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NORMAL FUNCTIONS: L^p ESTIMATES

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ABSTRACT. For a meromorphic (or harmonic) function f, let us call the dilation of f at z the ratio of the (spherical) metric at $f(z)$ and the (hyperbolic) metric at z. Inequalities are known which estimate the sup norm of the dilation in terms of its L^p norm, for $p > 2$, while capitalizing on the symmetries of f. In the present paper we weaken the hypothesis by showing that such estimates persist even if the L^p norms are taken only over the set of z on which f takes values in a fixed spherical disk. Naturally, the bigger the disk, the better the estimate. Also, We give estimates for holomorphic functions without zeros and for harmonic functions in the case that $p = 2$.

1. **Introduction.** Let $\mathbb{C}$ denote the complex plane, let $D = \{z \in \mathbb{C} : |z| < 1\}$, and let $D_r = \{z \in \mathbb{C} : |z| < r\}$. For a meromorphic function f, let

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

denote its spherical derivative. A function f meromorphic in D is called a normal function if the family $\{f \circ \gamma : \gamma \in \mathrm{Aut}(D)\}$ is a normal family in the sense of Montel, where $\mathrm{Aut}(D)$ is the group of Möbius transformations of D onto itself. A harmonic function h is called a normal function if for every sequence $\{h \circ \gamma_n\}$, $\gamma_n \in \mathrm{Aut}(D)$ for $n = 1, 2, \cdots$, there exists a subsequence $\{h \circ \gamma_{n_k}\}$ which locally uniformly converges to a harmonic function, to $+\infty$ or to $-\infty$ identically. It is known that a meromorphic function f is normal if and only if

$$\sup_{z\in D}(1 - |z|^2) f^{\#}(z) = \sup_{z\in D}(1 - |z|^2)\frac{|f'(z)|}{1 + |f(z)|^2} < \infty, \tag{1}$$

and a harmonic function h is normal if and only if

$$\sup_{z\in D}(1 - |z|^2)\frac{|\operatorname{grad} h(z)|}{1 + h^2(z)} < \infty. \tag{2}$$

For the definitions and general properties of normal functions see for example [6], [7] and [8].

The following theorem, proved by Pommerenke [12] for $p = 2$ and by Aulaskari, Hayman, and Lappan [2] for $p > 2$, gives an integral condition for an automorphic meromorphic function to be normal.

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THEOREM. *Let f be a function meromorphic in D and automorphic with respect to a Fuchsian group Γ. If*

$$I = \iint_F (1-|z|^2)^{p-2}\{f^\#(z)\}^p \, dx\, dy < \infty \tag{3}$$

for some $p \geq 2$, where F is a fundamental region of Γ, then f is normal and, furthermore, for $p > 2$,

$$\sup_{z \in D}(1-|z|^2) f^\#(z) \leq 3 \max(I^{1/p}, I^{1/(p-2)}). \tag{4}$$

In [4], we strengthened the conclusion of the above theorem by proving that the assumption (3) implies the strong normality of the function f with respect to the group Γ. *Strong normality* means that

$$(1-|z|^2) f^\#(z) \longrightarrow 0, \quad z \longrightarrow \partial D, z \in F.$$

At the same time, we obtained a similar result for harmonic functions. Now, in Section 3 of this paper, we weaken the assumption (3) by taking the integral only on a subset F_δ of F in which f assumes values in a fixed spherical disk of angular radius δ only. Under this weaker assumption, we prove that f is still normal and that, for $p > 2$, we have

$$M = \sup_{z \in F_\delta}(1-|z|^2) f^\#(z) \leq C_\delta \max(I^{1/p}, \quad I^{1/(p-2)}),$$
$$\sup_{z \in D}(1-|z|^2) f^\#(z) \leq M(1 + 1/R^2) + 1/R,$$

where, C_δ is a constant depending on δ only and $R = \tan(\delta/2)$. In general, there is no estimate like (4) for $p = 2$. However, in Section 4, we prove that such an estimate does exist for holomorphic functions without zeros and $F_\delta = D$, and we give examples to show that our restriction is quite reasonable. As applications of the above results, we obtain, in Section 5, corresponding theorems for harmonic functions, which improve a theorem of Aulaskari and Lappan [3]. In addition, we give some necessary and sufficient conditions for a harmonic function to be normal.

2. **Some lemmas.** The following version of the Ahlfors Lemma is similar to that formulated by Pommerenke [11] and Ahlfors [1]. The proof is almost the same as in [1].

AHLFORS LEMMA. *Let $\rho(z)|dz|$ be a continuous Riemannian metric in D such that for every $z \in D$, either $\rho(z) \leq 1/(1-|z|^2)$ or $\rho(z)|dz|$ is smooth and has constant Gaussian curvature -4 in a neighbourhood of z. Then, in fact $\rho(z) \leq 1/(1-|z|^2)$ for every $z \in D$*

LEMMA 1. *Let h be a real-valued function harmonic in D, then h is normal if and only if for every conjugate harmonic function $\tilde{h}$ of h, the holomorphic function (without zeros) $g = \exp(h + i\tilde{h})$ is normal.*

PROOF. Assume that h is normal. For any sequence $\{\gamma_n\} \subset \mathrm{Aut}(D)$ we can choose a subsequence $\{\gamma_{n_k}\}$ such that $\{h \circ \gamma_{n_k}\}$ locally uniformly converges to a harmonic function

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h_0, to $+\infty$ or to $-\infty$ identically. In the former case we have $|g \circ \gamma_{n_k}| \longrightarrow \exp h_0$. Consequently, by a theorem of Montel about sequences of holomorphic functions bounded locally uniformly, we can choose again a subsequence of $\{g \circ \gamma_{n_k}\}$ which converges locally uniformly to a holomorphic function g_0 with $|g_0| = \exp h_0$. If the latter case happens, then $g \circ \gamma_{n_k} \longrightarrow \infty$ or 0 locally uniformly. This argument is reversible. The lemma is proved.

