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RICH PROXIMITIES AND COMPACTIFICATIONS

STEPHAN C. CARLSON

1. Introduction. Each Hausdorff compactification of a given Tychonoff space is the Smirnov compactification associated with a compatible proximity on the space. Also each realcompactification of a given Tychonoff space is the underlying topological space of the completion of a compatible uniformity on the space. But if T is a realcompactification of a Tychonoff space X which is contained in a particular compactification Z of X, then it is not always possible to find a compatible uniformity $\mathscr{U}$ on X such that T is the underlying topological space of the completion of $(X, \mathscr{U})$ and $\mathscr{U}$ induces the proximity on X associated with Z. We shall call a Hausdorff compactification Z of a Tychonoff space X a *rich compactification* of X (and the associated proximity on X a *rich proximity*) if every realcompactification of X contained in Z can be obtained as the underlying topological space of the completion of a compatible uniformity on X which induces the proximity on X associated with Z. Questions concerning the rich compactifications of Tychonoff spaces were originally communicated by Marlon Rayburn of the University of Manitoba.

For any Tychonoff space X the Stone-Čech compactification of X is a rich compactification of X. Since a realcompact and pseudocompact space is compact, every Hausdorff compactification of a pseudocompact Tychonoff space is a rich compactification. But when X is a realcompact, noncompact Tychonoff space, the existence of rich compactifications of X besides the Stone-Čech compactification is not clear. In this paper we shall construct such compactifications for spaces belonging to a certain class of locally compact, noncompact spaces.

In fact, this construction occurs in a more general setting. Realcompactness is a special case of E-compactness in the sense of Engelking and Mrówka [**7**]. A Hausdorff compactification Z of a Tychonoff space X is called an *E-rich compactification* of X (and the associated proximity on X an *E-rich proximity*) if every E-compactification of X contained in Z can be obtained as the underlying topological space of the uniform completion of a compatible uniformity on X which induces the proximity on X associated with Z. For a certain class of Tychonoff spaces E: (1) we show that every E-completely regular space has an E-completely regular, E-rich Hausdorff compactification which turns out to be the projective maximum among all its E-completely regular Hausdorff compactifica-

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tions, and (2) we construct an E-completely regular, E-rich Hausdorff compactification (which is not the projective maximum of (1)) for each member of a certain class of locally compact, noncompact E-completely regular spaces. When E is the real line, the projective maximum of (1) is the Stone-Čech compactification, and, when E is the countably infinite discrete space, it is the Banaschewski zero-dimensional compactification [**2**]. We note here that uniformities on E-completely regular spaces have been discussed (especially when E is zero-dimensional) in [**1**], [**2**], and [**5**], and E-completely regular Hausdorff compactifications for general spaces E are discussed in [**11**], [**14**], and [**15**].

By *uniformity* we shall mean separated diagonal uniformity, and the collection of pseudometrics associated with a uniformity (which is called a uniform structure in [**9**]) shall be called the *gauge structure* associated with the uniformity. We shall use $\mathbf{R}$ to denote the set of real numbers and $\mathbf{N}$ to denote the set of positive integers, and, when used as topological spaces, they shall be assumed to possess their usual topologies. If $f: X \to \mathbf{R}$ is a real-valued function on a set X, then

$$\psi_f(x, y) = |f(x) - f(y)| \quad (x, y \in X)$$

defines a pseudometric on X. For $\emptyset \neq D \subseteq \mathbf{R}^X$, $\{\psi_f: f \in D\}$ is a subbase for a gauge structure $\mathscr{D}(D)$ on X. A gauge structure $\mathscr{D}$ (and its associated uniformity) on X is called *functionally determined* if for some $\emptyset \neq D \subseteq \mathbf{R}^X$, $\mathscr{D} = \mathscr{D}(D)$. As usual, for a Tychonoff space X, $\mathscr{C}(X)$ will denote $\mathscr{D}(C(X))$ and $\mathscr{C}^*(X)$ will denote $\mathscr{D}(C^*(X))$.

A uniform space $(X, \mathscr{U})$ may be completed as follows. Let $\mathscr{U}X$ denote the set of minimal $\mathscr{U}$-Cauchy filters on X and identify $x \in X$ with the neighborhood filter $\mathscr{N}_x \in \mathscr{U}X$. For $U \in \mathscr{U}$, set

$$U^* = \{(\mathscr{F}, \mathscr{G}) \in \mathscr{U}X \times \mathscr{U}X: \text{for some } F \in \mathscr{F} \cap \mathscr{G},\ F \times F \subseteq U\},$$

and let $\mathscr{U}^*$ be the uniformity on $\mathscr{U}X$ generated by the uniform base $\{U^*: U \in \mathscr{U}\}$. Then $(\mathscr{U}X, \mathscr{U}^*)$ is a complete, separated uniform space, $\mathscr{U}^*|_X = \mathscr{U}$, and X is $\tau(\mathscr{U}^*)$-dense in $\mathscr{U}X$. If $\mathscr{D}$ is the gauge structure on X associated with $\mathscr{U}$, we shall denote the gauge structure on $\mathscr{U}X$ associated with $\mathscr{U}^*$ by $\mathscr{D}^*$, and $\mathscr{U}X$ may be denoted by $\mathscr{D}X$.

By *proximity* we shall mean separated Efremovič proximity. If δ is a proximity on X, we let $\Pi(\delta)$ denote the set of uniformities on X which induce δ. There is a one-to-one correspondence between the compatible proximities on a Tychonoff space X and the Hausdorff compactifications of X. Given a compatible proximity δ on X the associated Hausdorff compactification δX (called the *Smirnov compactification* of (X, δ)) may be constructed as follows. Let δX denote the set of maximal δ-round filters on X, and identify $x \in X$ with $\mathscr{N}_x \in \delta X$. For $A \subseteq X$, set

$$O(A) = \{\mathscr{F} \in \delta X: A \in \mathscr{F}\}$$

and declare (for $E_1, E_2 \subseteq \delta X$) $E_1 \not\delta^* E_2$ if and only if there are $A_1, A_2 \subseteq X$ with $A_1 \not\delta A_2$ and $E_i \subseteq O(A_i)$ $(i = 1, 2)$. Then δ^* is a separated proximity on δX, $\delta^*|_X = \delta$, X is $\tau(\delta^*)$-dense in δX, and $\tau(\delta^*)$ is compact and Hausdorff. Since a compact Hausdorff space admits a unique proximity, we have (for $A_1, A_2 \subseteq X$) $A_1 \delta A_2$ if and only if

$$\mathrm{cl}_{\delta X} A_1 \cap \mathrm{cl}_{\delta X} A_2 \neq \emptyset.$$

Note also that for $\mathcal{F} \in \delta X$, $\{O(A): A \in \mathcal{F}\}$ is a $\tau(\delta^*)$-neighborhood base at $\mathcal{F}$.

If δ is a proximity on X and $\mathcal{U} \in \Pi(\delta)$, then every minimal $\mathcal{U}$-Cauchy filter is a maximal δ-round filter. So $X \subseteq \mathcal{U}X \subseteq \delta X$. Moreover, $\delta(\mathcal{U}^*) = \delta^*|_{\mathcal{U}X}$ and so $\tau(\mathcal{U}^*) = \tau(\delta^*)|_{\mathcal{U}X}$. It is a consequence of Shirota's Theorem [**9**, p. 229] that, assuming the nonexistence of measurable cardinals, $\mathcal{U}X$ is realcompact as a topological space.

Also, if $\mathcal{U} \in \Pi(\delta)$ is functionally determined by $D \subseteq \mathbf{R}^X$, then a filter $\mathcal{F}$ on X is $\mathcal{U}$-Cauchy if and only if for each $\epsilon > 0$ and finite subset $\{f_1, \ldots, f_n\} \subseteq D$, there is $F \in \mathcal{F}$ such that whenever $x, y \in F$ and $k \in \{1, \ldots, n\}$,

$$|f_k(x) - f_k(y)| \leqq \epsilon;$$

for $A, B \subseteq X$, $A \not\delta B$ if and only if there is $n \in \mathbf{N}$, $f_1, \ldots, f_n \in D$, and $\epsilon > 0$ such that if $x \in A$ and $y \in B$ then for some $k \in \{1, \ldots, n\}$,

$$|f_k(x) - f_k(y)| \geqq \epsilon.$$

If Z_1 and Z_2 are Hausdorff compactifications of a Tychonoff space X, we write $Z_1 \geqq_X Z_2$ and say Z_1 is *projectively larger* than Z_2 if there is a continuous surjection $f: Z_1 \to Z_2$ such that $f(x) = x$ for all $x \in X$. We write $Z_1 =_X Z_2$ and say Z_1 is *isomorphic* to Z_2 if there is a homeomorphism $h: Z_1 \to Z_2$ such that $h(x) = x$ for all $x \in X$. We let $\mathcal{K}(X)$ denote the set of all isomorphism classes of Hausdorff compactifications of X. Then $(\mathcal{K}(X), \geqq_X)$ is a complete upper semilattice and is a complete lattice when X is locally compact. When δ_1 and δ_2 are compatible proximities on X, $\delta_1 \subseteq \delta_2$ if and only if $\delta_1 X \geqq_X \delta_2 X$. We write $\delta_1 \geqq \delta_2$ when $\delta_1 \subseteq \delta_2$ so that the set of compatible proximities on X, partially ordered by $\geqq$, is order-isomorphic to $(\mathcal{K}(X), \geqq_X)$.

Some other concepts will be recalled in later sections. The reader may find references to [**16**], [**9**], [**17**], or [**18**] helpful. The work presented in this paper was initiated in the author's Ph.D. dissertation [**4**], and the author wishes to thank Jack R. Porter for invaluable aid and encouragement during its preparation.

2. Proximities on E-completely regular spaces. Let δ be a compatible proximity on a Tychonoff space X. If T is a realcompactification of X, then $\mathcal{C}(T)$ corresponds to a complete compatible uniformity $\mathcal{V}$ on

T, in which case $(T, \mathscr{V})$ is the completion of $(X, \mathscr{V}|_X)$. We can ask further: when is there a compatible complete uniformity $\mathscr{V}$ on T such that $\delta(\mathscr{V}|_X) = \delta$? Of course, if $\Pi(\delta)$ contains only the totally bounded member, then the answer is: only when $T =_X \delta X$, the Smirnov compactification of (X, δ).

Definition. Let δ be a compatible proximity on a Tychonoff space X, and let T be a Tychonoff extension of X. We say that X is *δ-completable* to T if there is a compatible complete uniformity $\mathscr{V}$ on T such that $\delta(\mathscr{V}|_X) = \delta$.

We note that if X is δ-completable to T, then (without loss of generality) T is an extension of X contained in δX and (assuming the nonexistence of measurable cardinals) T is realcompact.

We call a filter $\mathscr{F}$ on a topological space X *fixed* if

$$\bigcap \{\text{cl}_X F \colon F \in \mathscr{F}\} \neq \emptyset,$$

and *free* otherwise. The fixed minimal $\mathscr{U}$-Cauchy filters on a uniform space $(X, \mathscr{U})$ are the $\tau(\mathscr{U})$-neighborhood filters.

