + \frac{\delta_t}{2}\right) = \frac{f_{i+1}(x) - f_i(x)}{\delta_t}.$$

The big advantage of this method is that it is explicit and unconditionally stable: we can choose the number of discretization points in x -space and t -space independently. This is not true in general explicit finite-difference methods, where we would solve the Fokker–Planck equation by discretizing the Markov generator and the time derivative, since δ_t and δ_x have to lie in a certain subset in order for the methods to be stable.

Figure 2 summarizes the numerical results for the PDF $p^*(x_0; s)$ over the time interval $[0, 5]$ for the VG model with the following two sets of parameters.

Set I. $x_0 = 0.5$, $\beta = 0.2$, and $\nu = 1$.

Set II. $x_0 = 0.5$, $\beta = -0.2$, and $\nu = 2$.

The number of grid points used was $N_t = 50$ and $N_x = 10$. The circles correspond to the solution obtained by a high-resolution finite-difference PIDE method as described above

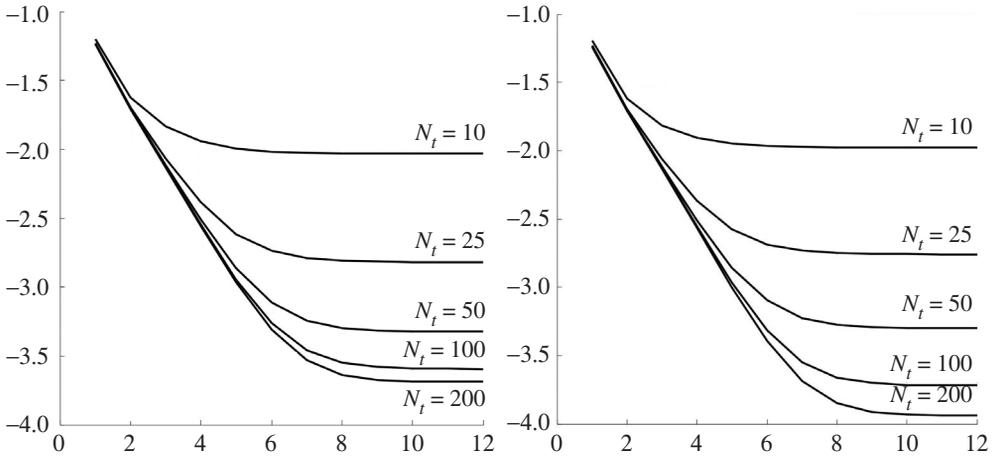


FIGURE 3: The error, $\log_{10}(\|p^* - p_i^*\|_{L_1})$, for the new approach plotted against the number of iterations. The left-hand plot has $N_x = 10$, the right-hand plot has $N_x = 20$, and $N_t \in \{10, 25, 50, 100, 200\}$.

TABLE 1: Computation time (seconds) for the new approach.

		N_t				
		N_x	10	25	50	100
Precomputing time	10	0.0313	0.0259	0.0324	0.0461	0.0687
Each iteration		0.0006	0.0008	0.0011	0.0021	0.0046
Precomputing time	20	0.0645	0.0612	0.0745	0.0868	0.1298
Each iteration		0.0037	0.0045	0.0066	0.0120	0.0269

(with $n_t = 1000$ and $n_x = 10\,000$), and the black lines show successive iterations $p_i^*(x_0, t)$ converging to $p^*(x_0, t)$. As we see, three iterations of (3) provide a visually acceptable accuracy in a running time of less than 0.1 seconds (on a 2.5GHz laptop).

Figure 3 illustrates the convergence of our method and Table 1 shows the computation times (on the same 2.5GHz laptop). We used the parameters of set II for the VG process, and the PIDE method with $n_t = 1000$ and $n_x = 10\,000$ to compute the ‘exact’ solution $p^*(x_0, t)$. Figure 3 shows the \log_{10} of the error

$$\|p^* - p_i^*\|_{L_1} = \int_0^T |p^*(x_0, t) - p_i^*(x_0, t)| dt$$

on the vertical axis and the number of iterations on the horizontal axis; different curves correspond to a different number of discretization points in t -space. The number of discretization points in x -space is fixed at $N_x = 10$ for the left-hand plot and $N_x = 20$ for the right-hand plot. We see that initially the error decreases exponentially and then flattens out. The flattening indicates that our method converges to the wrong target (which is to be expected since there is always a discretization error coming from N_x and N_t being finite). However, increasing N_t and N_x brings us closer to the ‘target’. In Table 1 we show the precomputing time needed

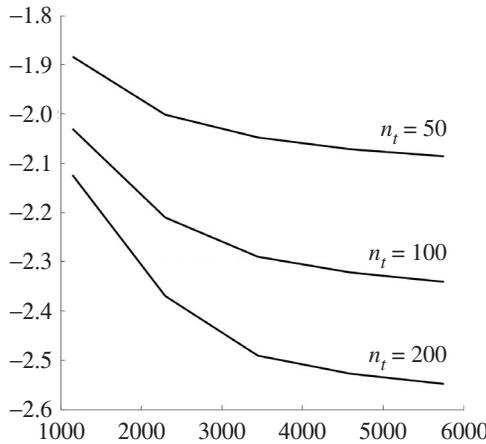


FIGURE 4: The error, $\log_{10}(\|p^* - \hat{p}^*\|_{L_1})$, for the finite-difference method is plotted against the number of grid points n_x for the values $n_t = 50, 100$, and 200 .

TABLE 2: Computation time (seconds) for the finite-difference approach.

N_t	n_x				
	1150	2300	3450	4600	5750
50	0.0756	0.2935	0.9582	1.8757	2.9967
100	0.1456	0.5821	1.9397	3.7409	6.0026
200	0.2870	1.1478	3.8833	7.4768	11.9935

to compute the three-dimensional array $p_1^*(x_i; t_j, x_k)$ and the time needed to perform each iteration (3).

To put these results into perspective, Figure 4 and Table 2 show similar results for the finite-difference method. In Figure 4 we show the same logarithm of the error on the vertical axis and the number of discretization points n_x on the horizontal axis. Different curves correspond to $n_t \in \{50, 100, 200\}$. The running time presented in Table 2 includes only the time needed to perform n_t convolutions (11) using the FFT. As we can see by comparing Figures 3 and 4, even with a relatively large number of discretization points, $n_x = 5750$ and $n_t = 200$, the accuracy produced by a finite-difference method is an order of magnitude worse than the accuracy produced by our method (with much fewer discretization points). Moreover, we can see that the running times of the PIDE method are consistently orders of magnitude larger.

7. Conclusions

First passage times are an important modeling tool in finance and other areas of applied mathematics. The main result of this paper is the theoretical connection between two distinct notions of first passage time that arise for LSBMs. This relation leads to a new way to compute true first passage for these processes that is apparently less expensive than finite-difference methods for a given level of accuracy. Our paper opens up many avenues for further theoretical

and numerical work. For example, the methods we describe are certainly applicable for a much broader class of time-changed Brownian motions and time-changed diffusions. Finally, it will be worthwhile to explore the use of the first passage of the second kind as a modeling alternative to the usual first passage time.

Acknowledgement

The authors would like to thank Professor Martin Barlow for providing the proof of Theorem 4.

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