LEMMA 2. *Let f be a function holomorphic in D without zeros. If* $(1-|z|^2)\,|f'(z)| \leq M$ *for $z \in D$ such that* $|f(z)| = 1$, *then*

$$(1 - |z|^2)|f'(z)| \leq |f(z)|(2|\log|f(z)|| + M)$$

for every point $z \in D$.

PROOF. Set

$$\rho(z)|\,dz| = \frac{|f'(z)||\,dz|}{|f(z)|(2|\log|f(z)|| + M)}.$$

This continuous metric has constant Gaussian curvature -4 at every point $z \in D$ with $|f(z)| \neq 1$ and $f'(z) \neq 0$. In fact, if $|f(z)| > 1$, $\rho(z)|\,dz|$ is obtained from the Poincaré metric of $\mathbb{C} \setminus D_r$, $r = e^{-M/2}$, by the substitution $w = f(z)$. Also, $\rho(z)|\,dz|$ is obtained from the Poincaré metric of $D_{1/r} \setminus \{0\}$ if $|f(z)| < 1$. At a point z with $|f(z)| = 1$ or $f'(z) = 0$, we have $\rho(z) \leq 1/(1 - |z|^2)$ by the assumption of the lemma or $\rho(z) = 0$ respectively. Thus, applying the Ahlfors Lemma gives $\rho(z) \leq 1/(1-|z|^2)$ for every point $z \in D$. This proves the lemma.

LEMMA 3. *Let f be a function meromorphic in D. If* $(1 - |z|^2)|f'(z)| \leq M$ *for $z \in D$ with* $|f(z)| \leq R$, *then*

$$(1 - |z|^2)|f'(z)| \leq \beta|f(z)|^2 - 1/\beta$$

for $z \in D$ with $|f(z)| \geq R$, where

$$\beta = \frac{M + \sqrt{M^2 + 4R^2}}{2R^2} \leq \frac{M}{R^2} + \frac{1}{R}.$$

PROOF. Set

$$\rho(z)|\,dz| = \frac{\beta|f'(z)||\,dz|}{\beta^2|f(z)|^2 - 1}, \text{ if } |f(z)| \geq R,$$

$$\rho(z)|\,dz| = \frac{1}{M}|f'(z)||\,dz|, \text{ if } |f(z)| \leq R.$$

This time, the metric is obtained from the Poincaré metric $\beta|\,dw|/(\beta^2|w|^2-1)$ of $\overline{\mathbb{C}} \setminus D_{\beta^{-1}}$ for z with $|f(z)| > R$. We have $\rho(z) \leq 1/(1 - |z|^2)$ for every $z \in D$ with $|f(z)| \leq R$ by hypothesis. The Ahlfors Lemma gives the conclusion of the lemma.

The following lemma is due to J. Dufresnoy [5].

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LEMMA 4. *Let f be a function meromorphic in the disk D_r and let A denote the spherical area of $f(D_r)$, counted without consideration of multiplicity. If $A \leq \sigma\pi$ with $0 \leq \sigma < 1$, then*

$$f^{\#}(0) \leq \frac{1}{r}\left\{\frac{\sigma}{1-\sigma}\right\}^{1/2}.$$

We will use a result of Hayman [6] on a covering property of meromorphic functions in D, which is stated as follows.

LEMMA 5. *Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ be a function meromorphic in D and let E denote the set of all positive numbers r such that the circle $\{w \in \mathbb{C} : |w| = r\}$ meets $\mathbb{C} \setminus f(D)$. Then*

$$|a_1| \int_E \frac{dr}{(|a_0| + r)^2} \leq 4.$$

LEMMA 6. *Let $f(z)$ and E be defined as in Lemma 5 and let $G = (0,1) \setminus E$. If $a_0 = 0$, then*

$$2\pi \int_G r\,dr \geq \pi\left(\frac{|a_1|}{4 + |a_1|}\right)^2.$$

PROOF. Suppose that G consists of intervals $l_1, l_2, \cdots$, where $l_1 = (0, \delta)$. The value $A = 2\pi \int_G r\,dr$ denotes the total area of the annuli $\{w \in D : |w| \in l_i\}$, $i = 1, 2, \cdots$. Given $\epsilon > 0$, choose $l_1, l_2, \cdots, l_n$ such that

$$\int_{E'} \frac{dr}{r^2} < \int_E \frac{1}{r^2}\,dr + \epsilon,$$

where,

$$E' = (0,1) \setminus \bigcup_{i=1}^{n} l_i.$$

Thus, by Lemma (5),

$$\int_{E'} \frac{dr}{r^2} < \frac{4}{|a_1|} + \epsilon. \tag{5}$$

Moving the finite number of intervals $l_1, l_2, \cdots, l_n$ to the left to form a single interval $(0, r')$ so that they lie one after another without gaps nor overlaps, we have