2.1. PROPOSITION. *Let δ be a compatible proximity on a Tychonoff space X, and let $X \subseteq T \subseteq \delta X$. The following are equivalent:*

(a) *X is δ-completable to T.*

(b) *There is a compatible complete uniformity $\mathscr{V}$ on T such that $\delta(\mathscr{V}) = \delta^*|_T$, where δ^* is the unique compatible proximity on δX.*

(c) *There is $\mathscr{U} \in \Pi(\delta)$ such that T is the set of minimal $\mathscr{U}$-Cauchy filters on X.*

(d) *There is $\mathscr{U} \in \Pi(\delta)$ such that $T \backslash X$ is the set of free minimal $\mathscr{U}$-Cauchy filters on X.*

Proof. We shall prove only (a) $\Rightarrow$ (b). Suppose there is a compatible complete uniformity $\mathscr{V}$ on T such that $\delta(\mathscr{V}|_X) = \delta$. Let $\mathscr{V}_1$ be the unique totally bounded uniformity on T such that $\delta(\mathscr{V}_1) = \delta(\mathscr{V})$, and let $\mathscr{V}_2$ be the unique totally bounded uniformity on T such that $\delta(\mathscr{V}_2) = \delta^*|_T$. Then $\delta(\mathscr{V}_1|_X) = \delta$ and $\delta(\mathscr{V}_2|_X) = \delta$. So, since $\mathscr{V}_1|_X$ and $\mathscr{V}_2|_X$ are both totally bounded members of $\Pi(\delta)$, $\mathscr{V}_1|_X = \mathscr{V}_2|_X$. Thus, $\mathscr{V}_1 = \mathscr{V}_2$ since both $\mathscr{V}_1$ and $\mathscr{V}_2$ are compatible on T and X is dense in T. Hence,

$$\delta(\mathscr{V}) = \delta(\mathscr{V}_1) = \delta(\mathscr{V}_2) = \delta^*|_T.$$

2.2. PROPOSITION. *Let δ be a compatible proximity on a Tychonoff space X. If T is a realcompactification of X contained in δX and $\delta X =_T \beta T$, then X is δ-completable to T.*

Proof. Let $\mathscr{V}$ be the complete compatible uniformity on T corresponding to the gauge structure $\mathscr{C}(T)$. For $A, B \subseteq T$, $A\ \delta(\mathscr{V})B$ if and only if

$$\text{cl}_{\beta T} A \cap \text{cl}_{\beta T} B \neq \emptyset$$

if and only if

$$\mathrm{cl}_{\delta X} A \cap \mathrm{cl}_{\delta X} B \neq \emptyset$$

if and only if

$$A \ \delta^*|_T B.$$

So $\delta(\mathscr{V}) = \delta^*|_T$. Thus, X is δ-completable to T by 2.1.

2.3. COROLLARY. *Let X be a Tychonoff space and let δ be the proximity on X corresponding to βX. Then X is δ-completable to every realcompactification of X contained in δX.*

Proof. Let $X \subseteq T \subseteq \delta X =_X \beta X$. Then $\delta X =_T \beta T$ [**9**, p. 89]. So X is δ-completable to T by 2.2 since T is realcompact.

Recall [**17**] that if E is a fixed topological space, then a topological space X is called *E-completely regular* if X is homeomorphic to a subspace of some product of copies of E, and X is *E-compact* if X is homeomorphic to a closed subspace of some product of copies of E. Thus, the [0, 1]-completely regular spaces are the Tychonoff spaces, and the [0, 1]-compact spaces are the compact Hausdorff spaces. The **R**-completely regular spaces are also the Tychonoff spaces, and the **R**-compact spaces are the realcompact spaces which have been studied extensively and characterized in terms of $C(X)$ ([**9**], [**17**]). The **N**-completely regular spaces are the zero-dimensional T_0 spaces. (A topological space is *zero-dimensional* if the clopen subsets of X are a basis for the open sets; a Tychonoff space X is *strongly zero-dimensional* if βX is zero-dimensional.) An *E-compactification* of a topological space X is an E-compact extension of X.

Definition. (a) Let E be a topological space and let δ be a compatible proximity on a Tychonoff space X. δ is an *E-rich proximity* if X is δ-completable to every E-compactification of X contained in δX.

(b) Let E be a topological space and let Z be a Hausdorff compactification of a Tychonoff space X. Z is called an *E-rich compactification* of X if the proximity induced on X by Z is an E-rich proximity.

(c) A compatible proximity on (respectively, a Hausdorff compactification of) a Tychonoff space X is a *rich proximity* (respectively, a *rich compactification* of X) if it is an **R**-rich proximity (respectively, an **R**-rich compactification of X).

2.4. COROLLARY. *For any realcompact space E and any Tychonoff space X, βX is an E-rich compactification of X. In particular, βX is a rich compactification of X.*

Proof. The second assertion follows from 2.3, and, since any E-compact space is realcompact, the first assertion follows from the second.

Every Hausdorff compactification of a pseudocompact Tychonoff space X is a rich compactification of X. Also if X is not E-completely regular, then X has no E-compactifications and, hence, every Hausdorff compactification of X is an E-rich compactification of X. Of course, in this case, no Hausdorff compactification of X can be E-completely regular. Even when X is E-completely regular, βX need not be E-completely regular, as the existence of zero-dimensional, not strongly zero-dimensional spaces shows [**12**]. It is of interest to determine, for a given space E, whether every E-completely regular space X has an E-completely regular Hausdorff compactification. This is not the case for every realcompact space E, as the following example shows.

Example. Let E be a realcompact space which is totally disconnected but not zero-dimensional ([**9**, 16L], [**18**, 29B]). Then every E-completely regular space is totally disconnected, and every compact E-completely regular space is zero-dimensional. So E can have no E-completely regular Hausdorff compactification. (This example was discussed in [**14**] where it was observed that βE is not E-completely regular.)

The next theorem concerns the existence of E-completely regular Hausdorff compactifications of E-completely regular spaces (without regard to their E-richness). The concepts involved are essentially those discussed in [**10**], [**3**], and [**19**].

2.5. THEOREM. *For a Tychonoff space E, the following are equivalent.*

(a) *E has a Hausdorff compactification which is E-completely regular.*

(b) *Every E-completely regular space has a Hausdorff compactification which is E-completely regular.*

(c) *Every E-completely regular space has a Hausdorff compactification which is E-completely regular and is projectively larger than each of its E-completely regular Hausdorff compactifications.*

Proof. We shall prove only (b) $\Rightarrow$ (c). Let X be E-completely regular and from each isomorphism class of E-completely regular compactifications of X choose a representative. Let $\{Z_i: i \in I\}$ be the nonempty set of these representatives. Let $h: X \to \Pi\{Z_i: i \in I\}$ be defined by $\pi_i(h(x)) = x$ $(x \in X, i \in I)$. Then h is an embedding. Let

$$Z = \mathrm{cl}_{\Pi_i Z_i}\, h(X).$$

Then Z is an E-completely regular compactification of $h(X)$ (which we can identify with X via h) and $\pi_i|_Z$ is a continuous surjection of Z onto Z_i such that $\pi_i(h(x)) = x$ for all $x \in X$. So $Z \geqq_X Z_i$.

The remainder of this section is devoted to showing that if E is a realcompact space which satisfies a stronger condition than (a) of 2.5, then the largest E-completely regular compactification of an E-completely regular space is an E-rich compactification. Throughout the remainder of this section E will denote a fixed Tychonoff space.

Definition. For a Tychonoff space X, set

$$D(X) = \{g \circ f: \text{for some } n \in \mathbf{N}, f \in C(X, E^n) \text{ and } g \in C(E^n)\}, \text{ and}$$
$$D^*(X) = D(X) \cap C^*(X).$$

Also define the gauge structures on X: $\mathscr{D}(X) = \mathscr{D}(D(X))$ and $\mathscr{D}^*(X) = \mathscr{D}(D^*(X))$.

Note that $D(X)$ is just the collection of continuous real-valued functions on X which factor continuously through some finite power of E, and also

$$D^*(X) = \{g \circ f: \text{for some } n \in \mathbf{N}, f \in C(X, E^n) \text{ and } g \in C^*(E^n)\}.$$

2.6. Proposition. *Let X be a Tychonoff space. $D(X)$ and $D^*(X)$ are vector sublattices of $C(X)$ which contain the constant functions.*

Proof. Let $h_i = g_i \circ f_i \in D(X)$ where $f_i \in C(X, E^{n_i})$ and $g_i \in C(E^{n_i})$ $(i = 1, 2)$. Let $m: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be continuous. Define $f \in C(X, E^{n_1} \times E^{n_2})$ by

$$f(x) = (f_1(x), f_2(x)) \quad (x \in X),$$

and $g \in C(E^{n_1} \times E^{n_2})$ by

$$g(y_1, y_2) = m(g_1(y_1), g_2(y_2)) \quad (y_i \in E^{n_i}, i = 1, 2).$$

Then $h = g \circ f \in D(X)$. It follows that $D(X)$ is closed under the necessary binary operations and clearly $D(X)$ contains the constant functions.

Since the above function g will be bounded if g_1 and g_2 are bounded, the proof is similar for $D^*(X)$.

2.7. Proposition. *For a Tychonoff space X, the following are equivalent.*

(a) *X is E-completely regular.*

(b) *$\mathscr{D}^*(X)$ is a compatible gauge structure on X.*

(c) *$\mathscr{D}(X)$ is a compatible gauge structure on X.*

Proof. Let τ denote the topology of X. First note that since $\mathscr{D}^*(X) \subseteq \mathscr{D}(X) \subseteq \mathscr{C}(X)$,

$$\tau(\mathscr{D}^*(X)) \subseteq \tau(\mathscr{D}(X)) \subseteq \tau(\mathscr{C}(X)) = \tau.$$

So (b) $\Rightarrow$ (c) is clear. To show (a) $\Rightarrow$ (b) we show $\tau \subseteq \tau(\mathscr{D}^*(X))$ when X is E-completely regular. Let $U \in \tau$ and suppose that $p \in U$. Since X is E-completely regular by [**17**, p. 16] there is an $n \in \mathbf{N}$ and $f \in C(X, E^n)$ such that

$$f(p) \notin \mathrm{cl}_{E^n} f(X \backslash U).$$

Since E^n is Tychonoff, there is $g \in C^*(E^n)$ such that

$$\{y \in E^n: |g(y) - g(f(p))| < 1\} \subseteq E^n \backslash \mathrm{cl}_{E^n} f(X \backslash U).$$

Now if $x \in X$ such that $|g(f(x)) - g(f(p))| < 1$, then

$$f(x) \notin \mathrm{cl}_{E^n} f(X \backslash U)$$

and so $x \notin X \backslash U$. Thus,

$$\{x \in X: |g \circ f(x) - g \circ f(p)| < 1\} \subseteq U.$$

So $U \in \tau(\mathscr{D}^*(X))$.

It remains to show (c) $\Rightarrow$ (a). Suppose that $\mathscr{D}(X)$ is compatible with τ. Let $p \in X$ and let A be a closed subset of X such that $p \notin A$. Then $X \backslash A$ is a neighborhood of p and, since $\mathscr{D}(X)$ is compatible with τ, there are $n_j \in \mathbf{N}, f_j \in C(X, E^{n_j}), g_j \in C(E^{n_j})$ $(j = 1, \ldots, k)$, and $\epsilon > 0$ such that whenever

$$|g_j \circ f_j(x) - g_j \circ f_j(p)| < \epsilon \quad \text{for all } j = 1, \ldots, k,$$

then $x \in X \backslash A$. Let

$$Y = \prod_{j=1}^{k} E^{n_j}$$

and define $f: X \rightarrow Y$ by

$$f(x) = (f_j(x))_j \quad (x \in X).$$

Then f is continuous. Note that the gauge structure $\mathscr{E}$ on Y functionally determined by

$$\bigcup_{j=1}^{k} \{g \circ \pi_j : g \in C(E^{n_j})\}$$

is compatible with the topology of Y [**6**, p. 200]. So

$$G = \{y \in Y: |g_j \circ \pi_j(y) - g_j \circ \pi_j(f(p))| < \epsilon \quad \text{for all } j = 1, \ldots, k\}$$

is a neighborhood of $f(p)$ in Y, and $G \cap f(A) = \emptyset$. So $f(p) \notin \mathrm{cl}_Y f(A)$. Since Y is a finite power of E, by [**17**, p. 16] X is E-completely regular.

2.8. Proposition. *Let X be an E-completely regular space, let $n \in \mathbf{N}$, and let $f \in C(X, E^n)$. Then*

(a) *$f: (X, \mathscr{D}(X)) \rightarrow (E^n, \mathscr{C}(E^n))$ is uniformly continuous, and*

(b) *$f: (X, \mathscr{D}^*(X)) \rightarrow (E^n, \mathscr{C}^*(E^n))$ is uniformly continuous.*

Proof. We must show that for every pseudometric $e \in \mathscr{C}(E^n)$ (respectively, $\mathscr{C}^*(E^n)$), $e \circ (f \times f) \in \mathscr{D}(X)$ (respectively, $\mathscr{D}^*(X)$). Let $e \in \mathscr{C}(E^n)$ (respectively, $\mathscr{C}^*(E^n)$). Let $\epsilon > 0$ be given. There are

$g_1, \dots, g_k \in C(E^n)$ (respectively, $C^*(E^n)$) and $\delta > 0$ such that

$$|g_i(y_1) - g_i(y_2)| < \delta \quad \text{for all } i = 1, \dots, k$$

implies

$$e(y_1, y_2) < \epsilon.$$

Now $g_i \circ f \in D(X)$ (respectively, $D^*(X)$) for $i = 1, \dots, k$. Suppose $x_1, x_2 \in X$ and

$$|g_i \circ f(x_1) - g_i \circ f(x_2)| < \delta \quad \text{for all } i = 1, \dots, k.$$

Then

$$e(f(x_1), f(x_2)) < \epsilon.$$

I.e.,

$$e \circ (f \times f)(x_1, x_2) < \epsilon.$$

So $e \circ (f \times f) \in \mathscr{D}(X)$ (respectively, $\mathscr{D}^*(X)$).

