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Composition Operators Induced by Analytic Maps to the Polydisk

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Abstract. We study properties of composition operators induced by symbols acting from the unit disk to the polydisk. This result will be involved in the investigation of weighted composition operators on the Hardy space on the unit disk and, moreover, be concerned with composition operators acting from the Bergman space to the Hardy space on the unit disk.

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane and $\mathbb{D}^n$ be the polydisk in the n-dimensional complex space. For analytic self-maps $\varphi_1, \dots, \varphi_n$ of $\mathbb{D}$ and $z \in \mathbb{D}$, denote an analytic map Φ from $\mathbb{D}$ to $\mathbb{D}^n$ by $\Phi(z) = (\varphi_1(z), \dots, \varphi_n(z))$. Then for analytic functions f on $\mathbb{D}^n$, we define the composition operator C_Φ by

$$C_\Phi f(z) = f \circ \Phi(z) = f(\varphi_1(z), \dots, \varphi_n(z)) \quad \text{for } z \in \mathbb{D}.$$

Recently, composition operators acting between Hardy and weighted Bergman spaces of the polydisk or the unit ball have been investigated by researchers. Some of them have considered when composition operators would naturally act between such spaces of the polydisk or the unit ball in the different dimensions. Consequently, Koo and Smith [4] studied composition operators between Hardy or weighted Bergman spaces on the unit ball in the different dimensions. Furthermore, Stessin and Zhu [8] characterized the polydisk case and gave necessary and sufficient conditions for composition operators from the Hardy space on the unit disk to the Hardy space on the polydisk to be compact. But there is the characterization left over for composition operators induced by analytic mappings from the polydisk to the polydisk in higher dimension. We consider this case here. More precisely, we will concentrate on the explicit question of which composition operators map boundedly or compactly from the Hardy space on the bidisk into the Hardy space on the unit disk. As a result, our problem gives new attention to the characterizations of weighted composition operators on the Hardy space on $\mathbb{D}$ and of composition operators acting from the Bergman space to the Hardy space on the unit disk.

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The Hardy space $H^2(\mathbb{D}^n)$ denotes the space of all analytic functions f on $\mathbb{D}^n$ such that

$$\|f\|^2_{H^2(\mathbb{D}^n)} = \int_{\partial\mathbb{D}^n} |f(\zeta)|^2 dm(\zeta) < \infty,$$

where dm is the normalized Lebesgue measure on the distinguished boundary $\partial\mathbb{D}^n$. Let $H^\infty(\mathbb{D}^n)$ be the Banach space of bounded analytic functions on $\mathbb{D}^n$ with the norm $\|f\|_{H^\infty(\mathbb{D}^n)} = \sup_{z\in\mathbb{D}^n} |f(z)|$, where we note simply $\|f\|_{H^\infty(\mathbb{D})} = \|f\|_\infty$. See [5] for basic information about Hardy spaces on the polydisk.

Here we notice the relationship between our problems and the study of composition operators on the Hardy space in one dimension. Let $\Phi(z) = (\varphi(z), \psi(z))$ with analytic self-maps φ and ψ of $\mathbb{D}$. Suppose that $C_\Phi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ is bounded. For g and $h \in H^2(\mathbb{D})$, put $f(z, w) = g(z)h(w)$. Then $f \in H^2(\mathbb{D}^2)$ and

$$\|C_\Phi f\|_{H^2(\mathbb{D})} \le C\|f\|_{H^2(\mathbb{D}^2)},$$

and so

$$\|(g \circ \varphi)C_\psi h\|_{H^2(\mathbb{D})} \le C\|g\|_{H^2(\mathbb{D})}\|h\|_{H^2(\mathbb{D})}.$$

Thus this inequality says that $(g \circ \varphi)C_\psi\colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is bounded. So this problem has a relation to the boundedness of weighted composition operators on $H^2(\mathbb{D})$. See [1, 6, 10] for more information about composition operators on the unit disk.

Also, we will see that the boundedness and compactness of $C_\Phi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ is involved in the investigation of composition operators acting from the Bergman space to the Hardy space on the unit disk. Let $L^2_a(\mathbb{D})$ be the Bergman space consisting of those analytic functions f on $\mathbb{D}$ such that

$$\|f\|^2_{L^2_a(\mathbb{D})} = \int_{\mathbb{D}} |f(z)|^2 dA(z) < \infty,$$

where dA is the normalized Lebesgue measure on $\mathbb{D}$.