$$\pi r'^2 = 2\pi \int_0^{r'} r\,dr \leq \sum_{i=1}^{n} 2\pi \int_{l_i} r\,dr \leq A, \tag{6}$$

since the integral $\int_{l_i} r\,dr$ decreases as l_i is moved to the left. On the other hand, E' is moved to the right when we move the l_i to the left, so

$$\int_{E'} \frac{dr}{r^2} \geq \int_{r'}^{1} \frac{dr}{r^2} = \frac{1}{r'} - 1, \tag{7}$$

since $\int_{E'} r^{-2}\,dr$ decreases when each of its intervals is moved to the right. Combining (5), (6) and (7), we obtain

$$A \ge \pi r'^2 > \pi\left(\frac{1}{4/|a_1| + 1 + \epsilon}\right)^2.$$

Since ϵ may be arbitrarily small, we have

$$A \ge \pi\left(\frac{|a_1|}{4 + |a_1|}\right)^2.$$

The lemma is proved.

As a consequence of Lemma 4, we have the following.

LEMMA 7. *Let h be a real-valued function harmonic in D_r. If*

$$\iint_{D_r} \left\{\frac{|\operatorname{grad} h(z)|}{1 + h^2(z)}\right\}^2 dx\,dy \le \sigma\pi,$$

with $0 \le \sigma < 1$, *then*

$$\frac{|\operatorname{grad} h(0)|}{1 + h^2(0)} \le \frac{1}{r}\left\{\frac{\sigma}{1-\sigma}\right\}^{1/2}.$$

PROOF. Let $f = h + i\tilde{h}$ be a holomorphic function and $\tilde{h}(0) = 0$. Since

$$f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2} \le \frac{|\operatorname{grad} h(z)|}{1 + h^2(z)},$$

we have

$$\iint_{D_r} \{f^{\#}(z)\}^2\,dx\,dy \le \sigma\pi.$$

Thus, Lemma 4 gives

$$f^{\#}(0) \le \frac{1}{r}\left\{\frac{\sigma}{1-\sigma}\right\}^{1/2}.$$

Since

$$f^{\#}(0) = \frac{|\operatorname{grad} h(0)|}{1 + h^2(0)},$$

the conclusion of Lemma 7 follows.

3. **Meromorphic functions.**

THEOREM 1. *Let $p \ge 2$, let f be a function meromorphic in D and automorphic with respect to a Fuchsian group Γ, let F be a fundamental region for Γ, and let K_δ be a spherical disk whose angular radius measured from the center of the sphere is δ. If*

(5) $$I = \iint_{F_\delta} (1 - |z|^2)^{p-2} \{f^{\#}(z)\}^p\,dx\,dy < \infty$$

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where $F_\delta = \{z \in F : f(z) \in K_\delta\}$*, then* f *is normal. Furthermore, if* $p > 2$*, set*

$$M = \sup_{z\in F_\delta}(1-|z|^2)f^\#(z), \quad R = \tan(\delta/2),$$

then we have

$$M \le \max(23I^{1/p}, \quad 7I^{1/(p-2)}) \text{ if } \delta \ge \pi/2, \tag{6}$$

$$M \le \max(46I^{1/p}, \quad 14R^{-2/(p-2)}I^{1/(p-2)}) \text{ if } \delta \le \pi/2, \tag{7}$$

$$\sup_{z\in D}(1-|z|^2)f^\#(z) \le M(1+1/R^2)+1/R. \tag{8}$$

PROOF. If $p = 2$, the value of the integral I denotes the spherical area of the part of the covering surface $f(D)$ over K_δ, and $I < \infty$ implies that, for almost every point $w \in K_\delta$, the inverse image $f^{-1}(w)$ has only finitely many points in F. Thus, according to a theorem of Pommerenke [12], f is normal.

The normality of f in the case that $p > 2$ is a consequence of (6), (7) and (8). However, we would like to give an independent proof. If f is not normal then, by a theorem of Lohwater and Pommerenke [10], there exists a sequence $\{z_n\} \subset D$ and a sequence of positive numbers $\{\rho_n\}$ such that $\rho_n = o(1-|z_n|^2)$ and $g_n(z) = f(z_n+\rho_n z)$ converges to a non-constant meromorphic function $g(z)$, spherically and locally uniformly in $\mathbb{C}$. Since g assumes every complex value with two possible exceptions, it is clear that there exists a positive number R' such that $g_n(D_{R'}) \cap K_\delta$ has a spherical area, without consideration of multiplicity, greater than $\pi(1-\cos\delta)/4$ for sufficiently large n. Set $\phi_n(z) = z_n + \rho_n z$ and $\Delta_n = \phi_n(D_{R'})$. Then $f(\Delta_n)\cap K_\delta$ has a spherical area $A_n \ge \pi(1-\cos\delta)/4$. For any n, let $E_n \subset \Delta_n$ be a measurable set such that $f(z) \in K_\delta$ for $z \in E_n$, no points in E_n are equivalent and, for every point z in Δ_n with $f(z) \in K_\delta$, there is a point $\zeta \in E_n$ equivalent to z. Since f is automorphic,

$$f(\Delta_n)\cap K_\delta = f(E_n), \quad \iint_{E_n}\{f^\#(z)\}^2\,dx\,dy = A_n \ge \pi(1-\cos\delta)/4.$$