2.9. Corollary. *Let X be an E-completely regular space, let $n \in \mathbf{N}$ and let $f \in C(X, E^n)$. Let β denote the proximity induced on E^n by $\beta(E^n)$. Then*
(a) *f: $(X, \delta(\mathscr{D}(X))) \to (E^n, \beta)$ is a p-map, and*
(b) *f: $(X, \delta(\mathscr{D}^*(X))) \to (E^n, \beta)$ is a p-map.*

Proof. $\delta(\mathscr{C}(E^n)) = \delta(\mathscr{C}^*(E^n)) = \beta$, and every uniformly continuous function is a p-map with respect to the induced proximities. So (a) and (b) follow from (a) and (b) of 2.8.

2.10. Proposition. *Let X be an E-completely regular space, and let B_1, $B_2 \subseteq X$. If $B_1 \delta(\mathscr{D}(X)) B_2$, then there is an $n \in \mathbf{N}$ and $f \in C(X, E^n)$ such that $f(B_1) \beta f(B_2)$ where β is the proximity induced on E^n by $\beta(E^n)$.*

Proof. Suppose $B_1 \not\delta(\mathscr{D}(X)) B_2$. Then there are $n_j \in \mathbf{N}, f_j \in C(X, E^{n_j})$, $g_j \in C(E^{n_j})$ $(j = 1, \dots, k)$, and $\epsilon > 0$ such that whenever $x_1 \in B_1$ and $x_2 \in B_2$ there is $j \in \{1, \dots, k\}$ with

$$|g_j(f_j(x_1)) - g_j(f_j(x_2))| \geqq \epsilon.$$

Set

$$Y = \prod_{j=1}^{k} E^{n_j}$$

and let $\mathscr{E}$ be the compatible gauge structure on Y functionally determined by

$$\bigcup_{j=1}^{k} \{g \circ \pi_j : g \in C(E^{n_j})\}$$

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[**6**, p. 200]. Then $\delta(\mathscr{E})$ is a compatible proximity on Y. Define $f: X \to Y$ by

$$f(x) = (f_j(x))_j \quad (x \in X).$$

Then $f \in C(X, Y)$. If $x_1 \in B_1$ and $x_2 \in B_2$, then there is a $j \in \{1, \ldots, k\}$ such that

$$|g_j \circ \pi_j(f(x_1)) - g_j \circ \pi_j(f(x_2))| \geqq \epsilon.$$

So $f(B_1)\not\delta(\mathscr{E})f(B_2)$. Denote the proximity induced on Y by βY as β. Since $\beta \subseteq \delta(\mathscr{E})$, $f(B_1)\not\beta f(B_2)$. Finally note that Y is homeomorphic to E^n where $n = n_1 + \ldots + n_k$.

2.11. PROPOSITION. *Let X be an E-completely regular space. Then*

$$\delta(\mathscr{D}(X)) = \delta(\mathscr{D}^*(X)).$$

Proof. Since $\mathscr{D}^*(X) \subseteq \mathscr{D}(X)$, $\delta(\mathscr{D}(X)) \subseteq \delta(\mathscr{D}^*(X))$. Suppose that $B_1, B_2 \subseteq X$ and $B_1\not\delta(\mathscr{D}(X))\, B_2$. By 2.10, there is an $n \in \mathbf{N}$ and $f \in C(X, E^n)$ such that $f(B_1)\not\beta f(B_2)$ where β is the proximity induced on E^n by $\beta(E^n)$. By 2.9 (b),

$$f: (X, \delta(\mathscr{D}^*(X))) \to (E^n, \beta)$$

is a p-map. So $B_1 \not\delta(\mathscr{D}^*(X))\, B_2$. Therefore, $\delta(\mathscr{D}^*(X)) \subseteq \delta(\mathscr{D}(X))$.

Definition. For an E-completely regular space X let $\delta_E X$ denote $\delta(\mathscr{D}^*(X))X$, the Smirnov compactification of $(X, \delta(\mathscr{D}^*(X)))$, and let δ_E denote $\delta(\mathscr{D}^*(X))$ when no confusion can arise about the domain of the proximity.

2.12 PROPOSITION. (a) *$\delta_E X$ is E-completely regular for all E-completely regular spaces X if and only if $\beta(E^n)$ is E-completely regular for all $n \in \mathbf{N}$.*

(b) *If X is an E-completely regular space and γ is a compatible proximity on X such that γX is E-completely regular, then $\delta_E \subseteq \gamma$.*

Proof. (a) Suppose that $\delta_E X$ is E-completely regular for every E-completely regular space X. Let $n \in \mathbf{N}$. E^n is certainly E-completely regular, and $D(E^n) = C(E^n)$. So $\beta(E^n) =_{E^n} \delta_E(E^n)$ is E-completely regular.

Conversely, suppose that $\beta(E^n)$ is E-completely regular for all $n \in \mathbf{N}$. Let X be an E-completely regular space, and let δ denote δ_E. Let A_1 and A_2 be disjoint closed subsets of δX. Then $A_1 \not\delta^* A_2$ where δ^* is the unique compatible proximity on δX. So there are subsets $B_1, B_2 \subseteq X$ such that $B_1 \not\delta B_2$ and $A_i \subseteq O(B_i)$ $(i = 1, 2)$. By 2.10 there is $n \in \mathbf{N}$ and $f \in C(X, E^n)$ such that $f(B_1)\not\beta f(B_2)$, where β is the proximity induced on E^n by $\beta(E^n)$. Now $f: (X, \delta) \to (E^n, \beta)$ is a p-map by 2.9. So there is a continuous function $\bar{f}: \delta X \to \beta(E^n)$ such that $\bar{f}|_X = f$. Now for $i = 1, 2$,

$$\bar{f}(A_i) \subseteq \bar{f}(\text{cl}_{\delta X} B_i) \subseteq \text{cl}_{\beta(E^n)} f(B_i),$$

and since $f(B_1) \not\beta f(B_2)$,

$$\text{cl}_{\beta(E^n)} f(B_1) \cap \text{cl}_{\beta(E^n)} f(B_2) = \emptyset.$$

So $\bar{f}(A_1) \cap \bar{f}(A_2) = \emptyset$. Now $\beta(E^n)$ is E-completely regular, and $\bar{f}(A_1)$ and $\bar{f}(A_2)$ are disjoint closed subsets of $\beta(E^n)$. By [**17**, p. 16] and the compactness of $\beta(E^n)$, there is $m \in \mathbf{N}$ and $g \in C(\beta(E^n), E^m)$ such that

$$g(\bar{f}(A_1)) \cap g(\bar{f}(A_2)) = \emptyset.$$

So

$$g \circ \bar{f} \in C(\delta X, E^m) \quad \text{and} \quad g \circ \bar{f}(A_1) \cap g \circ \bar{f}(A_2) = \emptyset.$$

Therefore, by [**17** (3.3, p. 16)], δX is E-completely regular.

(b) Let γ be a compatible proximity on an E-completely regular space X such that γX is E-completely regular. Let $B_1, B_2 \subseteq X$ and suppose that $B_1 \not\gamma B_2$. Set $K_i = \text{cl}_{\gamma X} B_i$ $(i = 1, 2)$. Then $K_1 \cap K_2 = \emptyset$. Since γX is E-completely regular and compact, by [**17**, (3.3, p. 16)] there is an $m \in \mathbf{N}$ and $h \in C(\gamma X, E^m)$ such that $h(K_1) \cap h(K_2) = \emptyset$. $h(K_1)$ and $h(K_2)$ are disjoint compact subsets of E^m. So there is $g \in C(E^m, [0, 1])$ such that

$$g(h(K_1)) \subseteq \{0\} \quad \text{and} \quad g(h(K_2)) \subseteq \{1\}.$$

Set $f = h|_X$. Then $g \circ f \in D^*(X)$. Also if $x_1 \in B_1$ and $x_2 \in B_2$ then

$$|g \circ f(x_1) - g \circ f(x_2)| = 1.$$

So $B_1 \not\delta_E B_2$. Therefore, $\delta_E \subseteq \gamma$.

It follows immediately from 2.12 that if $\beta(E^n)$ is E-completely regular for all $n \in \mathbf{N}$ and X is an E-completely regular space, then δ_E is the proximity induced on X by the largest E-completely regular compactification of X of 2.5.

2.13. Proposition. *Suppose that $\beta(E^n)$ is E-completely regular for all $n \in \mathbf{N}$. Let X be an E-completely regular space, and let $X \subseteq T \subseteq \delta_E X$. Then T is E-completely regular, and $\delta_E T =_T \delta_E X$.*

Proof. By 2.12 (a), $\delta_E X$ is E-completely regular. So T is E-completely regular. Also, by 2.12 (a), $\delta_E T$ is E-completely regular. So $\delta_E T$ is an E-completely regular compactification of X, and therefore $\delta_E X \geqq_X \delta_E T$. Also, $\delta_E X$ is an E-completely regular compactification of T. So $\delta_E T \geqq_T \delta_E X$. So $\delta_E T =_X \delta_E X$ and, since X is dense in T, $\delta_E T =_T \delta_E X$.

2.14. Proposition. *Let E be realcompact. If X is an E-compact space, then $\mathscr{D}(X)$ is a complete gauge structure on X.*

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Proof. Suppose $X \neq \mathscr{D}(X)X$, the completion of $(X, \mathscr{D}(X))$. Let $f \in C(X, E)$. By 2.8 (a), $f: (X, \mathscr{D}(X)) \to (E, \mathscr{C}(E))$ is uniformly continuous. Since E is realcompact, $\mathscr{C}(E)$ is complete [**17** (13.6, p. 146)]. So there is a uniformly continuous function

$$\bar{f}: (\mathscr{D}(X)X, \mathscr{D}(X)^*) \to (E, \mathscr{C}(E))$$

such that $\bar{f}|_X = f$. Thus every member of $C(X, E)$ extends to a member of $C(\mathscr{D}(X)X, E)$ which, by [**17** (4.5, p. 28)] contradicts the E-compactness of X. Therefore, $X = \mathscr{D}(X)X$ and, hence, $\mathscr{D}(X)$ is complete.

2.15. THEOREM. *Let E be a realcompact space such that $\beta(E^n)$ is E-completely regular for all $n \in \mathbf{N}$. If X is an E-completely regular space, then δ_E is an E-rich proximity on X and $\delta_E X$ is an E-rich, E-completely regular compactification of X.*

Proof. Let $X \subseteq T \subseteq \delta_E X$ where T is E-compact. $\mathscr{D}(T)$ is a compatible gauge structure on T and is complete by 2.14. By 2.11, $\delta(\mathscr{D}(T)) = \delta(\mathscr{D}^*(T))$, and the Smirnov compactification of $(T, \delta(\mathscr{D}(T)))$ is $\delta_E T$. By 2.13, $\delta_E T =_T \delta_E X$. Thus,

$$\delta(\mathscr{D}(T)) = \delta(\mathscr{D}^*(X))^*|_T$$

where $\delta(\mathscr{D}^*(X))^*$ is the unique compatible proximity on $\delta_E X$. So, by 2.1, X is $\delta(\mathscr{D}^*(X))$-completable to T. Therefore, $\delta_E = \delta(\mathscr{D}^*(X))$ is an E-rich proximity on X. It follows that $\delta_E X$ is an E-rich compactification of X, and $\delta_E X$ is E-completely regular by 2.12 (a).

Since the class of $\mathbf{R}$-completely regular spaces coincides with the class of Tychonoff spaces, certainly $\beta(\mathbf{R}^n)$ is $\mathbf{R}$-completely regular for all $n \in \mathbf{N}$, and of course $\mathbf{R}$ is realcompact. In this case, it is clear that $\delta_{\mathbf{R}} X =_X \beta X$, the Stone-Čech compactification of X, for any Tychonoff space X. So the results of 2.15 with $E = \mathbf{R}$ have already been proven in 2.4.

