Canad. Math. Bull. Vol. **48** (4), 2005 pp. 614–621

On Finite-to-One Maps

H. Murat Tuncali and Vesko Valov

Abstract. Let $f\colon X \to Y$ be a σ-perfect k-dimensional surjective map of metrizable spaces such that $\dim Y \leq m$. It is shown that for every positive integer p with $p \leq m + k + 1$ there exists a dense G_δ-subset $\mathcal{H}(k, m, p)$ of $C(X, \mathbb{I}^{k+p})$ with the source limitation topology such that each fiber of $f\triangle g$, $g \in \mathcal{H}(k, m, p)$, contains at most $\max\{k + m - p + 2, 1\}$ points. This result provides a proof the following conjectures of S. Bogatyi, V. Fedorchuk and J. van Mill. Let $f\colon X \to Y$ be a k-dimensional map between compact metric spaces with $\dim Y \leq m$. Then: (1) there exists a map $h\colon X \to \mathbb{I}^{m+2k}$ such that $f\triangle h\colon X \to Y \times \mathbb{I}^{m+2k}$ is 2-to-one provided $k \geq 1$; (2) there exists a map $h\colon X \to \mathbb{I}^{m+k+1}$ such that $f\triangle h\colon X \to Y \times \mathbb{I}^{m+k+1}$ is $(k + 1)$-to-one.

1 Introduction

This paper is inspired by the following hypotheses of S. Bogatyi, V. Fedorchuk and J. van Mill [1].

Let $f\colon X \to Y$ be a k-dimensional map between compact metric spaces with $\dim Y \leq m$. Then:

(1) there exists a map $h\colon X \to \mathbb{I}^{m+2k}$ such that $f\triangle h\colon X \to Y \times \mathbb{I}^{m+2k}$ is 2-to-one provided $k \geq 1$;
(2) there exists a map $h\colon X \to \mathbb{I}^{m+k+1}$ such that $f\triangle h\colon X \to Y \times \mathbb{I}^{m+k+1}$ is $(k + 1)$-to-one.

The next theorem provides a solution to these two problems.

Theorem 1.1 *Let $f\colon X \to Y$ be a σ-perfect k-dimensional surjective map of metrizable spaces such that $\dim Y \leq m$. For every integer $p \geq 1$, let $\mathcal{H}(k, m, p)$ consist of all $g \in C(X, \mathbb{I}^{k+p})$ such that each fiber of the map $f\triangle g\colon X \to Y \times \mathbb{I}^{k+p}$ contains at most $\max\{k + m - p + 2, 1\}$ points. Then $\mathcal{H}(k, m, p)$ is dense and G_δ in $C(X, \mathbb{I}^{k+p})$ with respect to the source limitation topology.*

Observe that stronger forms of Hypothesis 1 and Hypothesis 2 follow from Theorem 1.1 when $p = m + k$, respectively k arbitrary and $p = m + 1$. Moreover, if $p = m+k+1$, then $\mathcal{H}(k, m, p)$ consists of one-to-one maps and Theorem 1.1 implies [8, Theorem 7.3] and the metrizable case of [10, Theorem 1.1(i)]. When both X and Y are compact, $k = 0$ and $p = 1$, Theorem 1.1 was established by M. Levin and W. Lewis [5, Proposition 4.4]. This result is one of the ingredients of our proof, another is a selection theorem proven by V. Gutev and the second author [3, Theorem 1.2].

Received by the editors September 27, 2002; revised March 3, 2003.

The first author was partially supported by his NSERC grant RGPIN 141066-04. The second author was partially supported by his NSERC grant RGPIN 261914-03.

AMS subject classification: Primary: 54F45; secondary: 55M10, 54C65.

Keywords: Finite-to-one maps, dimension, set-valued maps.

Recall that, $\dim f = \sup\{\dim f^{-1}(y) : y \in Y\}$ is the dimension of f. We say that a surjective map $f\colon X \to Y$ is called σ-perfect if X is the union of countably many closed sets X_i such that each restriction $f|X_i\colon X_i \to f(X_i)$ is a perfect map. By $C(X, M)$ we denote the set of all continuous maps from X into M. If (M, d) is a metric space, then the source limitation topology on $C(X, M)$ is defined in the following way: a subset $U \subset C(X, M)$ is open in $C(X, M)$ with respect to the source limitation topology provided for every $g \in U$ there exists a continuous function $\alpha\colon X \to (0, \infty)$ such that $\overline{B}(g, \alpha) \subset U$. Here, $\overline{B}(g, \alpha)$ denotes the set $\{h \in C(X, M) : d(g(x), h(x)) \leq \alpha(x)$ for each $x \in X\}$. The source limitation topology does not depend on the metric d if X is paracompact [4], and $C(X, M)$ with this topology has the Baire property provided (M, d) is a complete metric space [7]. Moreover, if d is a bounded metric on M and X is compact, then the source limitation topology coincides with the uniform convergence topology generated by d.