Let $E'_n \subset F$ be a measurable set equivalent to E_n. Then $E'_n \subset F_\delta$ and

$$\iint_{E'_n}(1-|z|^2)^{p-2}\{f^\#(z)\}^p\,dx\,dy = \iint_{E_n}(1-|z|^2)^{p-2}\{f^\#(z)\}^p\,dx\,dy,$$

since f is automorphic. Now, we have

$$\begin{aligned}
&\iint_{F_\delta}(1-|z|^2)^{p-2}\{f^\#(z)\}^p\,dx\,dy\\
&\quad\ge \iint_{E'_n}(1-|z|^2)^{p-2}\{f^\#(z)\}^p\,dx\,dy = \iint_{E_n}(1-|z|^2)^{p-2}\{f^\#(z)\}^p\,dx\,dy\\
&\quad\ge \left(\iint_{E_n}(1-|z|^2)^{-2}\,dx\,dy\right)^{1-p/2}\left(\iint_{E_n}\{f^\#(z)\}^2\,dx\,dy\right)^{p/2}\\
&\quad\ge \delta_n^{1-p/2}(\pi(1-\cos\delta)/4)^{p/2},
\end{aligned}$$

where δ_n is the non-Euclidian area of Δ_n, which tends to zero since $\rho_n = o(1-|z_n|^2)$. This contradicts the assumption (5), since $\delta_n^{1-p/2} \longrightarrow \infty$. The normality of f is proved.

Now, we proceed to prove the second half of Theorem 1. To prove (6), choose a point $z_0 \in F_\delta$ arbitrarily. We want to prove that if $\delta \geq \pi/2$, then

$$(1-|z_0|^2)f^\#(z_0) \leq \max(23I^{1/p}, 7I^{1/(p-2)}).$$

Without loss of generality we may, by replacing $f(z)$ by $f\big((z+z_0)/(1+\bar{z}_0 z)\big)$, assume that $z_0 = 0$. Then, the above inequality becomes

$$f^\#(0) \leq \max(23I^{1/p}, 7I^{1/(p-2)}). \tag{9}$$

Let $\alpha' > 0$ be the solution of the equation

$$I^{2/p}\left(\frac{4\pi}{3\alpha'^2}\right)^{1-2/p} = \frac{2}{5}\pi, \tag{10}$$

and let $\alpha = \max(\alpha', 2)$. Let $E \subset D_{1/\alpha}$ be a measurable set such that (i) $f(z) \in K_\delta$ for $z \in E$, (ii) no points in E are equivalent, and (iii) for every point z in $D_{1/\alpha}$ with $f(z) \in K_\delta$, there is a point $\zeta \in E$ equivalent to z. There is a measurable set $E' \subset F$ which is equivalent to E. Then, $f(D_{1/\alpha}) \cap K_\delta = f(E)$, $E' \subset F_\delta$ and

$$\iint_{E'}(1-|z|^2)^{p-2}\{f^\#(z)\}^p\,dx\,dy = \iint_E (1-|z|^2)^{p-2}\{f^\#(z)\}^p\,dx\,dy,$$

since f is automorphic.

There are two different cases $\alpha' \geq 2$ and $\alpha' < 2$ to be discussed separately. Note that $\alpha' \geq 2$ if and only if $I \geq (6/5)^{p/2}\pi/3$. If $\alpha' \geq 2$, then

$$\alpha = \alpha' = \left(\frac{5}{2\pi}\right)^{p/2(p-2)}\left(\frac{4\pi}{3}\right)^{1/2} I^{1/(p-2)}. \tag{11}$$

By Hölder's inequality for non-Euclidean area measure, noting (11) and (10), we have

$$\begin{aligned}
&\iint_E \{f^\#(z)\}^2\,dx\,dy \\
&\quad\leq \left(\iint_E (1-|z|^2)^{p-2}\{f^\#(z)\}^p\,dx\,dy\right)^{2/p}\left(\iint_E (1-|z|^2)^{-2}\,dx\,dy\right)^{1-2/p} \\
&\quad\leq \left(\iint_{E'} (1-|z|^2)^{p-2}\{f^\#(z)\}^p\,dx\,dy\right)^{2/p}\left(\iint_{D_{1/\alpha}} (1-|z|^2)^{-2}\,dx\,dy\right)^{1-2/p} \\
&\quad\leq I^{2/p}\left(\frac{4\pi}{3\alpha^2}\right)^{1-2/p} = I^{2/p}\left(\frac{4\pi}{3\alpha'^2}\right)^{1-2/p} = \frac{2}{5}\pi.
\end{aligned}$$