This paper is organized as follows. For an analytic map $\Phi = (\varphi, \psi)$ from $\mathbb{D}$ to $\mathbb{D}^2$, in Section 2 we study the boundedness of C_Φ acting from $H^2(\mathbb{D}^2)$ to $H^2(\mathbb{D})$ and consider a problem concerning with weighted composition operators on the Hardy space on $\mathbb{D}$. Specifically we let $\varphi = \psi$ and so $\Phi = (\varphi, \varphi)$. Then we will obtain that the boundedness of $C_\Phi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ is equivalent to the boundedness of $C_\varphi\colon L^2_a(\mathbb{D}) \to H^2(\mathbb{D})$. In Section 3, we characterize the compactness of C_Φ and the special case $\varphi = \psi$. These results give the new condition equivalent to the well-known characterization using the Nevanlinna counting function.

2 Boundedness

The aim of this section is to obtain function-theoretic characterizations for the boundedness of $C_\Phi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$.

Theorem 2.1 *Let $\Phi = (\varphi, \psi)$. If $\|\varphi\psi\|_\infty < 1$, then $C_\Phi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ is bounded.*

Proof Take a number σ as $\|\varphi\psi\|_\infty < \sigma < 1$. Then there are measurable subsets Γ_1, Γ_2 of $\partial\mathbb{D}$ such that $\Gamma_1 \cap \Gamma_2 = \varnothing$, $m(\Gamma_1 \cup \Gamma_2) = 1$, $|\varphi(\zeta)| < \sqrt{\sigma}$ a.e. on Γ_1 and $|\psi(\zeta)| < \sqrt{\sigma}$ a.e. on Γ_2. For any function $F \in H^2(\mathbb{D}^2)$, we may write

$$F(z,w) = \sum_{n=0}^{\infty} z^n F_n(w) = \sum_{n=0}^{\infty} w^n G_n(z),$$

where $F_n(w), G_n(z) \in H^2(\mathbb{D})$ satisfy

$$\|F\|^2_{H^2(\mathbb{D}^2)} = \sum_{n=0}^{\infty} \|F_n\|^2_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} \|G_n\|^2_{H^2(\mathbb{D})}.$$

We note that

$$\|C_\Phi F\|^2_{H^2(\mathbb{D})} = \int_{\Gamma_1} |C_\Phi F(\zeta)|^2 \, dm(\zeta) + \int_{\Gamma_2} |C_\Phi F(\zeta)|^2 \, dm(\zeta).$$

We have

$$\begin{aligned}
\int_{\Gamma_1} |C_\Phi F(\zeta)|^2 \, dm(\zeta) &= \int_{\Gamma_1} \Big| \sum_{n=0}^{\infty} \varphi^n(\zeta) C_\psi F_n(\zeta) \Big|^2 \, dm(\zeta) \\
&\le \int_{\Gamma_1} \Big(\sum_{n=0}^{\infty} |\varphi(\zeta)|^{2n} \Big) \Big(\sum_{n=0}^{\infty} |C_\psi F_n(\zeta)|^2 \Big) \, dm(\zeta) \\
&\le \frac{1}{1-\sigma} \int_{\partial\mathbb{D}} \sum_{n=0}^{\infty} |C_\psi F_n(\zeta)|^2 \, dm(\zeta) \\
&\le \frac{1}{1-\sigma} \sum_{n=0}^{\infty} \|C_\psi F_n\|^2_{H^2(\mathbb{D})} \le \frac{1}{1-\sigma} \|C_\psi\|^2 \|F\|^2_{H^2(\mathbb{D}^2)}.
\end{aligned}$$

Similarly we have

$$\int_{\Gamma_2} |C_\Phi F(\zeta)|^2 \, dm(\zeta) \le \frac{1}{1-\sigma} \|C_\varphi\|^2 \|F\|^2_{H^2(\mathbb{D}^2)}.$$

Thus we get the assertion. ■

As we saw in the introduction, this research has the relationship to the boundedness of weighted composition operators on $H^2(\mathbb{D})$.