The paper is organized as follows. In Section 2 we prove the special case of Theorem 1.1 when both X and Y are compact. The final proof is accomplished in Section 3.

All maps are assumed to be continuous and all function spaces, if not explicitly stated otherwise, are equipped with the source limitation topology. Everywhere in this paper, by an n-to-one map, where $n \geq 1$ is an integer, we mean a map with all fibers containing at most n points.

2 Proof of Theorem 1.1: The Compact Case

Let ω be an open cover of the space X, $m \in \mathbb{N}$ and $H \subset X$. We say that a map $g\colon H \to Z$ is an (m, ω)-map if every $z \in g(H)$ has a neighborhood V_z in Z such that $g^{-1}(V_z)$ can be covered by m elements of ω. We also agree to denote by $\operatorname{cov}(M)$ the family of all open covers of M.

Suppose $f\colon X \to Y$ is a surjective map, $\omega \in \operatorname{cov}(X)$ and $n, m \in \mathbb{N}$. Then we denote by $C(X, Y \times \mathbb{I}^n, f)$ the set of all maps $h\colon X \to Y \times \mathbb{I}^n$ such that $\pi_Y \circ h = f$, where $\pi_Y\colon Y \times \mathbb{I}^p \to Y$ is the projection. For any $K \subset X$, $C_{(m,\omega)}(X|K, Y \times \mathbb{I}^n, f)$ stands for the set of all $h \in C(X, Y \times \mathbb{I}^n, f)$ with $h|K$ being an (m, ω)-map, and $C_{(m,\omega)}(X|K, \mathbb{I}^n)$ consists of all $g \in C(X, \mathbb{I}^n)$ such that $f \triangle g \in C_{(m,\omega)}(X|K, Y \times \mathbb{I}^n, f)$. In case $K = X$ we simply write $C_{(m,\omega)}(X, Y \times \mathbb{I}^n, f)$ (resp., $C_{(m,\omega)}(X, \mathbb{I}^n)$) instead of $C_{(m,\omega)}(X|X, Y \times \mathbb{I}^n, f)$ (resp., $C_{(m,\omega)}(X|X, \mathbb{I}^n)$).

Proposition 2.1 *Let $f\colon X \to Y$ be a surjection between metrizable spaces and $\{X_i\}$ a sequence of closed subsets of X such that each restriction $f|X_i$ is a perfect map. Then for any positive integers m and p the set*

$$\mathcal{A}(m, p) = \{g \in C(X, \mathbb{I}^p) : (f \triangle g)\Big(\bigcup_{i=1}^{\infty} X_i\Big) \text{ is } m\text{-to-one}\}$$

is G_δ in $C(X, \mathbb{I}^p)$.

Proof We need a few lemmas. In all these lemmas we suppose that X, Y and f are as in Proposition 2.1 and $\omega \in \operatorname{cov}(X)$.

https://doi.org/10.4153/CMB-2005-057-x Published online by Cambridge University Press

Lemma 2.2 *Let f be a perfect map and $g \in C_{(m,\omega)}(X|f^{-1}(y), \mathbb{I}^p)$ for some $y \in Y$. Then there exists a neighborhood U_y of y in Y such that the restriction $g|f^{-1}(U_y)$ is an (m,ω)-map.*