Finally we consider the case where $E = \mathbf{N}$. The class of $\mathbf{N}$-completely regular spaces coincides with the class of zero-dimensional Tychonoff spaces. If $n \in \mathbf{N}$, then $\mathbf{N}^n$ is a countable discrete space, and so $\beta(\mathbf{N}^n)$ is zero-dimensional and, hence, $\mathbf{N}$-completely regular. Also, of course, $\mathbf{N}$ is realcompact. In this case, for a zero-dimensional Tychonoff space X, $\delta_{\mathbf{N}} X$ is the Banaschewski zero-dimensional compactification of X [**2**]. Since there exist zero-dimensional Tychonoff spaces which are not strongly zero-dimensional, $\delta_{\mathbf{N}} X$ does not coincide with βX in general. In fact there exists an $\mathbf{N}$-compact space X which is not strongly zero-dimensional [**12**]. For any such space X, one of the following must hold:

(1) $\delta_{\mathbf{N}} X$ is a rich compactification of the realcompact, noncompact space X which is not the Stone-Čech compactification of X, or

(2) there is a realcompact space T such that $X \subseteq T \subseteq \delta_{\mathbf{N}} X$ and T is not $\mathbf{N}$-compact. In the next section we shall see that many realcompact,

noncompact spaces X have rich compactifications besides βX. The existence of realcompact zero-dimensional spaces which are not **N**-compact has been shown by Nyikos [**13**].

3. More E-rich proximities. Throughout this section E will denote a fixed realcompact space with at least two points such that $\beta(E^n)$ is E-completely regular for all $n \in \mathbf{N}$. According to 2.15, for every E-completely regular space X, $\delta_E X$ is an E-completely regular, E-rich compactification of X. In this section we shall show that there are E-completely regular spaces X which admit E-completely regular, E-rich compactifications besides $\delta_E X$.

We shall begin with a specific construction. Let (Y, τ) be a locally compact, noncompact, E-completely regular space such that the one-point compactification αY of Y is E-completely regular. Let $X = Y \oplus Y$ be the topological sum of two copies of Y, and let $\tau \oplus \tau$ denote the topology of X. We set $X_i = Y \times \{i\}$ $(i = 0, 1)$ so that

$$X = Y \times \{0, 1\} = X_0 \cup X_1$$

where X_i is homeomorphic to Y $(i = 0, 1)$. Let $p: X \rightarrow Y$ be defined by

$$p(y, i) = y \quad (y \in Y, i \in \{0, 1\}).$$

Then p is continuous.

3.1. Proposition. *The one-point compactification αX of X is E-completely regular.*

Proof. Let x_0 (respectively, y_0) denote the point at infinity in αX (respectively, αY). p: $X \rightarrow Y$ extends to the continuous function p^α: $\alpha X \rightarrow \alpha Y$ such that $p^\alpha(x_0) = y_0$. Let $x_1, x_2 \in \alpha X$, $x_1 \neq x_2$. If $p^\alpha(x_1) \neq p^\alpha(x_2)$, then since αY is E-completely regular, by [**17** (3.3, p. 16)], there there is $f \in C(\alpha Y, E)$ such that $f(p^\alpha(x_1)) \neq f(p^\alpha(x_2))$. So

$$f \circ p^\alpha \in C(\alpha X, E) \quad \text{and} \quad f \circ p^\alpha(x_1) \neq f \circ p^\alpha(x_2).$$

If $p^\alpha(x_1) = p^\alpha(x_2)$, then $\{x_1, x_2\} = \{(y, 0), (y, 1)\}$ for some $y \in Y$. Since αY is E-completely regular, by [**17** (3.3, p. 16)], there is $f \in C(\alpha Y, E)$ such that $f(y) \neq f(y_0)$. Define g: $\alpha X \rightarrow E$ by $g(x_0) = f(y_0)$, and (for $t \in Y, i \in \{0, 1\}$)

$$g(t, i) = \begin{cases} f(t) & \text{if } i = 0 \\ f(y_0) & \text{if } i = 1. \end{cases}$$

Then $g \in C(\alpha X, E)$ and $g(x_1) \neq g(x_2)$. Therefore, αX is E-completely regular by [**17** (3.3, p. 16)].

Let

$$\mathscr{L} = \{(G, H) \in \tau \times \tau : \mathrm{cl}_Y G \subseteq H \quad \text{and} \quad \mathrm{cl}_Y H \text{ is compact}\}.$$

Let $(G, H) \in \mathscr{L}$. Then $\text{cl}_Y G$ and $Y \backslash H$ are disjoint closed subsets of Y and $\text{cl}_Y G$ is compact. Since Y is E-completely regular, by [**17** (3.3, p. 16)], there is an $n \in \mathbf{N}$ and $g \in C(Y, E^n)$ such that

$$g(\text{cl}_Y G) \cap \text{cl}_{E^n} g(Y \backslash H) = \emptyset.$$

Since $g(\text{cl}_Y G)$ is compact, there is $h \in C(E^n, [0, 1])$ such that

$$h(g(\text{cl}_Y G)) \subseteq \{1\} \quad \text{and} \quad h(\text{cl}_{E^n} g(Y \backslash H)) \subseteq \{0\}.$$

Set $f_{(G,H)} = h \circ g$. Then

$$f_{(G,H)} \in D^*(Y), f_{(G,H)}(Y) \subseteq [0, 1], f_{(G,H)}(\text{cl}_Y G) \subseteq \{1\}, \quad \text{and} \quad f_{(G,H)}(Y \backslash H) \subseteq \{0\}.$$

For $g \in D(Y)$, $i \in \{0, 1\}$, and $(G, H) \in \mathscr{L}$, define $[g; (G, H); i] : X \to \mathbf{R}$ (for $y \in Y, j \in \{0, 1\}$) by

$$[g; (G, H); i](y, j) = \begin{cases} g(y) & \text{if } j \neq i \\ g(y) + f_{(G,H)}(y) & \text{if } j = i. \end{cases}$$

Then $[g; (G, H); i] \in C(X)$.

For $D^*(Y) \subseteq D \subseteq D(Y)$, let

$$\tilde{D} = \{[g; (G, H); i] : i \in \{0, 1\}, g \in D, (G, H) \in \mathscr{L}\}.$$

Then $\mathscr{D}(\tilde{D})$ is a gauge structure on X.

3.2. Proposition. *For $D^*(Y) \subseteq D \subseteq D(Y)$, $\mathscr{D}(\tilde{D})$ is compatible with the topology $\tau \oplus \tau$ on X.*

Proof. Since each member of $\tilde{D}$ is continuous, we have

$$\tau(\mathscr{D}(\tilde{D})) \subseteq \tau \oplus \tau.$$

Let $(x, i) \in X$. A typical basic open neighborhood of (x, i) in X is of the form $H \times \{i\}$ where $x \in H \in \tau$ and $\text{cl}_Y H$ is compact. Let $x \in G \in \tau$ with $\text{cl}_Y G \subseteq H$. Let

$$h = [f_{(G,H)}; (G, H); i].$$

We claim that

$$\{(y, j) \in X : |h(y, j) - h(x, i)| < 1\} \subseteq H \times \{i\}.$$

Note that $h(x, i) = 2$. If $j \neq i$, then

$$h(y, j) = f_{(G,H)}(y) \leqq 1;$$

if $y \notin H$, then $h(y, j) = 0$. So if $|h(y, j) - h(x, i)| < 1$, then $h(y, j) > 1$, and we must have $(y, j) \in H \times \{i\}$, as claimed. Thus, $\tau \oplus \tau \subseteq \tau(\mathscr{D}(\tilde{D}))$.

3.3. Proposition. *If $D^*(Y) \subseteq D_i \subseteq D(Y)$ $(i = 1, 2)$, then*

$$\delta(\mathscr{D}(\tilde{D}_1)) = \delta(\mathscr{D}(\tilde{D}_2)).$$

Proof. It suffices to show that

$$\delta(\mathscr{D}(\widetilde{D^*(Y)})) = \delta(\mathscr{D}(\widetilde{D(Y)})).$$

Since $\mathscr{D}(\widetilde{D^*(Y)}) \subseteq \mathscr{D}(\widetilde{D(Y)})$ we have

$$\delta(\mathscr{D}(\widetilde{D(Y)})) \subseteq \delta(\mathscr{D}(\widetilde{D^*(Y)})).$$

Let $A_1, A_2 \subseteq X$ and suppose $A_1 \not\delta(\mathscr{D}(\widetilde{D(Y)}))\, A_2$. Then there are $g_1, \ldots, g_n \in D(Y)$, $(G_1, H_1), \ldots, (G_n, H_n) \in \mathscr{L}$, $i_1, \ldots, i_n \in \{0, 1\}$, and $\epsilon > 0$ such that $z \in A_1$, $z' \in A_2$ implies for some $k \in \{1, \ldots, n\}$,

$$|[g_k; (G_k, H_k); i_k](z) - [g_k; (G_k, H_k); i_k](z')| \geqq \epsilon.$$

Set $G = \bigcup \{G_k: k = 1, \ldots, n\}$, and $H = \bigcup \{H_k: k = 1, \ldots, n\}$. Then $(G, H) \in \mathscr{L}$.

We claim that $p(A_1)\backslash H \not\delta_E\, p(A_2)\backslash H$. ($\delta_E = \delta(\mathscr{D}(Y)) = \delta(\mathscr{D}^*(Y))$ is the proximity on Y defined in Section 2.) For suppose that $y \in p(A_1)\backslash H$ and $y' \in p(A_2)\backslash H$. Then there are $i, j \in \{0, 1\}$ such that $(y, i) \in A_1$ and $(y', j) \in A_2$. Thus, for some $k \in \{1, \ldots, n\}$,

$$|[g_k; (G_k, H_k); i_k](y, i) - [g_k; (G_k, H_k); i_k](y', j)| \geqq \epsilon.$$

Since $y \notin H$ and $y' \notin H$, we have

$$[g_k; (G_k, H_k); i_k](y, i) = g_k(y) \quad \text{and}$$

$$[g_k; (G_k, H_k); i_k](y', j) = g_k(y').$$

So

$$|g_k(y) - g_k(y')| \geqq \epsilon.$$

Since $g_1, \ldots, g_n \in D(Y)$, this shows that $p(A_1)\backslash H \not\delta_E p(A_2)\backslash H$, as claimed.

Since δ_E is the proximity induced on Y by $\mathscr{D}(D^*(Y))$, there are $a_1, \ldots, a_m \in D^*(Y)$ and $\rho \in (0, 1)$ such that $y \in p(A_1)\backslash H$ and $y' \in p(A_2)\backslash H$ imply for some $l \in \{1, \ldots, m\}$,

$$|a_l(y) - a_l(y')| \geqq \rho.$$

Since Y is locally compact, there is $U \in \tau$ with $\mathrm{cl}_Y H \subseteq U$ and $\mathrm{cl}_Y U$ compact. $(H, U) \in \mathscr{L}$. If $k \in \{1, \ldots, n\}$, then g_k is bounded on $\mathrm{cl}_Y U$. Since $D(Y)$ is closed under infs and sups and contains the constant functions (by 2.6), there is $b_k \in D^*(Y)$ such that

$$b_k|_{\mathrm{cl}_Y U} = g_k|_{\mathrm{cl}_Y U}.$$

Let $\eta = \min\{\epsilon, \rho, 1 - \rho\}$. The functions $[b_k; (G_k, H_k); i_k]$ ($k = 1, \ldots, n$); $[a_l; (G, H); t]$, $[a_l; (H, U); t]$, ($l = 1, \ldots, m$; $t = 0, 1$) form a finite subset of $\widetilde{D^*(Y)}$. We claim that if $z \in A_1$ and $z' \in A_2$, then for one of these functions, h, we have

$$|h(z) - h(z')| \geqq \eta.$$

Let $z = (y, i) \in A_1$ and $z' = (y', j) \in A_2$. First suppose that $y \in U$ and $y' \in U$. Then for some $k \in \{1, \ldots, n\}$,

$$|[g_k; (G_k, H_k); i_k](y, i) - [g_k; (G_k, H_k); i_k](y', j)| \geqq \epsilon.$$

So

$$|[b_k; (G_k, H_k); i_k](y, i) - [b_k; (G_k, H_k); i_k](y', j)| \geqq \eta.$$

Next suppose that $y \in Y\backslash H$ and $y' \in Y\backslash H$. Then

$$y \in p(A_1)\backslash H \quad \text{and} \quad y' \in p(A_2)\backslash H.$$

So, for some $l \in \{1, \ldots, m\}$,

$$|a_l(y) - a_l(y')| > \rho.$$

Now

$$[a_l; (G, H); 0](y, i) = a_l(y) \quad \text{and}$$
$$[a_l; (G, H); 0](y', j) = a_l(y').$$

So

$$|[a_l; (G, H); 0](y, i) - [a_l; (G, H); 0](y', j)| \geqq \rho \geqq \eta.$$

Finally, suppose $y \in H$ and $y' \in Y\backslash U$. If, for some $l \in \{1, \ldots, m\}$,

$$|a_l(y) - a_l(y')| \geqq \rho,$$

then take $t \in \{0, 1\}\backslash\{i\}$. Then

$$[a_l; (H, U); t](y, i) = a_l(y) \quad \text{and}$$
$$[a_l; (H, U); t](y', j) = a_l(y').$$

So

$$|[a_l; (H, U); t](y, i) - [a_l; (H, U); t](y', j)| \geqq \rho \geqq \eta.$$

Otherwise, for all $l \in \{1, \ldots, m\}$,

$$|a_l(y) - a_l(y')| < \rho.$$

Now

$$[a_1; (H, U); i](y, i) = a_1(y) + f_{(H,U)}(y) = a_1(y) + 1$$

and

$$[a_1; (H, U); i](y', j) = a_1(y').$$

Thus,

$$|[a_1; (H, U); i](y, i) - [a_1; (H, U); i](y', j)| = |(1 + a_1(y)) - a_1(y')|$$
$$= |1 + (a_1(y) - a_1(y'))| \geqq 1 - \rho \geqq \eta.$$

This proves the claim.