However,

$$f(D_{1/\alpha}) \subset (\mathbb{C} \setminus K_\delta) \cup \big(f(D_{1/\alpha}) \cap K_\delta\big) = (\mathbb{C} \setminus K_\delta) \cup f(E),$$

so the spherical area, without consideration of multiplicity, of $f(D_{1/\alpha})$ is not greater than

$$\frac{\pi}{2} + \iint_E \{f^\#(z)\}^2\,dx\,dy \leq \frac{\pi}{2} + \frac{2}{5}\pi = \frac{9}{10}\pi,$$

since $\delta \geq \pi/2$. Thus, it follows from Lemma 4 and (11) that

$$f^{\#}(0) \leq \alpha\left(\frac{9/10}{1-9/10}\right)^{1/2} = 3\alpha \tag{12}$$
$$= 3\left(\frac{5}{2\pi}\right)^{p/2(p-2)}\left(\frac{4\pi}{3}\right)^{1/2} I^{1/(p-2)}.$$

If $\alpha' < 2$, then $\alpha = 2$ and, since the equation (10) has a solution $\alpha' < 2$, the left side of (10) will be less than $2\pi/5$ when α' is replaced by 2. Thus, in this case,

$$\iint_E \{f^{\#}(z)\}^2\,dx\,dy \leq (\pi/3)^{1-2/p} I^{2/p} < 2\pi/5.$$

Consequently, by Lemma 4 and the definition of E, we have

$$f^{\#}(0) \leq 2\left(\frac{(\pi/3)^{1-2/p}I^{2/p} + \pi/2}{\pi - (\pi/3)^{1-2/p}I^{2/p} - \pi/2}\right)^{1/2} \tag{13}$$
$$\leq 2(10/\pi)^{1/2}\left((\pi/3)^{1-2/p}I^{2/p} + \pi/2\right)^{1/2}.$$

The estimate (13) is not good for small I, since the upper bound for $f^{\#}(0)$ tends to a constant $2 \cdot 5^{1/2}$ as $I \longrightarrow 0$. To get a better bound for $f^{\#}(0)$, we assume that $f^{\#}(0) \leq 6$. By a rotation of the w-sphere which carries $w = f(0)$ to $w = 0$, we may assume that $f(0) = 0$. Of course, the spherical disk K_δ is also carried to another one which is still denoted by K_δ and which now contains 0. Now, we have $f(0) = 0$ and $|f'(0)| \leq 6$. Set $g(z) = f(z/2)$ for $z \in D$. Let G denote the set of all positive numbers $r < 1$ such that the circle $\{w \in D : |w| = r\}$ is contained in $g(D) = f(D_{1/2})$ completely. Let $H = \{w \in D : |w| \in G\}$ and let A be the Euclidean area of H. Then, applying Lemma 6 to the function $g(z)$, we know that

$$A = 2\pi \int_G r\,dr \geq \pi\left(\frac{|g'(0)|}{4 + |g'(0)|}\right)^2 = \pi\left(\frac{|f'(0)|}{8 + |f'(0)|}\right)^2.$$

The spherical area of H is not less than $A/4$. Since $H \subset D$ consists of annuli with center $w = 0$, $0 \in K_\delta$ and $\delta \geq \pi/2$, it is clear that $H \cap K_\delta$ has a spherical area not less than $A/8$. Define E and E' in $D_{1/2}$ just as above. Since $H \cap K_\delta \subset f(D_{1/2}) \cap K_\delta = f(E)$, the spherical area of $f(E)$ is not less than $A/8$. Thus,

$$\iint_E \{f^{\#}(z)\}^2\,dx\,dy \geq \frac{A}{8} \geq \frac{\pi}{8}\left(\frac{|f'(0)|}{8 + |f'(0)|}\right)^2 \geq \frac{\pi}{1568}|f'(0)|^2.$$

From the preceding paragraph, we have

$$\iint_E \{f^{\#}(z)\}^2\,dx\,dy \leq (\pi/3)^{1-2/p} I^{2/p}.$$

Therefore, for $I < (6/5)^{p/2}\pi/3$ and $f^{\#}(0) < 6$,

$$|f'(0)|^2 \leq \frac{1568}{\pi}\iint_E \{f^{\#}(z)\}^2\,dx\,dy \leq \frac{1568}{\pi}\left(\frac{\pi}{3}\right)^{1-2/p} I^{2/p} < 523(3/\pi)^{2/p} I^{2/p}, \tag{14}$$
$$f^{\#}(0) = |f'(0)| \leq 23(3/\pi)^{1/p} I^{1/p}.$$

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Let us return to the estimates for $f^\#(0)$ we have obtained earlier. If $I \geq (6/5)^{p/2}\pi/3$, then, by (12), we have

$$f^\#(0) \leq 3\left(\frac{5}{2\pi}\right)^{p/2(p-2)}\left(\frac{4\pi}{3}\right)^{1/2} I^{1/(p-2)} < 7I^{1/(p-2)}.$$

If $I < (6/5)^{p/2}\pi/3$, then (13) is valid. However, the right side of (13) is less than 6 as $I < (6/5)^{p/2}\pi/3$. By (14), we have

$$f^\#(0) \leq 23(3/\pi)^{1/p} I^{1/p} < 23I^{1/p} \text{ for } I < (6/5)^{p/2}\pi/3.$$

Hence, (9) and, consequently, (6) is proved.