Proof Obviously, $g \in C_{(m,\omega)}(X|f^{-1}(y), \mathbb{I}^p)$ implies that $g|f^{-1}(y)$ is an (m,ω)-map. Hence, for every $x \in f^{-1}(y)$ there exists an open neighborhood $V_{g(x)}$ of $g(x)$ in $\mathbb{I}^p$ such that $g^{-1}(V_{g(x)}) \cap f^{-1}(y)$ can be covered by m elements of ω whose union is denoted by W_x. Therefore, for every $x \in f^{-1}(y)$ we have $(f\triangle g)^{-1}(f(x), g(x)) = f^{-1}(y) \cap g^{-1}(g(x)) \subset W_x$ and, since $f\triangle g$ is a closed map, there exists an open neighborhood $H_x = U_y^x \times G_x$ of $(y, g(x))$ in $Y \times \mathbb{I}^p$ with $S_x = (f\triangle g)^{-1}(H_x) \subset W_x$. Next, choose finitely many points $x(i) \in f^{-1}(y)$, $i = 1, 2, \dots n$, such that $f^{-1}(y) \subset \bigcup_{i=1}^{i=n} S_{x(i)}$. Using that f is a closed map we can find a neighborhood U_y of y in Y such that $U_y \subset \bigcap_{i=1}^{i=n} U_y^{x(i)}$ and $f^{-1}(U_y) \subset \bigcup_{i=1}^{i=n} S_{x(i)}$. Let us show that $g|f^{-1}(U_y)$ is an (m,ω)-map. Indeed, if $z \in f^{-1}(U_y)$, then $z \in S_{x(j)}$ for some j and $g(z) \in G_{x(j)}$ because $S_{x(j)} = f^{-1}(U_y^{x(j)}) \cap g^{-1}(G_{x(j)})$. Consequently, $f^{-1}(U_y) \cap g^{-1}(G_{x(j)}) \subset S_{x(j)} \subset W_{x(j)}$. Therefore, $G_{x(j)}$ is a neighborhood of $g(z)$ such that $f^{-1}(U_y) \cap g^{-1}(G_{x(j)})$ is covered by m elements of ω. ∎

Corollary 2.3 *If f is perfect and $g \in C_{(m,\omega)}(X|f^{-1}(y), \mathbb{I}^p)$ for every $y \in Y$, then $g \in C_{(m,\omega)}(X, \mathbb{I}^p)$.*

Proof By Lemma 2.2, for any $x \in X$ there exists a neighborhood U_y of $y = f(x)$ in Y such that $g|f^{-1}(U_y)$ is an (m,ω)-map. So, we can find a neighborhood G_x of $g(x)$ in $\mathbb{I}^p$ with $f^{-1}(U_y) \cap g^{-1}(G_x)$ being covered by m elements of ω. But $f^{-1}(U_y) \cap g^{-1}(G_x)$ equals $(f\triangle g)^{-1}(U_y \times G_x)$. Hence, $f\triangle g$ is an (m,ω)-map. ∎

Lemma 2.4 *For any closed $K \subset X$ the set $C_{(m,\omega)}(X|K, \mathbb{I}^p)$ is open in $C(X, \mathbb{I}^p)$ provided f is perfect.*

Proof The proof of this lemma follows the same scheme as the proof of [9, Lemma 2.5], we now apply Lemma 2.2 instead of [9, Lemma 2.3]. ∎

Let us finish the proof of Proposition 2.1. We can suppose that the sequence $\{X_i\}$ is increasing and fix a sequence $\{\omega_i\} \subset \operatorname{cov}(X)$ such that $\operatorname{mesh}(\omega_i) \le i^{-1}$ for every i. Denote by $\pi_i \colon C(X, \mathbb{I}^p) \to C(X_i, \mathbb{I}^p)$, $\pi_i(g) = g|X_i$, the restriction maps. By Lemma 2.4, every set $\mathcal{B}_{ij}$, $i, j \in \mathbb{N}$, consisting of all $h \in C(X_i, \mathbb{I}^p)$ with $(f|X_i)\triangle h$ being an (m,ω_j)-map is open in $C(X_i, \mathbb{I}^p)$. So are the sets $\mathcal{A}_{ij} = (\pi_i)^{-1}(\mathcal{B}_{ij})$ in $C(X, \mathbb{I}^p)$, because each π_i is continuous. It is easily seen that the intersection of all $\mathcal{A}_{ij}$ is exactly the set $\mathcal{A}(m,p)$. Hence, $\mathcal{A}(m,p)$ is a G_δ-subset of $C(X, \mathbb{I}^p)$. ∎

Corollary 2.5 *Theorem* 1.1 *follows from the validity of its special case when $p \le k + m + 1$*

https://doi.org/10.4153/CMB-2005-057-x Published online by Cambridge University Press