Corollary 2.2 *Let φ be an analytic self-map of $\mathbb{D}$. If φ is not inner, then there is a function $u \in H^2(\mathbb{D}) \setminus H^\infty(\mathbb{D})$ such that $uC_\varphi \colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is bounded.*

Proof Since φ is not inner, there exists an analytic self-map ψ of $\mathbb{D}$ such that $\|\psi\|_\infty = 1$ and $\|\varphi\psi\|_\infty < 1$. Let $\Phi = (\varphi, \psi)$. By Theorem 2.1, $C_\Phi \colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ is bounded. Since $\|\psi\|_\infty = 1$, there is a function $g \in H^2(\mathbb{D}) \setminus H^\infty(\mathbb{D})$

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such that $u := g \circ \psi \in H^2(\mathbb{D}) \setminus H^\infty(\mathbb{D})$. For any function $f \in H^2(\mathbb{D})$, let $F(z,w) = f(z)g(w)$. Then

$$\|uC_\varphi f\|_{H^2(\mathbb{D})} = \|C_\Phi F\|_{H^2(\mathbb{D})} \le \|C_\Phi\|\|g\|_{H^2(\mathbb{D})}\|f\|_{H^2(\mathbb{D})}.$$

This shows that uC_φ is a bounded operator on $H^2(\mathbb{D})$. ∎

We now come to a full characterization for the boundedness of C_Φ.

Definition 2.3 A sequence $\{\lambda_j\}$ in $\mathbb{D}$ is called a *uniqueness sequence* for $H^2(\mathbb{D})$ if any $f \in H^2(\mathbb{D})$ vanishing on $\{\lambda_j\}$ must be zero.

Examples of uniqueness sequences are a sequence converging to a point in $\mathbb{D}$ and a sequence $\{\lambda_j\}$ satisfying $\sum(1 - |\lambda_j|) = \infty$. Another example is the sampling set for $H^2(\mathbb{D})$ (refer to [9]).

For $a = (z_0, w_0) \in \mathbb{D}^2$, we let

$$K_a(z,w) = \frac{1}{(1-\overline{z_0}z)(1-\overline{w_0}w)}.$$

Then K_a is the Cauchy kernel of $H^2(\mathbb{D}^2)$. Obviously $K_a \in H^\infty(\mathbb{D}^2)$.

Theorem 2.4 *Let $\{\lambda_j\}$ be a uniqueness sequence in $\mathbb{D}$. Let $\Phi = (\varphi, \psi)$ be an analytic map from $\mathbb{D}$ to $\mathbb{D}^2$. Then the following assertions are equivalent.*

(i) $C_\Phi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ *is bounded.*

(ii) *There exists a positive constant C such that for every $k \ge 1$ and complex numbers $\alpha_1, \dots, \alpha_k$,*

$$\sum_{m,n=1}^{k} \frac{\alpha_m\overline{\alpha_n}}{(1-\varphi(\lambda_n)\overline{\varphi(\lambda_m)})(1-\psi(\lambda_n)\overline{\psi(\lambda_m)})} \le C\sum_{m,n=1}^{k} \frac{\alpha_m\overline{\alpha_n}}{1-\lambda_n\overline{\lambda_m}}.$$

Condition (ii) can be rephrased as follows: There exists a positive constant C such that for any $k \ge 1$, the Hermitian matrix

$$\Big(\frac{C}{1-\lambda_n\overline{\lambda_m}} - \frac{1}{(1-\varphi(\lambda_n)\overline{\varphi(\lambda_m)})(1-\psi(\lambda_n)\overline{\psi(\lambda_m)})}\Big)_{1\le m,n\le k}$$

is positive semi-definite.

We need the following simple fact.

Lemma 2.5 *For a uniqueness sequence $\{\lambda_j\}$, the linear span $\mathcal{A}$ of the set*

$$\Big\{K_{\lambda_j}(z) := \frac{1}{1-\overline{\lambda_j}z},\ j \ge 1\Big\}$$

is dense in $H^2(\mathbb{D})$.

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Proof Let f be an element in $H^2(\mathbb{D})$ that is orthogonal to $\mathcal{A}$. By the reproducing property of Cauchy kernels and the uniqueness property of $\{\lambda_j\}$, we deduce that f must be identically zero. ■