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It follows from the claim that $A_1 \, \delta(\mathscr{D}(\widetilde{D^*(Y)}))\, A_2$. Therefore,

$$\delta(\mathscr{D}(\widetilde{D(Y)})) = \delta(\mathscr{D}(\widetilde{D^*(Y)})).$$

We shall let γ denote $\delta(\mathscr{D}(\widetilde{D^*(Y)}))$.

3.4. PROPOSITION. *γ is not the proximity induced on X by the compactification $\delta_E X$ of X.*

Proof. Pick $e_0, e_1 \in E$ such that $e_0 \neq e_1$. Define $f: X \to E$ by $f(x) = e_i$ if $x \in X_i$ $(i = 0, 1)$. Then $f \in C(X, E)$. There is $g \in C(E, [0, 1])$ such that $g(e_i) = i$ $(i = 0, 1)$. $g \circ f \in D^*(X)$, and, if $x_i \in X_i$ $(i = 0, 1)$, then

$$|g \circ f(x_0) - g \circ f(x_1)| = 1.$$

So $X_0 \not\delta_E X_1$. We claim that $X_0 \, \gamma \, X_1$. For let $g_1, \ldots, g_n \in D^*(Y)$, (G_1, H_1), $\ldots$, $(G_n, H_n) \in \mathscr{L}$, and $i_1, \ldots, i_n \in \{0, 1\}$. Let

$$y \in Y \backslash \bigcup \{H_k: k = 1, \ldots, n\}.$$

Then, for $k = 1, \ldots, n$,

$$[g_k; (G_k, H_k); i_k](y, i) = g_k(y) \quad (i = 0, 1).$$

Thus,

$$|[g_k; (G_k, H_k); i_k](y, 0) - [g_k; (G_k, H_k); i_k](y, 1)| = 0$$

for all $k = 1, \ldots, n$. So $X_0 \, \gamma \, X_1$ as claimed.

Before we discuss the compactification γX of X we need a few results about γ-round filters on X.

3.5. PROPOSITION. *A $\tau \oplus \tau$-free filter $\mathscr{F}$ on X is γ-round if and only if for each $F \in \mathscr{F}$ there is an $F' \in \mathscr{F}$ with $p^{-1}(p(F')) \not\gamma \, X \backslash F$.*

Proof. Let $\mathscr{F}$ be a $\tau \oplus \tau$-free filter on X. Suppose that $\mathscr{F}$ has the prescribed property. If $F \in \mathscr{F}$, then there is $F' \in \mathscr{F}$ such that

$$p^{-1}(p(F')) \not\gamma \, X \backslash F.$$

Now $F' \subseteq p^{-1}(p(F'))$ and so $p^{-1}(p(F')) \in \mathscr{F}$. Thus, $\mathscr{F}$ is γ-round.

Conversely, suppose $\mathscr{F}$ is γ-round. Let $F \in \mathscr{F}$. Then there is $F_1 \in \mathscr{F}$ such that $F_1 \not\gamma \, X \backslash F$. So there are $g_1, \ldots, g_n \in D^*(Y)$, (G_1, H_1), $\ldots$, $(G_n, H_n) \in \mathscr{L}$, $i_1, \ldots, i_n \in \{0, 1\}$, and $\epsilon > 0$ such that $z \in F_1$, $z' \in X \backslash F$ implies for some $k \in \{1, \ldots, n\}$,

$$|[g_k; (G_k, H_k); i_k](z) - [g_k; (G_k, H_k); i_k](z')| \geqq \epsilon.$$

Set

$$H = \bigcup \{H_k: k = 1, \ldots, n\}.$$

$\mathrm{cl}_Y H$ is compact, whence $p^{-1}(\mathrm{cl}_Y H)$ is compact. Since $\mathscr{F}$ is $\tau \oplus \tau$-free,

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there is $F_2 \in \mathscr{F}$ such that

$$F_2 \cap p^{-1}(\mathrm{cl}_Y H) = \emptyset.$$

Set $F' = F_1 \cap F_2$. Now suppose that

$$(y, i) \in p^{-1}(p(F')) \quad \text{and} \quad (y', j) \in X \backslash F.$$

$(y, l) \in F'$ for some $l \in \{0, 1\}$. So for some $k \in \{1, \ldots, n\}$,

$$|[g_k; (G_k, H_k); i_k](y, l) - [g_k; (G_k, H_k); i_k](y', j)| \geqq \epsilon.$$

Now $(y, l) \in F_2$. So $p(y, l) = y \in Y \backslash H$. Thus

$$[g_k; (G_k, H_k); i_k](y, i) = g_k(y) = [g_k; (G_k, H_k); i_k](y, l).$$

So

$$|[g_k; (G_k, H_k); i_k](y, i) - [g_k; (G_k, H_k); i_k](y', j)| \geqq \epsilon.$$

Therefore, $p^{-1}(p(F')) \not\gamma X \backslash F$.

3.6. COROLLARY. *A $\tau \oplus \tau$-free, γ-round filter $\mathscr{F}$ is generated by the filterbase $\mathscr{B} = \{p^{-1}(p(F)): F \in \mathscr{F}\}$.*

Proof. That $\mathscr{B}$ is a filterbase on X and that $\mathscr{B}$ generates $\mathscr{F}$ both follow from 3.5.

If $\mathscr{F}$ is a filter on X, define $\mathscr{F}^* = \{p(F): F \in \mathscr{F}\}$.

3.7. PROPOSITION. (a) *If $\mathscr{F}$ is a $\tau \oplus \tau$-free, γ-round filter on X, then $\mathscr{F}^*$ is the unique τ-free, δ_E-round filter on Y such that $\mathscr{F}$ is generated by*

$$\{p^{-1}(G): G \in \mathscr{F}^*\}.$$

(b) *If $\mathscr{F}_1$ and $\mathscr{F}_2$ are two distinct $\tau \oplus \tau$-free, γ-round filters on X, then $\mathscr{F}_1^* \neq \mathscr{F}_2^*$.*

(c) *If $\mathscr{G}$ is a τ-free, δ_E-round filter on Y, then there is a $\tau \oplus \tau$-free, γ-round filter $\mathscr{F}$ on X such that $\mathscr{G} = \mathscr{F}^*$.*

(d) *Let $\mathscr{F}$ be a $\tau \oplus \tau$-free, γ-round filter on X. Then $\mathscr{F}$ is a maximal γ-round filter if and only if $\mathscr{F}^*$ is a maximal δ_E-round filter.*

Proof. (a) Clearly $\mathscr{F}^*$ is a filter on Y. If $y \in \bigcap \{\mathrm{cl}_Y p(F): F \in \mathscr{F}\}$, then

$$\emptyset \neq p^{-1}(y) \subseteq \bigcap \{p^{-1}(\mathrm{cl}_Y p(F)): F \in \mathscr{F}\}$$
$$= \bigcap \{\mathrm{cl}_X p^{-1}(p(F)): F \in \mathscr{F}\} = \bigcap \{\mathrm{cl}_X F: F \in \mathscr{F}\},$$

which contradicts the freedom of $\mathscr{F}$. So $\mathscr{F}^*$ is τ-free. To see that $\mathscr{F}^*$ is δ_E-round, let $F \in \mathscr{F}$. We want to find $F' \in \mathscr{F}$ such that

$$p(F') \not\delta_E Y \backslash p(F).$$

There is $F_1 \in \mathscr{F}$ such that $F_1 \not\gamma X \backslash F$. So there are $g_1, \ldots, g_n \in D^*(Y)$, $(G_1, H_1), \ldots, (G_n, H_n) \in \mathscr{L}$, $i_1, \ldots, i_n \in \{0, 1\}$, and $\epsilon > 0$ such that

$z \in F_1$ and $z' \in X \backslash F$ imply for some $k \in \{1, \ldots, n\}$,

$$|[g_k; (G_k, H_k); i_k](z) - [g_k; (G_k, H_k); i_k](z')| \geqq \epsilon.$$

Let $H = \bigcup \{H_k: k = 1, \ldots, n\}$. Then $\mathrm{cl}_Y H$ is compact and, hence, $p^{-1}(\mathrm{cl}_Y H)$ is compact. So there is $F_2 \in \mathscr{F}$ such that

$$F_2 \cap p^{-1}(\mathrm{cl}_Y H) = \emptyset.$$

Set $F' = F_1 \cap F_2$. Let $y \in p(F')$ and $y' \in Y \backslash p(F)$. For some $i \in \{0, 1\}$, $(y, i) \in F'$; $(y', 0) \in X \backslash F$. So for some $k \in \{1, \ldots, n\}$, we have

$$|[g_k; (G_k, H_k); i_k](y, i) - [g_k; (G_k, H_k); i_k](y', 0)| \geqq \epsilon.$$

Since $y \notin H$,

$$[g_k; (G_k, H_k); i_k](y, i) = g_k(y).$$

If $i_k = 0$, then set $a_k = g_k + f_{(G_k, H_k)}$. Then

$$[g_k; (G_k, H_k); i_k](y, i) = a_k(y) \quad \text{and}$$
$$[g_k; (G_k, H_k); i_k](y', 0) = a_k(y').$$

So

$$|a_k(y) - a_k(y')| \geqq \epsilon.$$

If $i_k \neq 0$, then

$$[g_k; (G_k, H_k); i_k](y', 0) = g_k(y'),$$

and so

$$|g_k(y) - g_k(y')| \geqq \epsilon.$$

Thus, the functions g_k, $g_k + f_{(G_k, H_k)}$ $(k = 1, \ldots, n)$ form a finite subset of $D^*(Y)$ such that $y \in p(F')$, $y' \in Y \backslash p(F)$ imply for one of these functions, h, $|h(y) - h(y')| \geqq \epsilon$. So

$$p(F') \not\delta_E Y \backslash p(F),$$

as desired. So $\mathscr{F}^*$ is δ_E-round.

By 3.6, $\mathscr{F}$ is generated by $\{p^{-1}(G): G \in \mathscr{F}^*\}$. Suppose $\mathscr{G}$ is any τ-free, δ_E-round filter on Y such that $\mathscr{F}$ is generated by $\{p^{-1}(G): G \in \mathscr{G}\}$. If $G \in \mathscr{G}$, then $p^{-1}(G) \in \mathscr{F}$. So $p(p^{-1}(G)) \in \mathscr{F}^*$. Thus, $\mathscr{G} \subseteq \mathscr{F}^*$. If $G \in \mathscr{F}^*$, then $G = p(F)$ for some $F \in \mathscr{F}$. There is $H \in \mathscr{G}$ such that $p^{-1}(H) \subseteq F$. So

$$H = p(p^{-1}(H)) \subseteq p(F) = G,$$

and hence, $G \in \mathscr{G}$. So $\mathscr{F}^* \subseteq \mathscr{G}$. Therefore, $\mathscr{F}^* = \mathscr{G}$.