Proof Suppose $p = m + k + 1 + n$, where $n \geq 1$. Then $C(X, \mathbb{I}^{k+p})$ is homeomorphic to the product $C(X, \mathbb{I}^{2k+m+1}) \times C(X, \mathbb{I}^n)$; let $\pi\colon C(X, \mathbb{I}^{k+p}) \to C(X, \mathbb{I}^{2k+m+1})$ denote the projection. According to our assumption, the set $\mathcal{A} = \{h \in C(X, \mathbb{I}^{2k+m+1}) : f \triangle h \text{ is one-to-one}\}$ is dense in $C(X, \mathbb{I}^{2k+m+1})$, and so is the set $\pi^{-1}(\mathcal{A})$ in $C(X, \mathbb{I}^{k+p})$. Since $\max\{k + m - p + 2, 1\} = 1$, $\mathcal{H}(k, m, p)$ consists of one-to-one maps. Hence $\pi^{-1}(\mathcal{A}) \subset \mathcal{H}(k, m, p)$. The last inclusion yields that $\mathcal{H}(k, m, p)$ is dense in $C(X, \mathbb{I}^{k+p})$. It only remains to observe that, by Proposition 2.1, $\mathcal{H}(k, m, p)$ is G_δ in $C(X, \mathbb{I}^{k+p})$. ■

The remaining part of this section is devoted to the proof of the next proposition which, in combination with Proposition 2.1 and Corollary 2.5, provides a proof of Theorem 1.1 when both X and Y are compact.

Proposition 2.6 *Under the hypotheses of Theorem* 1.1*, the set $\mathcal{H}(k, m, p)$ is dense in $C(X, \mathbb{I}^{k+p})$ provided both X and Y are compact metric spaces and $p \leq m + k + 1$.*

Proof Let us first show that the proof of this proposition can be reduced to the proof of its special case when $k = 0$. Indeed, suppose Proposition 2.6 is valid for $k = 0$ and every positive p with $p \leq m + 1$. Fix $\epsilon > 0$ and $h \in C(X, \mathbb{I}^{k+p})$, where $k \geq 0$ and $1 \leq p \leq m + k + 1$. Then $h = h_1 \triangle h_2$ with $h_1 \in C(X, \mathbb{I}^k)$ and $h_2 \in C(X, \mathbb{I}^p)$. By [8], there exists $g_1 \in C(X, \mathbb{I}^k)$ such that $f \triangle g_1 \colon X \to Y \times \mathbb{I}^k$ is a 0-dimensional map and g_1 is $\frac{\epsilon}{2}$-close to h_1. Then, applying our assumption to the map $f \triangle g_1$, we can find $g_2 \in C(X, \mathbb{I}^p)$ which is $\frac{\epsilon}{2}$-close to h_2 and such that $(f \triangle g_1) \triangle g_2$ is a $(k + m - p + 2)$-to-one map. It remains only to observe that the map $g = g_1 \triangle g_2 \in C(X, \mathbb{I}^{k+p})$ is ϵ-close to h and $f \triangle g$ is a $(k + m - p + 2)$-to-one map.

So, the following statement will complete the proof:

$\Sigma(m, p)$: Let $f\colon X \to Y$ be a 0-dimensional surjection between compact metrizable spaces with $\dim Y \leq m$. Then for every positive integer $p \leq m + 1$, the set $\mathcal{H}(0, m, p) = \{g \in C(X, \mathbb{I}^p) : f \triangle g \text{ is } (m - p + 2)\text{-to-one}\}$ is dense in $C(X, \mathbb{I}^p)$.