Proof of Theorem 2.4 (i)⇒(ii) Considering the adjoint map of C_Φ, we suppose that $C_\Phi^*\colon H^2(\mathbb{D}) \to H^2(\mathbb{D}^2)$ is bounded. For a given set of complex numbers $\alpha_1, \dots, \alpha_k$, we set

$$K(z) = \sum_{m=1}^{k} \alpha_m K_{\lambda_m}(z).$$

For $a = (z, w) \in \mathbb{D}^2$, we obtain

$$\begin{aligned}(C_\Phi^* K)(z,w) &= \sum_{m=1}^{k} \alpha_m \langle C_\Phi^* K_{\lambda_m}, K_a\rangle = \sum_{m=1}^{k} \alpha_m \langle K_{\lambda_m}, C_\Phi K_a\rangle \\ &= \sum_{m=1}^{k} \frac{\alpha_m}{(1-\overline{\varphi(\lambda_m)}z)(1-\overline{\psi(\lambda_m)}w)}.\end{aligned}$$

It follows that

$$\|C_\Phi^* K\|^2_{H^2(\mathbb{D}^2)} = \sum_{m,n=1}^{k} \frac{\alpha_m \overline{\alpha_n}}{(1-\varphi(\lambda_n)\overline{\varphi(\lambda_m)})(1-\psi(\lambda_n)\overline{\psi(\lambda_m)})}.$$

Similarly we have

$$\|K\|^2_{H^2(\mathbb{D})} = \sum_{m,n=1}^{k} \frac{\alpha_m \overline{\alpha_n}}{1-\overline{\lambda_m}\lambda_n}.$$

Since C_Φ^* is bounded, we get the desired positive constant C (independent of k).

(ii)⇒(i) Let $\mathcal{A}$ be as in Lemma 2.5. For any $a = (z, w) \in \mathbb{D}^2$, define the linear map $T\colon \mathcal{A} \to H^2(\mathbb{D}^2)$ by

$$TK_{\lambda_j}(a) = \langle K_{\lambda_j}, C_\Phi K_a\rangle = \frac{1}{(1-\overline{\varphi(\lambda_j)}z)(1-\overline{\psi(\lambda_j)}w)}.$$

Then, by the proof of the first part, there exists a positive constant C such that

$$\|Th\|_{H^2(\mathbb{D}^2)} \le C\|h\|_{H^2(\mathbb{D})}$$

for all $h \in \mathcal{A}$. Since $\mathcal{A}$ is dense in $H^2(\mathbb{D})$, the map T is extended to a bounded linear map $T\colon H^2(\mathbb{D}) \to H^2(\mathbb{D}^2)$. Note that C_Φ is a *closed* densely defined operator, so the *formal* adjoint

$$C_\Phi^*\colon H^2(\mathbb{D}) \to H^2(\mathbb{D}^2)$$

is also defined on a linear *dense* subspace $\mathcal{B}$ of $H^2(\mathbb{D})$. We will show that $\mathcal{B} = H^2(\mathbb{D})$. To see this, fix $h \in \mathcal{B}$ and $a \in \mathbb{D}^2$. By Lemma 2.5 there exists a sequence $\{h_j\} \subset \mathcal{A}$ such that $h_j \to h$ in $H^2(\mathbb{D})$. Then we have

$$C_\Phi^* h(a) = \langle C_\Phi^* h, K_a \rangle = \langle h, C_\Phi K_a \rangle = \lim_{j\to\infty} \langle h_j, C_\Phi K_a \rangle$$
$$= \lim_{j\to\infty} (Th_j)(a) = (Th)(a).$$

Here, the fourth equality follows from the fact that each h_j is a finite linear combination of K_{λ_j}. Thus we obtain $C_\Phi^* h = Th$. Since $\mathcal{B}$ is dense in $H^2(\mathbb{D})$, we infer that $C_\Phi^* = T$ on $H^2(\mathbb{D})$ and $\mathcal{B} = H^2(\mathbb{D})$. So C_Φ must be bounded. ■

It should be remarked that we do not need the "uniqueness" of λ_j for the implication (i)⇒(ii). By taking $\lambda_j = \lambda$, $j \geq 1$ for an arbitrary $\lambda \in \mathbb{D}$, we obtain the following simple consequence.

