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# NON-CONVEXITY IN BEST COMPLEX CHEBYSHEV APPROXIMATION BY RATIONAL FUNCTIONS

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In real Chebyshev approximation by generalized rational functions, constraining denominators to be positive guarantees that the set of best coefficients is convex [1, 181]. We show by means of an example that denominators must be constrained to be of constant argument for such a convexity result to hold in complex Chebyshev approximation.

Let $X$ be a set of 3 points $\{x, y, z\}$ and approximations be of the form

$$R(A, t) = P(A, t)/Q(A, t) = a_1/(a_2\psi_1(t)+a_3\psi_2(t)),$$

where

$$\psi_1(x) = \psi_1(y) = 1 \qquad \psi_1(z) = 0$$
$$\psi_2(x) = \psi_2(y) = 0 \qquad \psi_2(z) = 1$$

and $a_1$, $a_2$, $a_3$ are complex. Let $f(x)=0$, $f(y)=2$, and $f(z)$ be chosen later. Let $\mu$, $\nu$ be given, $-\pi \le \mu < \nu \le \pi$ and

$$P = \{A: Q(A, t) \neq 0,\ \mu < \arg(Q(A, t)) < \nu,\ t \in X\}.$$

The approximation problem is to find $A^* \in P$ for which

$$e(A) = \max\{|f(t)-R(A, t)|: t \in X\}$$

is minimized over $A \in P$. Such a parameter $A^*$ is called best.

In [2] is considered the general problem of this type with $(\mu, \nu)=(-\pi/2, \pi/2)$ and $(\mu, \nu)=(-\pi/2, 0)$.

Since $R(\alpha A, .)=R(A, .)$ for all $\alpha\neq 0$, it is not difficult to see that there is no loss of generality in having $(\mu, \nu)$ of the form $(-\gamma, \gamma)$, where $2\gamma=\nu-\mu$. We assume this is the case.

THEOREM. *Let $\gamma > 0$ then there is a value for $f(z)$ such that the set of coefficients which are best to $f$ is not convex.*

**Proof.** By choice of $\psi_1$, $\psi_2$, $R(A, .)$ equals $a_1/a_2$ on $\{x, y\}$ and so approximations are constant on $\{x, y\}$. It follows from standard arguments in complex linear approximation that the best approximation to $f$ on $\{x, y\}$ is the constant 1, with absolute value of error$=1$. From this it follows that $A$ is best if $e(A)=1$. Let $\gamma>0$ then there exists $\eta>0$ such that $-\gamma<\arg(1-\eta i)<\arg(1+\eta i)<\gamma$. Now

$$\frac{1}{1+\eta i} = \frac{1-\eta i}{1+\eta^2} \qquad \frac{1}{1-\eta i} = \frac{1+\eta i}{1+\eta^2}$$



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so both of these have absolute value $<1$. There exists real $\alpha<0$ such that

$$(1) \qquad \left|\alpha-\frac{1}{1+\eta i}\right| < 1 \qquad \left|\alpha-\frac{1}{1-\eta i}\right| < 1.$$

Let $f(z)=\alpha$ and $A=(1, 1, 1+\eta i)$, $B=(1, 1, 1-\eta i)$. We have by (1) $e(A)=e(B)=1$ and $A$, $B$ are best. Let $C=(A+B)/2=(1, 1, 1)$ then $|f(z)-R(C, z)|=|\alpha|+1$ and $e(C)=1+|\alpha|$. Thus the set of best coefficients is non-convex.

If we instead define

$$P = \{A:Q(A, t) \neq 0, \mu \leq \arg(Q(A, t)) \leq \nu, t \in X\}$$

exactly the same thing happens if $\mu<\nu$.

## References

1. B. Brosowski, *Über die Eindeutigkeit der rationalen Tschebysheff-Approximationen*, Numer. Math. **7** (1965), 176–186.

2. R. L. Dolganov, *The approximation of continuous complex-valued functions by generalized rational functions*, Siberian Math. J. **11** (1970), 932–942.

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