We are going to prove $\Sigma(m, p)$ by induction with respect to p. The statement $\Sigma(m, 1)$ was proved by M. Levin and W. Lewis [5, Proposition 4.4]. Assume that $\Sigma(m, p)$ holds for any $p \leq n$ and $m \geq p - 1$, where $n \geq 1$, and let us prove the validity of $\Sigma(m, n+1)$. We need to show that for fixed m with $n \leq m$, $h^* \in C(X, \mathbb{I}^{n+1})$ and $\epsilon > 0$ there exists $g^* \in \mathcal{H}(0, m, n+1)$ which is ϵ-close to h^*. To this end, we represent h^* as $h_1^* \triangle h_2^*$, where $h_1^* \in C(X, \mathbb{I}^n)$ and $h_2^* \in C(X, \mathbb{I})$. Next, we use an idea from the proof of [1, Theorem 5]. By Urysohn's decomposition theorem (see [2, Theorem 1.5.7]), there exists an F_σ-subset $Y_0 \subset Y$ such that $\dim Y_0 \leq m - 1$ and $\dim(Y \backslash Y_0) = 0$. Let Y_0 be the union of an increasing sequence of closed sets $Y_i \subset Y$, $i \geq 1$, and $X_i = f^{-1}(Y_i)$, $i \geq 0$. Obviously, $\dim Y_i \leq m - 1$, $i \geq 1$ and $n \leq (m - 1) + 1$. Thus, according to our inductive hypothesis we can apply $\Sigma(m - 1, n)$ for the maps $f_i = f|X_i\colon X_i \to Y_i$, $i \geq 1$, to conclude that each set $\mathcal{B}_i = \{g \in C(X_i, \mathbb{I}^n) : f_i \triangle g \text{ is } (m - n + 1)\text{-to-one}\}$ is dense in $C(X_i, \mathbb{I}^n)$. Also, the sets $\mathcal{A}_i = (\pi_i)^{-1}(\mathcal{B}_i)$ are dense in $C(X, \mathbb{I}^n)$ because the restriction maps $\pi_i\colon C(X, \mathbb{I}^n) \to C(X_i, \mathbb{I}^n)$ are open and surjective. On the other hand, by Proposition 2.1, each of the sets $\mathcal{A}_i$ is G_δ in $C(X, \mathbb{I}^n)$. Hence, the set $\mathcal{A}_0 = \bigcap_{i=1}^{\infty} \mathcal{A}_i$ is dense and G_δ in $C(X, \mathbb{I}^n)$ and obviously, it consists of all maps $q \in C(X, \mathbb{I}^n)$ such that $(f \triangle q)|X_0$ is $(m-n+1)$-to-one. Therefore, there exists $g_1^* \in \mathcal{A}_0$ which is $\frac{\epsilon}{2}$-close to h_1^*. Consider the map $f \triangle g_1^* \colon X \to Y \times \mathbb{I}^n$ and the set $D = \{z \in Y \times \mathbb{I}^n : |(f \triangle g_1^*)^{-1}(z)| \geq m - n + 2\}$. By [2, Lemma 4.3.7], $D \subset Y \times \mathbb{I}^n$ is F_σ. Then $H = \pi_Y(D)$ does not meet Y_0 because of the choice of g_1^*, where $\pi_Y \colon Y \times \mathbb{I}^n \to Y$ denotes the projection. Hence, H is 0-dimensional and σ-compact, so is the set $K = f^{-1}(H)$ (0-dimensionality of K follows by the Hurewicz theorem on dimension-lowering mappings, see [2, Theorem 1.12.4]). Representing H as the union of an increasing sequence of closed sets $H_i \subset Y$ and applying $\Sigma(0, 1)$ for any of the maps $f|K_i$, where $K_i = f^{-1}(H_i)$, we can conclude (as we did for the set $\mathcal{A}_0$ above) that the set $\mathcal{F}$ of all maps $q \in C(X, \mathbb{I})$ with $(f \triangle q)|K$ one-to-one is dense and G_δ in $C(X, \mathbb{I})$. Consequently, there exists $g_2^* \in \mathcal{F}$ which is $\frac{\epsilon}{2}$-close to h_2^*. Then $g^* = g_1^* \triangle g_2^*$ is ϵ-close to h^*. It follows from the definition of the set D and the choice of the maps g_1^*, g_2^* that $f \triangle g^*$ is $(m - n + 1)$-to-one, *i.e.,* $g^* \in \mathcal{H}(0, m, n + 1)$. This completes the induction. ■

3 Proof of Theorem 1.1: The General Case

By Corollary 2.5, we can assume that $p \leq m + k + 1$. Representing X as the union of an increasing sequence of closed sets $X_i \subset X$ such that each $f|X_i$ is perfect and using that all restriction maps $\pi_i \colon C(X, \mathbb{I}^{k+p}) \to C(X_i, \mathbb{I}^{k+p})$ are open and surjective, we can show that the proof of Theorem 1.1 is reduced to the case when f is a perfect map (see the proof of Proposition 2.6 for a similar situation). So, everywhere below we can suppose that the map f from Theorem 1.1 is perfect.