To prove (7) and (8), we may assume that $K_\delta = \{w \in \mathbb{C} : |w| < R = \tan(\delta/2)\}$. For an arbitrary $\delta \leq \pi/2$, set $g(z) = R^{-1}f(z)$. Then, $|g(z)| < 1$ for $z \in F_\delta$ and $|g(z)| \geq 1$ for $z \in F \setminus F_\delta$. Thus, we have

$$g^\#(z) = \frac{R^{-1}|f'(z)|}{1 + |f(z)|^2/R^2} \leq \frac{R^{-1}|f'(z)|}{1 + |f(z)|^2} = R^{-1}f^\#(z),$$
$$f^\#(z) = \frac{R|g'(z)|}{1 + R^2|g(z)|^2} \leq R|g'(z)| \leq \frac{2R|g'(z)|}{1 + |g(z)|^2} = 2Rg^\#(z)$$

for $z \in F_\delta$, and consequently,

$$\sup_{z\in F_\delta}(1 - |z|^2)f^\#(z) \leq 2R \sup_{z\in F_\delta}(1 - |z|^2)g^\#(z),$$
$$I' = \iint_{F_\delta}(1 - |z|^2)^{p-2}g^\#(z)^p\,dx\,dy \leq R^{-p}I.$$

Applying the result we have proved for $\delta \geq \pi/2$ to $g(z)$ and noting the above inequalities, we obtain

$$\begin{aligned}\sup_{z\in F_\delta}(1 - |z|^2)f^\#(z) &\leq 2R \sup_{z\in F_\delta}(1 - |z|^2)g^\#(z) \\ &\leq \max\Big(46R(I')^{1/p}, \quad 14R(I')^{1/(p-2)}\Big) \\ &\leq \max\Big(46I^{1/p}, \quad 14R^{-2/(p-2)}I^{1/(p-2)}\Big).\end{aligned}$$

This proves (7).

Let $z \in D$ be such that $|f(z)| < R$ and $\zeta \in F$ be the point equivalent to z, then $|f(\zeta)| < R$, $\zeta \in F_\delta$, and consequently,

$$(1 - |z|^2)|f^\#(z)| = (1 - |\zeta|^2)f^\#(\zeta) \leq M, \quad (1 - |z|^2)|f'(z)| \leq M(1 + R^2).$$

By continuity, for $|f(z)| \leq R$,

$$(1 - |z|^2)|f'(z)| \leq M(1 + R^2).$$

Thus, from Lemma 3,

$$(1 - |z|^2)|f'(z)| \leq (M + M/R^2 + 1/R)|f(z)|^2$$

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for $z \in D$ with $|f(z)| \geq R$, and

$$(1-|z|^2)f^\#(z) \leq M(1+1/R^2)+1/R \text{ for } z \in D.$$

This proves (8), and the proof of Theorem 1 is complete.

In the conclusion of Theorem 1, there is a factor $R^{-2/(p-2)}$ preceding $I^{1/(p-2)}$, which tends to ∞ as $\delta \longrightarrow 0$ for a fixed p. We show that the power $-2/(p-2)$ is best by the following example.

EXAMPLE 1. Let $f_n(z) = nz$ for $z \in D$ and $n = 1, 2, \cdots$, and let $K_\delta = \{w \in \mathbb{C} : |w| < R = \tan(\delta/2)\}$. Then, $F = D$ and $F_\delta = D_{R/n}$ for $f_n(z)$. We have

$$M = \sup_{z \in F_\delta}(1-|z|^2)f^\#(z) = f^\#(0) = n$$

and, for fixed p,

$$\begin{aligned} I &= \iint_{F_\delta}(1-|z|^2)^{p-2}\{f^\#(z)\}^p\,dx\,dy \\ &= \int_0^{2\pi}\int_0^{R/n}\frac{(1-r^2)^{p-2}n^p}{(1+n^2r^2)^p}r\,dr\,d\theta \approx \int_0^{2\pi}\int_0^{R/n}\frac{n^p}{(1+n^2r^2)^p}r\,dr\,d\theta \\ &= \frac{\pi n^{p-2}}{p-1}\left(1-\frac{1}{(1+R^2)^{p-1}}\right) \approx \pi R^2 n^{p-2}, \text{ as } R \longrightarrow 0, n \longrightarrow \infty. \end{aligned}$$

Thus,

$$M/I^{1/(p-2)} \approx \pi^{-1/(p-2)}R^{-2/(p-2)}.$$

4. **Holomorphic functions without zeros.** In the theorem formulated in the introduction, the estimate (4) is valid only for $p > 2$. Set $f_n(z) = nz$ $(n = 1, 2, \cdots)$. We have, for $p = 2$,

$$I_n = \iint_D \{f_n^\#(z)\}^2\,dx\,dy < \pi,$$

but $f_n^\#(0) \longrightarrow \infty$. This shows that it is, in general, impossible to bound $(1-|z|^2)f^\#(z)$ in terms of the integral I for $p = 2$. Note that the functions $f_n(z)$ do not assume ∞. The following example indicates that there need not be such an estimate for $p = 2$ even for functions which are automorphic with respect to a fixed group and omit two fixed complex values.