(b) If $\mathscr{F}_1^* = \mathscr{F}_2^*$, then $\mathscr{F}_1$ and $\mathscr{F}_2$ are both generated by $\{p^{-1}(G): G \in \mathscr{F}_1^*\}$, and, hence, $\mathscr{F}_1 = \mathscr{F}_2$.

(c) Let $\mathscr{B} = \{p^{-1}(G): G \in \mathscr{G}\}$. Clearly $\mathscr{B}$ is a filterbase on X. Let $G \in \mathscr{G}$. Then there is $G' \in \mathscr{G}$ such that $G' \not\delta_E Y\backslash G$. So there are $g_1, \ldots, g_n \in D^*(Y)$ and $\epsilon > 0$ such that $y \in G'$, $y' \in Y\backslash G$ imply for some $k \in \{1, \ldots, n\}$,

$$|g_k(y) - g_k(y')| \geqq \epsilon.$$

Let $(y, i) \in p^{-1}(G')$ and $(y', j) \in X\backslash p^{-1}(G)$. Then $y \in G'$ and $y' \in Y\backslash G$. So there is $k \in \{1, \ldots, n\}$ such that

$$|g_k(y) - g_k(y')| \geqq \epsilon.$$

Thus,

$$[g_k; (\emptyset, \emptyset); 0](y, i) - [g_k; (\emptyset, \emptyset); 0](y', j)| = |g_k(y) - g_k(y')| \geqq \epsilon.$$

Therefore, $p^{-1}(G') \gamma X\backslash p^{-1}(G)$. So $\mathscr{B}$ is a γ-round filterbase on X, and $\mathscr{F}$, the filter generated by $\mathscr{B}$, is a γ-round filter on X. If

$$x \in \bigcap \{\text{cl}_X\, p^{-1}(G): G \in \mathscr{G}\},$$

then

$$p(x) \in \bigcap \{p(\text{cl}_X p^{-1}(G)): G \in \mathscr{G}\} = \bigcap \{\text{cl}_Y G: G \in \mathscr{G}\},$$

which contradicts the freedom of $\mathscr{G}$. Thus, $\mathscr{F}$ is a $\tau \oplus \tau$-free filter. It is clear that $\mathscr{F}^* = \mathscr{G}$.

(d) If $\mathscr{F}_1$ and $\mathscr{F}_2$ are $\tau \oplus \tau$-free, γ-round filters on X, then $\mathscr{F}_1 \subseteq \mathscr{F}_2$ if and only if $\mathscr{F}_1^* \subseteq \mathscr{F}_2^*$. Thus, for a $\tau \oplus \tau$-free, γ-round filter, we have $\mathscr{F}$ is a maximal γ-round filter if and only if $\mathscr{F}^*$ is a maximal δ_E-round filter.

Define $p^\gamma: \gamma X \to \delta_E Y$ by

$$p^\gamma(x) = p(x) \ (x \in X) \quad \text{and} \quad p^\gamma(\mathscr{F}) = \mathscr{F}^* \ (\mathscr{F} \in \gamma X\backslash X).$$

3.8. PROPOSITION. *p^γ is a continuous surjection, and*

$$p^\gamma|_{\gamma X\backslash X}: \gamma X\backslash X \to \delta_E Y\backslash Y$$

is one-to-one (and, in fact, a homeomorphism).

Proof. Everything is clear except the continuity of p^γ. Since X and Y are both locally compact and $p^\gamma|_X = p$ is continuous, it suffices to check continuity of p^γ at points of $\gamma X\backslash X$. Let $\mathscr{F} \in \gamma X\backslash X$ and let T be an open subset of $\delta_E Y$ containing $\mathscr{F}^*$. There is an open $G \in \mathscr{F}^*$ such that $O(G) \subseteq T$. Let $F = p^{-1}(G)$. Then F is an open member of $\mathscr{F}$ and $p(F) = G$. We claim that $p^\gamma(O(F)) \subseteq O(G)$. First suppose that $x \in O(F) \cap X$. Then F is a neighborhood of x in X. So $G = p(F)$ is a neighborhood of $p(x) = p^\gamma(x)$ in Y. I.e., $p^\gamma(x) \in O(G)$. Next suppose that $\mathscr{G} \in O(F)\backslash X$. Then

$F \in \mathscr{G}$ and, hence,

$$G = p(F) \in \mathscr{G}^* = p^\gamma(\mathscr{G}).$$

I.e., $p^\gamma(\mathscr{G}) \in O(G)$. Thus $p^\gamma(O(F)) \subseteq (O(G)$, as claimed. So p^γ is continuous at $\mathscr{F}$.

It follows from 2.11 that if $D^*(Y) \subseteq D \subseteq D(Y)$, then $\mathscr{D}(D)$ is a gauge structure on Y and $\delta(\mathscr{D}(D)) = \delta_E$.

3.9. PROPOSITION. *Let $D^*(Y) \subseteq D \subseteq D(Y)$ and let $\mathscr{F}$ be a $\tau \oplus \tau$-free, γ-round filter on X. Then $\mathscr{F}$ is $\mathscr{D}(\tilde{D})$-Cauchy if and only if $\mathscr{F}^*$ is $\mathscr{D}(D)$-Cauchy.*

Proof. Suppose that $\mathscr{F}$ is $\mathscr{D}(\tilde{D})$-Cauchy. Let $g_1, \ldots, g_n \in D$ and $\epsilon > 0$. Since $\mathscr{F}$ is $\mathscr{D}(\tilde{D})$-Cauchy, there is $F \in \mathscr{F}$ such that $z, z' \in F$ implies

$$|[g_k; (\emptyset, \emptyset); 0](z) - [g_k; (\emptyset, \emptyset); 0](z')| \leqq \epsilon$$

for all $k = 1, \ldots, n$. Let $G = p(F)$, and let $y, y' \in G$. For some $i, j \in \{0, 1\}$, $(y, i), (y', j) \in F$. Thus, if $k \in \{1, \ldots, n\}$, then

$$|g_k(y) - g_k(y')| = |[g_k; (\emptyset, \emptyset); 0](y, i) - [g_k; (\emptyset, \emptyset); 0](y', j)| \leqq \epsilon.$$

So $\mathscr{F}^*$ is $\mathscr{D}(D)$-Cauchy.

Conversely, suppose that $\mathscr{F}^*$ is $\mathscr{D}(D)$-Cauchy. Let $g_1, \ldots, g_n \in D$, $(G_1, H_1), \ldots, (G_n, H_n) \in \mathscr{L}$, $i_1, \ldots, i_n \in \{0, 1\}$, and $\epsilon > 0$. Since $\mathscr{F}^*$ is $\mathscr{D}(D)$-Cauchy and τ-free, and $\bigcup \{\mathrm{cl}_Y H_k\colon k = 1, \ldots, n\}$ is compact, there is $G \in \mathscr{F}^*$ such that

$$G \cap (\bigcup \{\mathrm{cl}_Y H_k\colon k = 1, \ldots, n\}) = \emptyset$$

and $y, y' \in G$ implies

$$|g_k(y) - g_k(y')| \leqq \epsilon \quad \text{for all } k = 1, \ldots, n.$$

Let $F = p^{-1}(G)$. Then $F \in \mathscr{F}$. Let $(y, i), (y', j) \in F$ and let $k \in \{1, \ldots, n\}$. Since $y, y' \in Y \backslash H_k$, we have

$$[g_k; (G_k, H_k); i_k](y, i) = g_k(y) \quad \text{and}$$
$$[g_k; (G_k, H_k); i_k](y', j) = g_k(y').$$

Thus,

$$|[g_k; (G_k, H_k); i_k](y, i) - [g_k; (G_k, H_k); i_k](y', j)| \leqq \epsilon.$$

So $\mathscr{F}$ is $\mathscr{D}(\tilde{D})$-Cauchy.

3.10. PROPOSITION. *γX is E-completely regular.*

Proof. Since γX is compact, by [**17**, p. 16] it suffices to show that $C(\gamma X, E)$ separates the points of γX. Let $t_1, t_2 \in \gamma X$, $t_1 \neq t_2$. If $p^\gamma(t_1) \neq$

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$p^\gamma(t_2)$, then there is $f \in C(\delta_E Y, E)$ such that

$$f(p^\gamma(t_1)) \neq f(p^\gamma(t_2))$$

since $\delta_E Y$ is E-completely regular. Then

$$f \circ p^\gamma \in C(\gamma X, E) \quad \text{and} \quad f \circ p^\gamma(t_1) \neq f \circ p^\gamma(t_2).$$

If $p^\gamma(t_1) = p^\gamma(t_2)$ then $t_1, t_2 \in X$. Let $q: \gamma X \to \alpha X$ be the unique continuous surjection of γX onto the one-point compactification, αX, of X. By 3.1, αX is E-completely regular. So there is $f \in C(\alpha X, E)$ such that $f(t_1) \neq f(t_2)$. Then $f \circ q \in C(\gamma X, E)$; $f \circ q(t_1) \neq f \circ q(t_2)$. Therefore, $C(\gamma X, E)$ separates the points of γX.

3.11. Proposition. *Let $X \subseteq T \subseteq \gamma X$. Then T is E-compact if and only if $p^\gamma(T)$ is E-compact.*

Proof. Suppose $p^\gamma(T)$ is E-compact. Since γX is E-compact,

$$T = (p^\gamma)^{-1}(p^\gamma(T))$$

is E-compact by [**17**, p. 24].

Conversely, suppose that T is E-compact. Set $A = X_0 \cup (\gamma X \backslash X)$. Then $A = \gamma X \backslash X_1$ is closed in γX since X_1 is open in γX. So A is compact. Now $p^\gamma|_A: A \to \delta_E Y$ is a one-to-one, continuous surjection, hence, a homeomorphism. $T \cap A$ is a closed subset of T, so $T \cap A$ is an E-compact subspace of A. Thus, $p^\gamma(T \cap A)$ is an E-compact subspace of $\delta_E Y$. But

$$p^\gamma(T \cap A) = p^\gamma(T).$$

So $p^\gamma(T)$ is E-compact.

3.12. Proposition. *γ is an E-rich proximity on X.*

Proof. Let $X \subseteq T \subseteq \gamma X$ where T is E-compact. By 3.11, $S = p^\gamma(T)$ is E-compact, and $Y \subseteq S \subseteq \delta_E Y$. δ_E is an E-rich proximity on Y. In fact, $\mathscr{D}(S) = \mathscr{D}(D(S))$ is a complete gauge structure on S whose restriction to Y induces the proximity δ_E. $\mathscr{D}(S)|_Y$ is functionally determined by

$$D = D^*(Y) \cup \{g|_Y: g \in D(S)\}.$$

Clearly, $D^*(Y) \subseteq D \subseteq D(Y)$. So $S \backslash Y$ is the set of τ-free minimal $\mathscr{D}(D)$-Cauchy filters on Y. Therefore, by 3.9, $(p^\gamma)^{-1}(S \backslash Y) = T \backslash X$ is the set of $\tau \oplus \tau$-free minimal $\mathscr{D}(\tilde{D})$-Cauchy filters. By 3.3, $\delta(\mathscr{D}(\tilde{D})) = \gamma$. So, by 2.1, X is γ-completable to T.

Therefore, γ is an E-rich proximity on X.

The main results obtained thus far are summarized in the following theorem.

3.13. Theorem. *Let E be a realcompact space with at least two points such that $\beta(E^n)$ is E-completely regular for all $n \in \mathbf{N}$. Let X be a noncompact*

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locally compact Hausdorff space which is the topological sum of two homeomorphic subspaces, X_0 and X_1, and whose one-point compactification is E-completely regular. Then X is an E-completely regular space which admits a compatible E-rich proximity γ such that

(a) *γX is E-completely regular,*

(b) *$\gamma X \neq_X \delta_E X$, and*

(c) *if $A_i \subseteq X_i$ and $\mathrm{cl}_X A_i$ is not compact $(i = 0, 1)$, then $A_0 \gamma X_1$ and $X_0 \gamma A_1$.*

Proof. Without loss of generality, $X = Y \oplus Y$ for some space Y, $X_i = Y \times \{i\}$, and we can take $\gamma = \delta(\mathscr{D}(\widetilde{D^*(Y)}))$. Then everything has been proven except (c) in its full generality. The proof of the fact that $X_0 \gamma X_1$ (in 3.4) may be easily modified to prove (c).

Now consider the case where $E = \mathbf{N}$. We already know that $\beta(\mathbf{N}^n)$ is $\mathbf{N}$-completely regular for all $n \in \mathbf{N}$ and that $\mathbf{N}$ is realcompact. (Recall that a space is $\mathbf{N}$-completely regular if and only if it is zero-dimensional and Tychonoff.) The straightforward proof of the following proposition is omitted.