Another reduction of Theorem 1.1 is provided by the following observation. By Lemma 2.4, the set $\mathcal{H}_\omega(k, m, p) = C_{(m+k-p+2,\omega)}(X, \mathbb{I}^{k+p})$ is open in $C(X, \mathbb{I}^{k+p})$ for every $\omega \in \operatorname{cov}(X)$. Since $\mathcal{H}(k, m, p) = \bigcap_{i=1}^{\infty} \mathcal{H}_{\omega_i}(k, m, p)$, where $\{\omega_i\} \subset \operatorname{cov}(X)$ is a sequence with $\operatorname{mesh}(\omega_i) < 2^{-i}$, it suffices to show that $\mathcal{H}_\omega(k, m, p)$ is dense in $C(X, \mathbb{I}^{k+p})$ for every $\omega \in \operatorname{cov}(X)$. The remaining part of this section is devoted to the proof of this fact. We need a few lemmas. In all these lemmas we suppose that X, Y, f and the numbers m, k, p are as in Theorem 1.1 with f perfect. We also fix $\omega \in \operatorname{cov}(X)$.

Lemma 3.1 *If $C(X, \mathbb{I}^{k+p})$ is equipped with the uniform convergence topology, then the set-valued map ψ from Y into $C(X, \mathbb{I}^{k+p})$, defined by the formula*

$$\psi(y) = C(X, \mathbb{I}^{k+p}) \backslash C_{(m+k-p+2,\omega)}(X|f^{-1}(y), \mathbb{I}^{k+p}),$$

has a closed graph.

Proof We can prove this lemma by following the arguments from the proof of [9, Lemma 2.6], but in the present situation there exists a shorter proof.

Let $G = \bigcup\{y \times \psi(y) : y \in Y\} \subset Y \times C(X, \mathbb{I}^{k+p})$ be the graph of ψ and $\{(y_n, g_n)\}$ a sequence in G converging to $(y_0, g_0) \in Y \times C(X, \mathbb{I}^{k+p})$. It suffices to show that $(y_0, g_0) \in G$. Assuming $(y_0, g_0) \notin G$, we conclude that $g_0 \notin \psi(y_0)$,

https://doi.org/10.4153/CMB-2005-057-x Published online by Cambridge University Press

so $g_0 \in C_{(m+k-p+2,\omega)}(X|f^{-1}(y_0), \mathbb{I}^{k+p})$. Then, by Lemma 2.2, there exists a neighborhood U of y_0 in Y with $g_0|f^{-1}(U)$ being an $(m+k-p+2,\omega)$-map. We can suppose that $f^{-1}(y_n) \subset f^{-1}(U)$ for every n because $\lim y_n = y_0$. Consequently, $g_0|K$ is also an $(m+k-p+2,\omega)$-map, where K denotes the union of all $f^{-1}(y_n)$, $n = 0, 1, 2, \ldots$. Obviously, K is compact and, according to Lemma 2.4 (applied to the constant map $q\colon K \to \{0\}$), the set W of all $(m+k-p+2,\omega)$-maps $h \in C(K, \mathbb{I}^{k+p})$ is open in $C(K, \mathbb{I}^{k+p})$. Since the sequence $\{g_n|K\}$ converges to $g_0|K$ in $C(K, \mathbb{I}^{k+p})$ and $g_0|K \in W$, $g_n|K \in W$ for almost all n. Therefore, there exists j such that $g_j|f^{-1}(y_j)$ is an $(m+k-p+2,\omega)$-map. The last conclusion contradicts the observation that $(y_j, g_j) \in G$ implies $g_j \notin C_{(m+k-p+2,\omega)}(X|f^{-1}(y_j), \mathbb{I}^{k+p})$. Thus, $(y_0, g_0) \in G$. ∎

Recall that a closed subset F of the metrizable space M is said to be a Z_n-set in M, where n is a positive integer or 0, if the set $C(\mathbb{I}^n, M\backslash F)$ is dense in $C(\mathbb{I}^n, M)$ with respect to the uniform convergence topology.

Lemma 3.2 *Let $\alpha\colon X \to (0,\infty)$ be a positive continuous function and $g_0 \in C(X, \mathbb{I}^{k+p})$. Then $\psi(y) \cap \overline{B}(g_0,\alpha)$ is a Z_m-set in $\overline{B}(g_0,\alpha)$ for every $y \in Y$, where $\overline{B}(g_0,\alpha)$ is considered as a subspace of $C(X, \mathbb{I}^{k+p})$ with the uniform convergence topology.*