EXAMPLE 2. We consider the functions $f_n(z) = ne^z$ $(n = 1, 2, \cdots)$ in the left half-plane $L = \{z \in \mathbb{C} : \Re z < 0\}$. They do not assume 0 and ∞, and are automorphic with respect to the group generated by the mapping $\gamma(z) = z + 2\pi i$, which has the strip $F = \{z \in \mathbb{C} : \Re z < 0, 0 \leq \Im z < 2\pi\}$ as its fundamental region. Recall that the Poincaré metric on L is $-(2\Re z)^{-1}|dz|$. It is obvious that

$$\iint_F \{f_n^\#(z)\}^2\,dx\,dy < \pi, \text{ for } n = 1, 2, \cdots.$$

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However, letting $z_n = -\log n \in F$ for $n = 2, 3, \cdots$, we have

$$f_n^{\#}(z_n) = n|e^{z_n}|/(1 + n^2|e^{z_n}|^2) = 1/2,$$
$$(-2\Re z_n)f^{\#}(z_n) = \log n \longrightarrow \infty.$$

Nevertheless, for all meromorphic functions which omit two fixed complex values, we do have an estimate like (4), in which the integral I is taken over the whole unit disk D

THEOREM 2. *Let f be a function holomorphic in D without zeros, and let*

$$M = \sup_{z\in D}(1 - |z|^2)f^{\#}(z),$$
$$I = \iint_D \{f^{\#}(z)\}^2\, dx\, dy.$$

If $I < \infty$, then f is normal and

$$M \le C \max(I^{1/2}, I), \tag{15}$$

where, C is an absolute constant.

PROOF. By the result of Pommerenke [12], we know that f is normal, *i.e.*, $M < \infty$. Let $z_0 \in D$ be a point such that $(1 - |z_0|^2)f^{\#}(z_0) \ge 14M/15$. We may assume that $z_0 = 0$ and $|f(0)| \le 1$. Then, $|f'(0)| \ge 14M/15$. We have

$$\sup_{|f(z)|=1} (1 - |z|^2)|f'(z)| \le 2M$$

and, by Lemma 2,

$$(1 - |z|^2)|f'(z)| \le 2|f(z)|(|\log|f(z)|| + M) \text{ for } z \in D. \tag{16}$$

Thus, for every θ, since $\{r \in [0, 1-\delta) : |f(re^{i\theta})| = 1\}$ consists of finitely many points and segments,

$$\frac{\partial}{\partial r} \log(\log^+ |f(re^{i\theta})| + M) \le \frac{2}{1 - r^2}$$

for all $r \in [0, 1)$ with a countable number of exceptional values r, and consequently

$$\log^+ |f(z)| \le \frac{2M|z|}{1 - |z|} \text{ for } z \in D.$$

To prove (15), first assume that $M \ge 2$. Set $g(z) = \{f(z)\}^{2/M}$. Since $f(z) \ne 0, \infty$, $g(z)$ is a single-valued function. Then, if $z \in D$ is a point such that $|f(z)| < 1/3$, we have, from (16),

$$\begin{aligned}(1 - |z|^2)f^{\#}(z) &\le \frac{2|f(z)|\log(|f(z)|^{-1}) + 2M|f(z)|}{1 + |f(z)|^2}\\ &\le \sup_{[0,1/3]} 2 \cdot \frac{x \log x^{-1} + Mx}{1 + x^2} < \frac{14}{15}M.\end{aligned}$$

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Therefore, we conclude that $1/3 \le |f(0)| \le 1$, and

$$3^{-2/M} \le |g(0)| \le 1, \quad |g'(0)| = \frac{2}{M}|f(0)|^{2/M-1}|f'(0)| \ge 20/15,$$

and

$$\log^+ |g(z)| = \frac{2}{M}\log^+ |f(z)| \le \frac{4|z|}{1-|z|} < 4 \text{ for } z \in D_{1/2}.$$

It is well-known that $g(D_{1/2})$ contains a disk $\Delta_1 = \{w \in \mathbb{C} : |w - g(0)| < C_1\}$, where $C_1^{-1} = q(1+e^4)$. Set $\Delta' = \{w \in \Delta_1 : 3^{-2/M} < |w| < 1\}$. Then, $1/3 < |f(z)| < 1$ and $3^{-2/M} < |g(z)| < 1$ for $z \in g^{-1}(\Delta')$. The area A of Δ' tends to zero as $M \longrightarrow \infty$, since Δ' is thinner and thinner when $M \longrightarrow \infty$. However, it is clear that there exists an absolute constant C' such that $A \ge (C'M)^{-1}$. Now, we have

$$\begin{aligned} I &\ge \iint_{g^{-1}(\Delta')} \{f^{\#}(z)\}^2\,dx\,dy \ge \frac{1}{4}\iint_{g^{-1}(\Delta')} |f'(z)|^2\,dx\,dy \\ &= \frac{M^2}{16}\iint_{g^{-1}(\Delta')} |g(z)|^{M-2}|g'(z)|^2\,dx\,dy \\ &\ge \frac{M^2}{144}\iint_{g^{-1}(\Delta')} |g'(z)|^2\,dx\,dy \\ &\ge \frac{AM^2}{144} \ge \frac{M}{144C'}, \end{aligned}$$

$$M \le (144C')I.$$

If $M < 2$, then we have, for $z \in D_{1/2}$,

$$\log^+ |f(z)| \le \frac{4|z|}{1-|z|} < 4, \quad |f(z)| < e^4.$$

Thus,

$$\begin{aligned} I &\ge \iint_{D_{1/2}} \{f^{\#}(z)\}^2\,dx\,dy \\ &\ge (1+e^8)^{-2}\iint_{D_{1/2}} |f'(z)|^2\,dx\,dy \ge \frac{\pi}{4}(1+e^8)^{-2}|f'(0)|^2, \end{aligned}$$

$$\frac{14}{15}M \le |f'(0)| \le \frac{2}{\pi^{1/2}}(1+e^8)I^{1/2},$$

$$M \le \frac{15}{7\pi^{1/2}}(1+e^8)I^{1/2} < 6040I^{1/2}.$$

This completes the proof of Theorem 2.