3.14. PROPOSITION. *If Y is a zero-dimensional, noncompact, locally compact Hausdorff space, then the one-point compactification of Y is zero-dimensional.*

3.15. COROLLARY. *If X is a noncompact, locally compact, zero-dimensional Tychonoff space which is the topological sum of two homeomorphic subspaces, then X admits a compatible, $\mathbf{N}$-rich proximity γ such that $\gamma X \neq_X \delta_{\mathbf{N}} X$.*

Proof. This result follows from 3.13 and 3.14.

Next consider the case where $E = \mathbf{R}$. Since $\mathbf{R}$-completely regular coincides with Tychonoff, it is clear that $\beta(\mathbf{R}^n)$ is $\mathbf{R}$-completely regular for all $n \in \mathbf{N}$ and that the one-point compactification of any locally compact Tychonoff space is $\mathbf{R}$-completely regular.

3.16. COROLLARY. *If X is a noncompact, locally compact Hausdorff space which is the topological sum of two homeomorphic subspaces, then X admits a compatible rich proximity γ such that $\gamma X \neq_X \beta X$.*

Proof. This result follows from the previous remarks, 3.13, and the fact that $\delta_{\mathbf{R}} X =_X \beta X$.

We note here that in [**11**] Marin presents a condition on a Hausdorff space E which guarantees that the one-point compactification of every non-compact, locally compact, E-compact space be E-completely regular.

It is of interest to find compatible rich proximities, besides that induced by the Stone-Čech compactification, on spaces which are not necessarily the topological sum of two homeomorphic subspaces. We shall now obtain some results in this direction. In the remainder of this section we will be concerned with rich (**R**-rich) proximities and compactifications of Tychonoff (**R**-completely regular) spaces.

Throughout we will let (X, τ) be a fixed Tychonoff space and let A be a fixed closed subset of (X, τ) such that $K = \mathrm{cl}_X (X\backslash A)$ is compact.

3.17. PROPOSITION. *Let δ_1 be a compatible proximity on $(A, \tau|_A)$. Then there is a unique compatible proximity δ on (X, τ) such that $\delta|_A = \delta_1$.*

Proof. Let (A^*, σ) be a Hausdorff compactification of A such that $A^* =_A \delta_1 A$ and $(A^*\backslash A) \cap X = \emptyset$. Let $X^* = A^* \cup X$ have the weak topology τ^* induced by $\{A^*, K\}$ [**6**]. Then (X^*, τ^*) is a Hausdorff compactification of (X, τ) and $\sigma = \tau^*|_{A^*}$. Let δ^* be the unique compatible proximity on (X^*, τ^*). Then $\delta^*|_{A*}$ is the unique compatible proximity on (A^*, σ). Let $\delta = \delta^*|_X$. Then δ is a compatible proximity on (X, τ), and $\delta|_A = \delta_1$.

Now suppose that δ' is any compatible proximity on (X, τ) such that $\delta'|_A = \delta_1$. Let $(X^\#, \tau^\#)$ be the Smirnov compactification of (X, δ'), and let $\delta^\#$ denote the unique compatible proximity on $(X^\#, \tau^\#)$. Let $A^\# = (X^\#\backslash X) \cup A$. Then $(A^\#, \tau^\#|_{A^\#})$ is a Hausdorff compactification of $(A, \tau|_A)$. Thus, $\delta^\#|_{A^\#}$ is the unique compatible proximity on $(A^\#, \tau|_{A^\#})$. Moreover, $(\delta^\#|_{A^\#})|_A = \delta_1$. Therefore, there is a homeomorphism

$$h_1: (A^\#, \tau^\#|_{A^\#}) \rightarrow (A^*, \sigma)$$

such that $h_1(x) = x$ for all $x \in A$. Define $h: X^\# \rightarrow X^*$ by

$$h(x) = h_1(x) \ (x \in A^\#), \ h(x) = x \ (x \in X\backslash A).$$

Now $\tau^\#$ is the weak topology on $X^\#$ induced by $\{A^\#, K\}$. Thus, since $h|_{A^\#}$ and $h|_K$ are continuous, also h is continuous. Thus, h is a homeomorphism, and $h(x) = x$ for all $x \in X$. Therefore, $\delta' = \delta$.

If $\mathscr{F}$ is a filterbase on X, then we set

$$\hat{\mathscr{F}} = \{G \subseteq X: \quad \text{for some } F \in \mathscr{F}, F \subseteq G\}.$$

3.18. PROPOSITION. *Let δ_1 be a compatible proximity on $(A, \tau|_A)$ and let δ be the unique compatible proximity on (X, τ) such that $\delta|_A = \delta_1$.*

(a) *If $\mathscr{F}$ is a free δ_1-round filter on A, then $\hat{\mathscr{F}}$ is a free δ-round filter on X.*

(b) *If $\mathscr{G}$ is a free δ-round filter on X, then $\mathscr{F} = \mathscr{G} \cap A$ is the unique free δ_1-round filter on A such that $\hat{\mathscr{F}} = \mathscr{G}$.*

(c) *For a free δ_1-round filter $\mathscr{F}$ on A, $\mathscr{F}$ is a maximal δ_1-round filter on A if and only if $\hat{\mathscr{F}}$ is a maximal δ-round filter on X.*

Proof. (a) Clearly $\hat{\mathscr{F}}$ is a filter on X generated by the filterbase $\mathscr{F}$ on X; and since, for $F \in \mathscr{F}$, $\mathrm{cl}_A F = \mathrm{cl}_X F$, $\hat{\mathscr{F}}$ is τ-free. To see that $\hat{\mathscr{F}}$ is δ-round, it suffices to show that $\mathscr{F}$ is a δ-round filterbase on X. Let $F \in \mathscr{F}$. There is $F_1 \in \mathscr{F}$ such that $F_1 \not\delta_1 A\backslash F$. Since $\mathscr{F}$ is $\tau|_A$-free and

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$K \cap A$ is compact, there is $F_2 \in \mathscr{F}$ such that

$$\mathrm{cl}_A\, F_2 \cap (K \cap A) = \emptyset.$$

Now $\mathrm{cl}_A\, F_2 = \mathrm{cl}_X\, F_2 = (\mathrm{cl}_{\delta X} F_2) \cap X$, and $\mathrm{cl}_{\delta X} K = K$. So

$$\mathrm{cl}_{\delta X}\, F_2 \cap \mathrm{cl}_{\delta X}\, K = \emptyset.$$

Thus, $F_2 \not\delta\, K$. Since $X \backslash A \subseteq K$, $F_2 \not\delta\, X\backslash A$. Set $F' = F_1 \cap F_2$. Then $F' \not\delta\, A\backslash F$ and $F' \not\delta\, X\backslash A$. So

$$F' \not\delta\, (A\backslash F) \cup (X\backslash A) = X\backslash F,$$

and $F' \in \mathscr{F}$. Thus, $\mathscr{F}$ is a δ-round filterbase on X.

(b) Since $\mathscr{G}$ is free and K is compact, there is $G \in \mathscr{G}$ such that $G \cap K = \emptyset$, whence $G \subseteq A$. Thus,

$$\mathscr{F} = \mathscr{G} \cap A = \{G \in \mathscr{G} : G \subseteq A\},$$

$\mathscr{F}$ is a filter on A, and $\mathscr{F}$ is a filterbase on X which generates $\mathscr{G}$. For $F \in \mathscr{F}$, $\mathrm{cl}_A\, F = \mathrm{cl}_X\, F$. So $\mathscr{F}$ is $\tau|_A$-free since $\mathscr{G}$ is τ-free. Let $F \in \mathscr{F}$. Then there is $F' \in \mathscr{G}$ such that $F' \not\delta\, X\backslash F$, and clearly $F' \in \mathscr{F}$. So $F' \not\delta\, A\backslash F$ and, since $\delta_1 = \delta|_A$, $F' \not\delta_1\, A\backslash F$. So $\mathscr{F}$ is δ_1-round.

Now suppose that $\mathscr{H}$ is a $\tau|_A$-free, δ_1-round filter on A such that $\hat{\mathscr{H}} = \mathscr{G}$. If $H \in \mathscr{H}$, then $H \in \mathscr{G}$ and $H \subseteq A$. So $H \in \mathscr{F}$. If $F \in \mathscr{F}$, then $F \in \mathscr{G} = \hat{\mathscr{H}}$. So there is $H \in \mathscr{H}$ such that $H \subseteq F$, whence $F \in \mathscr{H}$. Thus, $\mathscr{H} = \mathscr{F}$.

(c) If $\mathscr{F}_1$ and $\mathscr{F}_2$ are two $\tau|_A$-free, δ_1-round filters on A, then $\mathscr{F}_1 \subseteq \mathscr{F}_2$ if and only if $\hat{\mathscr{F}}_1 \subseteq \hat{\mathscr{F}}_2$.

3.19. PROPOSITION. *Let δ_1 be a compatible proximity on A and let δ be the unique compatible proximity on X such that $\delta|_A = \delta_1$. Let $\mathscr{U}_1$ be a uniformity on A such that $\delta(\mathscr{U}_1) = \delta_1$, and suppose that there is a compatible uniformity $\mathscr{U}$ on X such that $\mathscr{U}|_A = \mathscr{U}_1$. Then*

(a) $\delta(\mathscr{U}) = \delta$, *and*

(b) *if $\mathscr{F}$ is a $\tau|_A$-free, δ_1-round filter on A, then $\mathscr{F}$ is $\mathscr{U}_1$-Cauchy if and only if $\hat{\mathscr{F}}$ is $\mathscr{U}$-Cauchy.*

Proof. (a) $\delta(\mathscr{U})$ is a compatible proximity on X such that $\delta(\mathscr{U})|_A = \delta_1$.

(b) Suppose $\mathscr{F}$ is $\mathscr{U}_1$-Cauchy. Let $U \in \mathscr{U}$. Then $V = U \cap (A \times A) \in \mathscr{U}_1$. So there is $F \in \mathscr{F}$ such that $F \times F \subseteq V \subseteq U$. So $F \in \hat{\mathscr{F}}$ and $F \times F \subseteq U$. Thus, $\hat{\mathscr{F}}$ is $\mathscr{U}$-Cauchy. Conversely, suppose that $\hat{\mathscr{F}}$ is $\mathscr{U}$-Cauchy. Let $V \in \mathscr{U}_1$. Then there is $U \in \mathscr{U}$ such that $V = U \cap (A \times A)$. There is $G \in \hat{\mathscr{F}}$ such that $G \times G \subseteq U$. There is $F \in \mathscr{F}$ such that $F \subseteq G$. Then $F \subseteq A$. So

$$F \times F \subseteq U \cap (A \times A) = V.$$

Thus $\mathscr{F}$ is $\mathscr{U}_1$-Cauchy.

3.20. PROPOSITION. *Let δ_1 be a compatible proximity on A and let δ be the* *unique compatible proximity on* X *such that* $\delta|_A = \delta_1$. *Let* $A \subseteq T \subseteq \delta_1 A$. *Then* T *is realcompact if and only if* $X \cup \{\hat{\mathscr{F}}: \mathscr{F} \in T\backslash A\}$ *is a realcompact subspace of* δX.

Proof. Define $h: \delta_1 A \to \delta X$ by

$$h(x) = x \ (x \in A) \quad \text{and} \quad h(\mathscr{F}) = \hat{\mathscr{F}} (\mathscr{F} \in \delta_1 A\backslash A).$$

Then h is a one-to-one function. We claim that h is continuous. First suppose that $x \in A$. A typical basic open neighborhood of $h(x)$ in δX is of the form

$$O(B) = B \cup \{\mathscr{G} \in \delta X\backslash X: B \in \mathscr{G}\}$$

where $x \in B \in \tau$. Let $M = B \cap A$. Then $x \in M \in \tau|_A$, so

$$O(M) = M \cup \{\mathscr{H} \in \delta_1 A\backslash A: M \in \mathscr{U}\}$$

is an open neighborhood of x in $\delta_1 A$. It is easily checked that $h(O(M)) \subseteq O(B)$, so that h is continuous at x. Next suppose that $\mathscr{F} \in \delta_1 A\backslash A$. A typical basic open neighborhood of $h(\mathscr{F}) = \hat{\mathscr{F}}$ in δX is of the form

$$O(G) = \text{int}_X G \cup \{\mathscr{G} \in \delta X\backslash X: G \in \mathscr{G}\},$$

where $G \in \hat{\mathscr{F}}$. There is $F \in \mathscr{F}$ such that $F \subseteq G$ and $F \cap K = \emptyset$. So

$$O(F) = \text{int}_A F \cup \{\mathscr{H} \in \delta_1 A\backslash A: F \in \mathscr{H}\}$$

is an open neighborhood of $\mathscr{F}$ in $\delta_1 A$. Noting that $\text{int}_A F$ is open in X, we can easily check that $h(O(F)) \subseteq O(G)$. So h is continuous at $\mathscr{F}$. Therefore, h is continuous.