Proof The proof of this lemma follows the proof of [9, Lemma 2.8]. For the sake of completeness we provide a sketch. In this proof all function spaces are equipped with the uniform convergence topology generated by the Euclidean metric d on $\mathbb{I}^{k+p}$. Since, by Lemma 3.1, ψ has a closed graph, each $\psi(y) \cap \overline{B}(g_0,\alpha)$ is closed in $\overline{B}(g_0,\alpha)$. We need to show that, for fixed $y \in Y$, $\delta > 0$ and a map $u\colon \mathbb{I}^m \to \overline{B}(g_0,\alpha)$ there exists a map $v\colon \mathbb{I}^m \to \overline{B}(g_0,\alpha)\backslash\psi(y)$ which is δ-close to u. Observe that u generates $h \in C(\mathbb{I}^m \times X, \mathbb{I}^{k+p})$, $h(z,x) = u(z)(x)$, such that $d(h(z,x), g_0(x)) \le \alpha(x)$ for any $(z,x) \in \mathbb{I}^m \times X$. Since $f^{-1}(y)$ is compact, take $\lambda \in (0,1)$ such that $\lambda \sup\{\alpha(x) : x \in f^{-1}(y)\} < \frac{\delta}{2}$ and define $h_1 \in C(\mathbb{I}^m \times f^{-1}(y), \mathbb{I}^{k+p})$ by $h_1(z,x) = (1-\lambda)h(z,x) + \lambda g_0(x)$. Then, for every $(z,x) \in \mathbb{I}^m \times f^{-1}(y)$, we have

$$d(h_1(z,x), g_0(x)) \le (1-\lambda)\alpha(x) < \alpha(x) \tag{1}$$

and

$$d(h_1(z,x), h(z,x)) \le \lambda\alpha(x) < \frac{\delta}{2}. \tag{2}$$

Let $q < \min\{r, \frac{\delta}{2}\}$, where $r = \inf\{\alpha(x) - d(h_1(z,x), g_0(x)) : (z,x) \in \mathbb{I}^m \times f^{-1}(y)\}$. Since $\dim f^{-1}(y) \le k$, by Proposition 2.6, applied to the projection

$$\mathrm{pr}\colon \mathbb{I}^m \times f^{-1}(y) \to \mathbb{I}^m,$$

there is a map $h_2 \in C(\mathbb{I}^m \times f^{-1}(y), \mathbb{I}^{k+p})$ such that $d(h_2(z,x), h_1(z,x)) < q$ and $h_2|(\{z\} \times f^{-1}(y))$ is an $(m+k-p+2,\omega)$-map for each $(z,x) \in \mathbb{I}^m \times f^{-1}(y)$. Then, by (1) and (2), for all $(z,x) \in \mathbb{I}^m \times f^{-1}(y)$ we have

$$d(h_2(z,x), h(z,x)) < \delta \quad \text{and} \quad d(h_2(z,x), g_0(x)) < \alpha(x). \tag{3}$$

https://doi.org/10.4153/CMB-2005-057-x Published online by Cambridge University Press

The equality $u_2(z)(x) = h_2(z,x)$ defines the map $u_2\colon \mathbb{I}^m \to C(f^{-1}(y), \mathbb{I}^{k+p})$. As in the proof [9, Lemma 2.8], we can show that the map $\pi\colon \overline{B}(g_0,\alpha) \to C(f^{-1}(y), \mathbb{I}^{k+p})$, $\pi(g) = g|f^{-1}(y)$, is continuous and open and $u_2(z) \in \pi(\overline{B}(g_0,\alpha))$ for every $z \in \mathbb{I}^m$. So, $\theta(z) = \overline{\pi^{-1}(u_2(z)) \cap B_\delta(u(z))}$ defines a convex-valued map from $\mathbb{I}^m$ into $\overline{B}(g_0,\alpha)$ which is lower semi-continuous. Here, $B_\delta(u(z))$ is the open ball in $C(X, \mathbb{I}^{k+p})$ (equipped with the uniform metric) having center $u(z)$ and radius δ. By the Michael selection theorem [6, Theorem 3.2], there is a continuous selection $v\colon \mathbb{I}^m \to C(X, \mathbb{I}^{k+p})$ for θ. Then v maps $\mathbb{I}^m$ into $\overline{B}(g_0,\alpha)$ and v is δ-close to u. Moreover, for any $z \in \mathbb{I}^m$ we have $\pi(v(z)) = u_2(z)$ and $u_2(z)$, being the restriction $h_2|(\{z\} \times f^{-1}(y))$, is an $(m+k-p+2,\omega)$-map. Hence, $v(z) \notin \psi(y)$ for any $z \in \mathbb{I}^m$, *i.e.*, $v\colon \mathbb{I}^m \to \overline{B}(g_0,\alpha)\backslash\psi(y)$. ∎

The next lemma will finally accomplish the proof of Theorem 1.1.