Corollary 2.6 *Let $\Phi = (\varphi, \psi)$ with analytic self-maps φ and ψ of $\mathbb{D}$. Suppose that $C_\Phi \colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ is bounded. Then*

$$\sup_{\lambda\in\mathbb{D}} \frac{1-|\lambda|^2}{(1-|\varphi(\lambda)|^2)(1-|\psi(\lambda)|^2)} < \infty.$$

So we obtain the next result. For an analytic self-map φ of $\mathbb{D}$, put

$$\Gamma(\varphi) = \Big\{ \zeta \in \partial\mathbb{D} : \limsup_{z\to\zeta, z\in\mathbb{D}} |\varphi(z)| = 1 \Big\}.$$

Corollary 2.7 *Let $\Phi = (\varphi, \psi)$ with analytic self-maps φ and ψ of $\mathbb{D}$. Suppose that $C_\Phi \colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ is bounded. Then φ has no finite angular derivative on $\Gamma(\psi)$ and ψ has no finite angular derivative on $\Gamma(\varphi)$.*

Inner functions having angular derivatives uniformly on $\partial\mathbb{D}$ are only finite Blaschke products.

Corollary 2.8 *Let $\Phi = (\varphi, \psi)$ with $\|\varphi\|_\infty = \|\psi\|_\infty = 1$, where either φ or ψ is a finite Blaschke product. Then $C_\Phi \colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ is unbounded.*

Let φ be an inner function that has a uniformly finite angular derivative at every point in $\partial\mathbb{D}$, that is,

$$\sup_{\lambda} \frac{1-|\varphi(\lambda)|^2}{1-|\lambda|^2} < \infty.$$

Then suppose that a weighted composition operator $uC_\varphi \colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is bounded for an analytic function u. Then its adjoint $(uC_\varphi)^* \colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is also bounded. Recall that $(uC_\varphi)^* K_\lambda = \overline{u(\lambda)} K_{\varphi(\lambda)}$, where $K_\lambda(z) = 1/(1-\overline{\lambda}z)$ for $\lambda \in \mathbb{D}$. So for the normalized kernel $k_\lambda(z) = \sqrt{1-|\lambda|^2}/(1-\overline{\lambda}z)$,

$$\|(uC_\varphi)^* k_\lambda\|_{H^2(\mathbb{D})}^2 \leq C \|k_\lambda\|_{H^2(\mathbb{D})}^2,$$

so

$$|u(\lambda)|^2 \frac{1-|\lambda|^2}{1-|\varphi(\lambda)|^2} \leq C.$$

As φ has a uniformly finite angular derivative at every point in $\partial\mathbb{D}$, we have $u \in H^\infty(\mathbb{D})$. This gives a fact that if a weighted composition operator $uC_\varphi \colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is bounded for a finite Blaschke product φ and analytic function u, then u is bounded. What about an inner function that is not a finite Blaschke product? We answer this question as follows.

Proposition 2.9 *Let φ be an inner function on $\mathbb{D}$ satisfying the following conditions.*

(a) *There exists a closed arc $I \subset \partial D$ with an end point ξ_0 such that φ' is extended to a non-vanishing analytic function near every point of $I \setminus \{\xi_0\}$.*

(b) $\displaystyle\lim_{z \to \xi_0, z \in I} |\varphi'(z)| = +\infty.$

Then there exists a function $u \in H^2(\mathbb{D}) \setminus H^\infty(\mathbb{D})$ such that $uC_\varphi \colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is bounded.

Let $S(z) = \exp((z+1)/(z-1))$. Then S satisfies the assumption of Proposition 2.9. We can also check that for every $\alpha \in \mathbb{D}$, the function

$$S_\alpha(z) := \frac{S(z) - \alpha}{1 - S(z)\overline{\alpha}}$$

also satisfies this assumption. By the Frostman theorem ([2]), for $\alpha \in \mathbb{D}$, S_α is an infinite Blaschke product except for a set of capacity zero.

Proof of Proposition 2.9 By conditions (a) and (b), there is a sequence of disjoint open subarcs $\{I_n\}$ of I satisfying that

(i) φ is a one-to-one map from I_n onto $\varphi(I_n)$;

(ii) $\displaystyle\sum_{n=1}^\infty \frac{1}{b_n^2} < \infty$, where $b_n = \inf_{z \in I_n} |\varphi'(z)|$.

We note that $\sum_{n=1}^\infty m(I_n) \leq m(I) < \infty$. By (ii), there is a sequence of positive numbers $\{a_n\}$ satisfying that

(iii) $a_n \to \infty$ and $a_n \geq 1$ for every n;

(iv) $\displaystyle\sum_{n=1}^\infty a_n^2 m(I_n) < \infty$;

(v) $\displaystyle\sum_{n=1}^\infty \frac{a_n^2}{b_n^2} < \infty$.