Now if $X \cup \{\hat{\mathscr{F}}: \mathscr{F} \in T\backslash A\}$ is realcompact, then

$$h^{-1}(X \cup \{\hat{\mathscr{F}}: \mathscr{F} \in T\backslash A\}) = T$$

is realcompact by [**17**, p. 24]. Note that h is a homeomorphism onto $h(\delta_1 A) = A \cup (\delta X\backslash X)$. So if T is realcompact, then $h(T) = A \cup \{\hat{\mathscr{F}}: \mathscr{F} \in T\backslash A\}$ is realcompact. Thus,

$$X \cup \{\hat{\mathscr{F}}: \mathscr{F} \in T\backslash A\} = h(T) \cup K$$

is realcompact since $h(T)$ is realcompact and K is compact [**17**, p. 87].

Definition. [**8**] Let S be a subspace of Tychonoff space Y.

(a) S is *u_0-embedded* in Y if every compatible uniformity on S has an extension to a compatible uniformity on Y.

(b) S is *u-embedded* (respectively, *u^*-embedded*) in Y if every compatible uniformity which is functionally determined by a collection of continuous (respectively, bounded continuous) real-valued functions on S has an extension to a compatible uniformity on Y.

3.21. PROPOSITION. *$(A, \tau|_A)$ is C-embedded in (X, τ). Thus, $(A, \tau|_A)$ is u-embedded in (X, τ).*

Proof. Let $f \in C(A)$. Then $f|_{A \cap K} \in C(A \cap K)$ and $A \cap K$ is C-embedded in K. So there is $g \in C(K)$ such that

$$g|_{A \cap K} = f|_{A \cap K}.$$

Define $\bar{f}: X \to \mathbf{R}$ by

$$\bar{f}(x) = x \ (x \in A), \bar{f}(x) = g(x) \ (x \in K).$$

Then $\bar{f} \in C(X)$ and $\bar{f}|_A = f$. Therefore, A is C-embedded in X, and it follows that A is u-embedded in X by results from [**8**].

3.22. THEOREM. *Let X be a Tychonoff space and let A be a closed u_0-embedded subset of X such that $\mathrm{cl}_X(X\backslash A)$ is compact. Let δ_1 be a compatible proximity on A, and let δ be the unique compatible proximity on X such that $\delta|_A = \delta_1$. Then δ_1 is a rich proximity on A if and only if δ is a rich proximity on X. Moreover, $\delta X =_X \beta X$ if and only if $\delta_1 A =_A \beta A$.*

Proof. First suppose that δ_1 is a rich proximity on A. Let $X \subseteq S \subseteq \delta X$ where S is realcompact. Let

$$T = A \cup \{\mathscr{G} \cap A: \mathscr{G} \in S\backslash X\}.$$

Then $A \subseteq T \subseteq \delta_1 A$, and $X \cup \{\hat{\mathscr{F}}: \mathscr{F} \in T\backslash A\} = S$ is realcompact. So, by 3.20, T is realcompact. Since δ_1 is a rich proximity on A, there is a uniformity $\mathscr{U}_1$ on A such that $\delta(\mathscr{U}_1) = \delta_1$ and $T\backslash A$ is the set of free minimal $\mathscr{U}_1$-Cauchy filters on A. Since A is u_0-embedded in X, there is a compatible uniformity $\mathscr{U}$ on X such that $\mathscr{U}|_A = \mathscr{U}_1$. By 3.19, $\delta(\mathscr{U}) = \delta$ and $S\backslash X = \{\hat{\mathscr{F}}: \mathscr{F} \in T\backslash A\}$ is the set of free minimal $\mathscr{U}$-Cauchy filters on X. So, by 2.1, X is δ-completable to S. Therefore, δ is a rich proximity on X.

Next suppose that δ is a rich proximity on X. Let $A \subseteq T \subseteq \delta_1 A$ where T is realcompact. Let

$$S = X \cup \{\hat{\mathscr{F}}: \mathscr{F} \in T\backslash A\}.$$

By 3.20, S is realcompact, and, since δ is a rich proximity on X, there is a compatible uniformity $\mathscr{U}$ on X such that $\delta(\mathscr{U}) = \delta$ and $S\backslash X$ is the set of free minimal $\mathscr{U}$-Cauchy filters on X. Then $\mathscr{U}_1 = \mathscr{U}|_A$ is a compatible uniformity on A and $\delta(\mathscr{U}_1) = \delta_1$. Also, by 3.19,

$$T\backslash A = \{\mathscr{F} \in \delta_1 A\backslash A: \ \hat{\mathscr{F}} \in S\backslash X\}$$

is the set of free minimal $\mathscr{U}_1$-Cauchy filters on A. Thus, by 2.1, A is δ_1-completable to T. Therefore, δ_1 is a rich proximity on A.

Now recall, from the proof of 3.20, that the subspace $A^* = A \cup (\delta X\backslash X)$ of δX has the property that $\delta_1 A =_A A^*$. Suppose that $\delta_1 A =_A \beta A$. Then $A^* =_A \beta A$. Let $f \in C^*(X)$. Then $f|_A \in C^*(A)$. So there is $g \in$

$C^*(A^*)$ such that $g|_A = f|_A$. Define $\bar{f}: \delta X \to \mathbf{R}$ by

$$\bar{f}(x) = f(x) \ (x \in X), \bar{f}(x) = g(x) \ (x \in A^*).$$

Then $\bar{f} \in C^*(\delta X)$ and $\bar{f}|_X = f$. Thus, X is C^*-embedded in δX, and so $\delta X =_X \beta X$.

Conversely, suppose that $\delta X =_X \beta X$. Let $f \in C^*(A)$. By 3.21, there is $g \in C^*(X)$ such that $g|_A = f$. Now X is C^*-embedded in δX. So there is $\bar{g} \in C^*(\delta X)$ such that $\bar{g}|_X = g$. Set $\bar{f} = \bar{g}|_{A^*}$. Then $\bar{f} \in C^*(A^*)$ and $\bar{f}|_A = f$. So A is C^*-embedded in A^*, and, hence, $\beta A =_A A^* =_A \delta_1 A$.

3.23. Corollary. *Let X be a collection-wise normal Hausdorff space and let A be a closed subset of X such that $\mathrm{cl}_X (X \backslash A)$ is compact. Then A admits a compatible rich proximity δ_1 such that $\delta_1 A \neq_A \beta A$ if and only if X admits a compatible rich proximity δ such that $\delta X \neq_X \beta X$.*

Proof. A is u_0-embedded in X [**8**]. So this result follows from 3.22.

The next result is a corollary to the proof of 3.22.

3.24. Corollary. *Let X be a noncompact, locally compact Hausdorff space containing a closed subset A such that*

(a) $\mathrm{cl}_X (X \backslash A)$ *is compact, and*

(b) *A is homeomorphic to $Y \oplus Y$ for some space Y.*

Then X admits a compatible rich proximity δ such that $\delta X \neq_X \beta X$.

Proof. A must be locally compact and Hausdorff, and so Y must be a noncompact, locally compact Hausdorff space. By 3.16, A admits a compatible rich proximity δ_1 such that $\delta_1 A \neq_A \beta A$. Moreover, it follows from the proof of 3.12 that δ_1 can be chosen so that if $A \subseteq T \subseteq \delta_1 A$ where T is realcompact, then there is a functionally determined uniformity $\mathcal{U}_1$ on A such that $\delta(\mathcal{U}_1) = \delta_1$ and $T \backslash A$ is the set of free minimal $\mathcal{U}_1$-Cauchy filters on A. Since, by 3.21, A is u-embedded in X, for any such uniformity $\mathcal{U}_1$, there is a compatible uniformity $\mathcal{U}$ on X such that $\mathcal{U}|_A = \mathcal{U}_1$. With this fact, the proof of 3.22 yields a compatible rich proximity δ on X such that $\delta X \neq_X \beta X$.

3.25. Corollary. *The real line $\mathbf{R}$ with the usual topology admits a compatible rich proximity δ such that $\delta \mathbf{R} \neq_{\mathbf{R}} \beta \mathbf{R}$.*

Proof. Take $A = (-\infty, -1] \cup [+1, +\infty)$ and $X = \mathbf{R}$. Then X and A satisfy the hypotheses of either 3.23 or 3.24.

We have seen that many Tychonoff spaces admit more than one rich proximity. The following question asks how far this result can be extended.

Question. Does every Tychonoff space which admits at least two distinct compatible proximities admit at least two compatible rich proximities?

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To see that, in certain instances, the number of compatible rich proximities on a noncompact, realcompact space may be large, we will conclude with an example.

Example. Let X be a countably infinite discrete space. For a subset A of X such that $|A| = |X\backslash A| = \aleph_0$ let $\delta(A)$ be a compatible rich proximity on X such that

(1) if B_1 and B_2 are disjoint subsets of A then $B_1 \not\delta(A)\, B_2$,
(2) if B_1 and B_2 are disjoint subsets of $X\backslash A$, then $B_1 \not\delta(A)\, B_2$,
(3) if B is an infinite subset of A, then $B\, \delta(A)\, X\backslash A$, and
(4) if B is an infinite subset of $X\backslash A$, then $A\, \delta(A)\, B$.

(The existence of $\delta(A)$ follows from 3.13 and its proof.)

Let $\mathscr{B}$ be an almost disjoint family of infinite subsets of X with $|\mathscr{B}| = c(= 2^{\aleph_0})$. (I.e., if $B_1, B_2 \in \mathscr{B}$ and $B_1 \neq B_2$, then $|B_1 \cap B_2| < \aleph_0$). Let $\mathscr{A} = \{X\backslash B: B \in \mathscr{B}\}$. Then $\mathscr{A}$ has these properties:

(a) $|\mathscr{A}| = c$,
(b) if $A \in \mathscr{A}$, then $|A| = |X\backslash A| = \aleph_0$, and
(c) if $A_1, A_2 \in \mathscr{A}$ and $A_1 \neq A_2$, then

$$|A_1 \cap A_2| = |A_1\backslash A_2| = |A_2\backslash A_1| = \aleph_0 \quad \text{and} \quad |X\backslash(A_1 \cup A_2)| < \aleph_0.$$

Now if $A_1, A_2 \in \mathscr{A}$ and $A_1 \neq A_2$, then we claim that $\delta(A_1) \neq \delta(A_2)$. To see this, note that $A_1 \cap A_2$ and $A_1\backslash A_2$ are disjoint subsets of A_1. Thus,

$$(A_1 \cap A_2) \not\delta(A_1)(A_1\backslash A_2).$$

Since $A_1 \cap A_2$ is an infinite subset of A_2,

$$(A_1 \cap A_2)\, \delta(A_2)(X\backslash A_2) = (A_1\backslash A_2) \cup [X\backslash(A_1 \cup A_2)].$$

Since $X\backslash(A_1 \cup A_2)$ is finite,

$$(A_1 \cap A_2) \not\delta(A_2)[X\backslash(A_1 \cup A_2)].$$

Thus, we must have $(A_1 \cap A_2)\, \delta(A_2)\, (A_1\backslash A_2)$. Therefore, $\delta(A_1) \neq \delta(A_2)$, as claimed.

Thus, $\{\delta(A): A \in \mathscr{A}\}$ is a collection of c distinct compatible proximities on X.

Also note that if A_1 and A_2 are two disjoint infinite subsets of X, then $A_1\, \delta(A_1)\, A_2$. Now the smallest member, α, in the lattice of compatible proximities on X is defined by (for $B_1, B_2 \subseteq X$) $B_1\, \alpha\, B_2$ if and only if $B_1 \cap B_2 \neq \emptyset$ or $|B_1| = |B_2| = \aleph_0$. So clearly α is the infimum of the rich proximities in the lattice of compatible proximities on X.

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University of North Dakota,
Grand Forks, North Dakota