Lemma 3.3 *The set $\mathcal{H}_\omega(k,m,p)$ is dense in $C(X, \mathbb{I}^{k+p})$.*

Proof Recall that by $\mathcal{H}_\omega(k,m,p)$ we denoted the set $C_{(m+k-p+2,\omega)}(X, \mathbb{I}^{k+p})$. It suffices to show that, for fixed $g_0 \in C(X, \mathbb{I}^{k+p})$ and a positive continuous function $\alpha\colon X \to (0,\infty)$, there exists $g \in \overline{B}(g_0,\alpha) \cap C_{(m+k-p+2,\omega)}(X, \mathbb{I}^{k+p})$. To this end, consider the space $C(X, \mathbb{I}^{k+p})$ with the uniform convergence topology as a closed and convex subset of the Banach space E consisting of all bounded maps from X into $\mathbb{R}^{k+p}$. We define the constant set-valued (and hence, lower semi-continuous) map ϕ from Y into $C(X, \mathbb{I}^{k+p})$, $\phi(y) = \overline{B}(g_0,\alpha)$. According to Lemma 3.2, $\overline{B}(g_0,\alpha) \cap \psi(y)$ is a Z_m-set in $\overline{B}(g_0,\alpha)$ for every $y \in Y$. So, we have a lower semi-continuous closed and convex-valued map ϕ from Y to E and a map $\psi\colon Y \to 2^E$ such that ψ has a closed graph (see Lemma 3.1) and $\phi(y) \cap \psi(y)$ is a Z_m-set in $\phi(y)$ for each $y \in Y$. Moreover, $\dim Y \leq m$, so we can apply [3, Theorem 1.2] to obtain a continuous map $h\colon Y \to E$ with $h(y) \in \phi(y)\backslash\psi(y)$ for every $y \in Y$. Observe that h is a map from Y into $\overline{B}(g_0,\alpha)$ such that $h(y) \notin \psi(y)$ for every $y \in Y$, *i.e.*, $h(y) \in \overline{B}(g_0,\alpha) \cap C_{(m+k-p+2,\omega)}(X|f^{-1}(y), \mathbb{I}^{k+p})$, $y \in Y$. Then $g(x) = h(f(x))(x)$, $x \in X$, defines a map $g \in \overline{B}(g_0,\alpha)$ such that $g \in C_{(m+k-p+2,\omega)}(X|f^{-1}(y), \mathbb{I}^{k+p})$ for every $y \in Y$. Hence, by virtue of Corollary 2.3, $g \in C_{(m+k-p+2,\omega)}(X, \mathbb{I}^{k+p})$. ∎

Acknowledgements The authors are grateful to the referee for helpful comments and suggestions.

References

[1] S. Bogatyi, V. Fedorchuk and J. van Mill, *On mappings of compact spaces into Cartesian spaces.* Topology Appl. **107**(2000), no. 1-2, 13–24.

[2] R. Engelking, *Theory of dimensions Finite and Infinite.* Sigma Series in Pure Mathematics 10, Heldermann Verlag, Lemgo, 1995.

[3] V. Gutev and V. Valov, *Dense families of selections and finite-dimensional spaces.* Set-Valued Anal. **11**(2003), no. 4, 373–391.

[4] N. Krikorian, *A note concerning the fine topology on function spaces.* Compositio. Math. **21**(1969), 343–348.

[5] M. Levin and W. Lewis, *Some mapping theorems for extensional dimension.* Israel J. Math. **133**(2003), 61–76.

[6] E. Michael, *Continuous selections. I*, Ann. of Math. **63**(1956), 361–382.
[7] J. Munkres, *Topology.* Prentice Hall, Englewood Cliffs, NY, 1975.
[8] B. Pasynkov, *On geometry of continuous maps of finite-dimensional metrizable compacta.* Tr. Mat. Inst. Steklova **212**(1996), no. 1, 147–172 (in Russian).
[9] H. M. Tuncali and V. Valov, *On dimensionally restricted maps*, Fund. Math. **175** (2002), no. 1, 35–52.
[10] ———, *On finite-dimensional maps*, Tsukuba J. Math. **28** (2004), no. 1, 155–167.

Department of Mathematics
Nipissing University
100 College Drive
P.O. Box 5002
North Bay, ON
P1B 8L7
e-mail: muratt@nipissingu.ca
e-mail: veskov@nipissingu.ca