Define a function g on $\partial\mathbb{D}$ as follows:

$$g(z) = \begin{cases} a_n & \text{for} \quad z \in I_n \\ 1 & \text{for} \quad z \in \partial\mathbb{D} \setminus \bigcup_{n=1}^\infty I_n. \end{cases}$$

By (iv), we have

$$0 < \int_{\partial\mathbb{D}} \log|g(\zeta)| dm(\zeta) \leq \int_{\partial\mathbb{D}} |g(\zeta)|^2 dm(\zeta) < \infty.$$

Thus we can find a function $u \in H^2(\mathbb{D})$ such that $|u| = g$ almost everywhere on $\partial\mathbb{D}$. By (iii), $u \notin H^\infty(\mathbb{D})$. For $f \in H^2(\mathbb{D})$, we have

$$\begin{aligned}
&\int_{\partial\mathbb{D}} |u(\zeta)|^2 |f(\varphi(\zeta))|^2 dm(\zeta) \\
&= \int_{\partial\mathbb{D}\setminus\bigcup_{n=1}^\infty I_n} |f(\varphi(\zeta))|^2 dm(\zeta) + \sum_{n=1}^\infty a_n^2 \int_{I_n} |f(\varphi(\zeta))|^2 dm(\zeta) \\
&\le \|C_\varphi\|^2 \|f\|_{H^2(\mathbb{D})}^2 + \sum_{n=1}^\infty a_n^2 \int_{\varphi(I_n)} \frac{|f(\xi)|^2}{\min_{\zeta\in I_n} |\varphi'(\zeta)|^2} dm(\xi) \\
&= \Big(\|C_\varphi\|^2 + \sum_{n=1}^\infty \frac{a_n^2}{b_n^2} \Big) \|f\|_{H^2(\mathbb{D})}^2 .
\end{aligned}$$

Here the first inequality comes from (i) and the change-of-variables theorem. By (v), $uC_\varphi \colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is bounded. ■

We have the following problem.

Problem 2.10 Suppose that φ is an inner function that is not a finite Blaschke product. Is there $u \in H^2(\mathbb{D}) \setminus H^\infty(\mathbb{D})$ such that $uC_\varphi \colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is bounded?

Next we study the special case $\varphi = \psi$. Let $\Phi = (\varphi, \varphi)$. For $f = \sum_{n=0}^\infty a_n z^n \in L_a^2(\mathbb{D})$, we define

$$Uf = \sum_{n=0}^\infty \frac{a_n}{n+1} \Big(\sum_{k=0}^n z^k w^{n-k} \Big).$$

Then $\|Uf\|_{H^2(\mathbb{D}^2)} = \|f\|_{L_a^2(\mathbb{D})}$. Let $N = \{Uf : f \in L_a^2(\mathbb{D})\}$. Then N is a closed subspace of $H^2(\mathbb{D}^2)$ and $C_\Phi U = C_\varphi$ on $L_a^2(\mathbb{D})$. It is known that $H^2(\mathbb{D}^2) \ominus N = \overline{(z-w)H^2(\mathbb{D}^2)}$ (see [3]). Hence $C_\Phi = 0$ on $H^2(\mathbb{D}^2) \ominus N$. Thus we get the following lemma.

Lemma 2.11 *For $\Phi = (\varphi, \varphi)$, $C_\Phi \colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ is bounded (compact) if and only if $C_\varphi \colon L_a^2(\mathbb{D}) \to H^2(\mathbb{D})$ is bounded (compact).*

We recall that the classical Nevanlinna counting function N_φ for an analytic self-map φ of $\mathbb{D}$ is defined by

$$N_\varphi(w) = \sum_{z\in\varphi^{-1}\{w\}} \log\frac{1}{|z|} \quad \text{for} \quad w \in \mathbb{D} \setminus \{\varphi(0)\}.$$

It is known that

$$\limsup_{|w|\to 1} \frac{N_\varphi(w)}{\log\frac{1}{|w|}} < \infty$$

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for every analytic self-map φ of $\mathbb{D}$, and that $C_\varphi\colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is compact if and only if

$$\lim_{|w|\to 1} \frac{N_\varphi(w)}{\log \frac{1}{|w|}} = 0$$

(see [6]). We have the following.

Proposition 2.12 *For $\Phi = (\varphi, \varphi)$ with an analytic self-map φ of $\mathbb{D}$, the following are equivalent:*

(i) $C_\Phi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ *is bounded;*

(ii) $C_\varphi\colon L^2_a(\mathbb{D}) \to H^2(\mathbb{D})$ *is bounded;*

(iii) *for a uniqueness sequence λ_j of $H^2(\mathbb{D})$, there exists a positive constant C such that for every $k \geq 1$ the matrix*

$$\Big(\frac{C}{1 - \lambda_n \overline{\lambda_m}} - \frac{1}{(1 - \varphi(\lambda_n)\overline{\varphi(\lambda_m)})^2} \Big)_{1 \leq m,n \leq k}$$

is positive semi-definite;

(iv) $\displaystyle \limsup_{|w|\to 1} \frac{N_\varphi(w)}{\left[\log \frac{1}{|w|}\right]^2} < \infty.$

Proof The equivalence of (i) and (iii) is due to Theorem 2.4. Smith [7] investigated the properties of $C_\varphi\colon L^2_a(\mathbb{D}) \to H^2(\mathbb{D})$ and gave the equivalence of conditions (ii) and (iv). The equivalence of (i) and (ii) follows from Lemma 2.11. ■

Suppose that $\|\varphi\|_\infty = 1$ and $C_\varphi\colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ is not compact. Then we have

$$\limsup_{|w|\to 1} \frac{N_\varphi(w)}{\left[\log \frac{1}{|w|}\right]^2} = \infty.$$

By Proposition 2.12, $C_\Phi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ is unbounded.

3 Compactness

In this section we study the compactness of C_Φ acting from $H^2(\mathbb{D}^2)$ to $H^2(\mathbb{D})$. If $C_\Phi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ is compact, then trivially both C_φ and C_ψ are compact operators from $H^2(\mathbb{D})$ to $H^2(\mathbb{D})$.

In analogy with Theorem 2.1, we have the following result.

Theorem 3.1 *Let $\Phi = (\varphi, \psi)$. If $\|\varphi\psi\|_\infty < 1$ and both operators $C_\varphi, C_\psi\colon H^2(\mathbb{D}) \to H^2(\mathbb{D})$ are compact, then $C_\Phi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ is compact.*

Proof Take a number σ as $\|\varphi\psi\|_\infty < \sigma < 1$. Then there are measurable subsets Γ_1, Γ_2 of $\partial\mathbb{D}$ such that $\Gamma_1 \cap \Gamma_2 = \varnothing$, $m(\Gamma_1 \cup \Gamma_2) = 1$, $|\varphi(\zeta)| < \sqrt{\sigma}$ a.e. on Γ_1 and $|\psi(\zeta)| < \sqrt{\sigma}$ a.e. on Γ_2. Let $\{F_k\}_k$ be a bounded sequence in $H^2(\mathbb{D}^2)$

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such that $F_k \to 0$ uniformly as $k \to \infty$ on any compact subset of $\mathbb{D}^2$. Let $M = \sup_k \|F_k\|_{H^2(\mathbb{D}^2)} < \infty$. We may write

$$F_k(z,w) = \sum_{n=0}^{\infty} z^n F_{n,k}(w) = \sum_{n=0}^{\infty} w^n G_{n,k}(z),$$

where $F_{n,k}(w), G_{n,k}(z) \in H^2(\mathbb{D})$ satisfying

$$\|F_k\|^2_{H^2(\mathbb{D}^2)} = \sum_{n=0}^{\infty} \|F_{n,k}\|^2_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} \|G_{n,k}\|^2_{H^2(\mathbb{D})}.$$

We note that

$$\|C_\Phi F_k\|^2_{H^2(\mathbb{D})} = \int_{\Gamma_1} |(C_\Phi F_k)(\zeta)|^2\, dm(\zeta) + \int_{\Gamma_2} |(C_\Phi F_k)(\zeta)|^2\, dm(\zeta).$$

We have

$$\begin{aligned}
&\int_{\Gamma_1} |(C_\Phi F_k)(\zeta)|^2\, dm(\zeta)\\
&= \int_{\Gamma_1} \Big| \sum_{n=0}^{\infty} \varphi^n(\zeta)(C_\psi F_{n,k})(\zeta)\Big|^2\, dm(\zeta)\\
&\le 2\int_{\Gamma_1} \Big(\Big| \sum_{n=0}^{N} \varphi^n(\zeta)(C_\psi F_{n,k})(\zeta)\Big|^2 + \Big| \sum_{n=N+1}^{\infty} \varphi^n(\zeta)(C_\psi F_{n,k})(\zeta)\Big|^2\Big)\, dm(\zeta)\\
&\le 2\int_{\partial\mathbb{D}} \Big(\Big| \sum_{n=0}^{N} \varphi(\zeta)(C_\psi F_{n,k})(\zeta)\Big|^2\\
&\qquad\qquad + \sigma^{2(N+1)} \Big| \sum_{n=N+1}^{\infty} \varphi^{n-N-1}(\zeta)(C_\psi F_{n,k})(\zeta)\Big|^2\Big)\, dm(\zeta).
\end{aligned}$$

Let $T_z^* F = (F - F(0,w))/z$ for $F \in H^2(\mathbb{D}^2)$. Then T_z^* is a contraction on $H^2(\mathbb{D}^2)$. We have

$$\sum_{n=N+1}^{\infty} \varphi^{n-N-1} C_\psi F_{n,k} = C_\Phi (T_z^*)^{N+1} F_k.$$

By Theorem 3.1,

$$\begin{aligned}
\int_{\Gamma_1} |(C_\Phi F_k)(\zeta)|^2\, dm(\zeta) \le 2\int_{\partial\mathbb{D}} \Big| \sum_{n=0}^{N} \varphi^n(\zeta)(C_\psi F_{n,k})(\zeta)\Big|^2\, dm(\zeta)&\\
+ 2\sigma^{2(N+1)}\|C_\Phi\|^2 M^2.&
\end{aligned}$$

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Take $\varepsilon > 0$ arbitrarily. Since $0 < \sigma < 1$, there exists a positive integer N_0 such that

$$2\sigma^{2(N_0+1)}\|C_\Phi\|^2 M^2 < \varepsilon.$$

We have

$$\int_{\Gamma_1} |(C_\Phi F_k)(\zeta)|^2\, dm(\zeta) \le 2(N_0+1)\sum_{n=0}^{N_0} \|C_\psi F_{n,k}\|^2_{H^2(\mathbb{D})} + \varepsilon.$$

For each $0 \le n \le N_0$, $\|F_{n,k}\|_{H^2(\mathbb{D})} \le M$ for every k. Since $F_k \to 0$ uniformly on any compact subset of $\mathbb{D}^2$, it is easy to see that $F_{n,k} \to 0$ uniformly as $k \to \infty$ on any compact subset of $\mathbb{D}$. Since C_ψ is compact on $H^2(\mathbb{D})$, $\|C_\psi F_{n,k}\|_{H^2(\mathbb{D})} \to 0$ as $k \to \infty$. Thus we get

$$\limsup_{k\to\infty} \int_{\Gamma_1} |(C_\Phi F_k)(\zeta)|^2\, dm(\zeta) \le \varepsilon.$$

Similarly we have

$$\limsup_{k\to\infty} \int_{\Gamma_2} |(C_\Phi F_k)(\zeta)|^2\, dm(\zeta) \le \varepsilon.$$

Therefore

$$\lim_{k\to\infty} \|C_\Psi F_k\|_{H^2(\mathbb{D})} = 0.$$

Thus we get the assertion. ■

In analogy with boundedness of C_Φ, we also have the following characterization for compactness in a special case.

Proposition 3.2 *For $\Phi = (\varphi, \varphi)$ with an analytic self-map φ of $\mathbb{D}$, the following are equivalent:*

(i) $C_\Phi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ *is compact;*
(ii) $C_\varphi\colon L^2_a(\mathbb{D}) \to H^2(\mathbb{D})$ *is compact;*
(iii) $\displaystyle\limsup_{|w|\to 1} \frac{N_\varphi(w)}{\left[\log \frac{1}{|w|}\right]^2} = 0.$

Proof Smith [7] gave the equivalence of conditions (ii) and (iii). The equivalence of conditions (i) and (ii) follows from Lemma 2.11. ■

About the Hilbert–Schmidtness, we can easily obtain the result using the orthogonal basis $\{z^n w^m\}$ in $H^2(\mathbb{D}^2)$.

Theorem 3.3 *Suppose that $C_\Phi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ is bounded. Then $C_\Phi\colon H^2(\mathbb{D}^2) \to H^2(\mathbb{D})$ is Hilbert–Schmidt if and only if*

$$\int_0^{2\pi} \frac{1}{(1-|\varphi(e^{i\theta})|^2)(1-|\psi(e^{i\theta})|^2)}\, d\theta < \infty.$$

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