The conclusion (15) of Theorem 2 states that $M \le CI$ for $I > 1$. One may expect that CI can be replaced by $CI^{1/2}$. However, this is impossible as the following example shows.

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EXAMPLE 3. Set $f_n(z) = nz^n$ for $z \in U = \{z \in \mathbb{C} : \Im z > 0\}$. Recall that the Poincaré metric of U is $(2\Im z)^{-1}|\,dz|$. Then,

$$\iint_U \{f_n^\#(z)\}^2\,dx\,dy = \frac{n\pi}{2}.$$

On the other hand, we have

$$(2\Im z)f_n^\#(z) = \frac{2n|z|^{n-1}\Im z}{1+|z|^{2n}} \le \frac{2n|z|^n}{1+|z|^{2n}} \le n,$$

and

$$(2\Im z)f_n^\#(z) = n \text{ for } z = i.$$

Thus,

$$\sup_{z\in U}(2\Im z)f^\#(z) = n.$$

Theorem 2 may be generalized so that 0 and ∞ are replaced by any two distinct complex values. We can also consider the situation where the integral I is taken over a subset of D, not the whole disk.

5. **Harmonic functions.** The following theorems on harmonic functions are direct consequences of Theorems 1 and 2.

THEOREM 3. *Let h be a real-valued function harmonic in D. If*

$$I = \iint_G (1-|z|^2)^{p-2}|\operatorname{grad} h(z)|^p\,dx\,dy < \infty,$$

where, $p \ge 2$, $G = \{z \in D : a < h(z) < b\}$, then h is normal.

PROOF. Set $f(z) = \exp\big(h(z) + i\tilde{h}(z)\big)$, where $\tilde{h}(z)$ is a harmonic function conjugate to $h(z)$. Then,

$$f^\#(z) = \frac{|\operatorname{grad} h(z)|}{\exp\big(-h(z)\big)+\exp\big(h(z)\big)} \le \frac{1}{2}|\operatorname{grad} h(z)|,$$

$$\iint_G (1-|z|^2)^{p-2}\{f^\#(z)\}^p\,dx\,dy \le 2^{-p}I < \infty.$$

Since $z \in G$ if and only if $e^a < |f(z)| < e^b$, it follows from Theorem 1 that f is normal. Consequently, h is also normal by Lemma 1. This proves Theorem 3.

THEOREM 4. *Let h be a real-valued function harmonic in D. If*

$$I = \iint_D \frac{|\operatorname{grad} h(z)|^2}{\Big(\exp\big(-h(z)\big)+\exp\big(h(z)\big)\Big)^2}\,dx\,dy < \infty, \tag{17}$$

then h is normal and

$$\sup_{z\in D}(1-|z|^2)\frac{|\operatorname{grad} h(z)|}{\exp\big(-h(z)\big)+\exp\big(h(z)\big)} \le M = C\max(I^{1/2}, I),$$

where C is the constant in Theorem 2.

Both Theorem 3 and 4 improve a result of Aulaskari and Lappan [3], which asserts the normality of a harmonic function having the property

$$\iint_D \frac{|\operatorname{grad} h(z)|^2}{\left(1+h^2(z)\right)^2}\,dx\,dy < \infty. \tag{18}$$

As consequences of Lemma 2 and Theorem 1, we have the following results.

THEOREM 5. *Let f be a holomorphic function in D without zeros. If*

$$(1-|z|^2)|f'(z)| \le M$$

for $z \in D$ with $|f(z)| = 1$, then

$$(1-|z|^2)f^{\#}(z) \le \frac{|f(z)|(2\big|\log|f(z)|\big| + M)}{1+|f(z)|^2} \le A + \frac{M}{2}$$

for every $z \in D$, where A is an absolute constant, and consequently f is normal.

THEOREM 6. *Let h be a real-valued function harmonic in D. If*

$$(1-|z|^2)|\operatorname{grad} h(z)| \le M$$

for $z \in D$ with $h(z) = a$, then h is normal and

$$(1-|z|^2)|\operatorname{grad} h(z)| \le 2|h(z)-a| + M$$

for every $z \in D$.

PROOF. Set $f(z) = \exp\big(h(z)-a+i\tilde{h}(z)\big)$, where $\tilde{h}(z)$ is a harmonic function conjugate to $h(z)$. Then,

$$(1-|z|^2)|f'(z)| = (1-|z|^2)|\operatorname{grad} h(z)| \le M$$

for $z \in D$ with $|f(z)| = 1$, *i.e.*, $h(z) = a$. By Theorem 5 and Lemma 1, h is normal. By Lemma 2, we have

$$(1-|z|^2)|f'(z)| \le |f(z)|(2\big|\log|f(z)|\big| + M) \text{ for } z \in D.$$

Thus,

$$(1-|z|^2)|\operatorname{grad} h(z)| \le 2|h(z)-a| + M \text{ for } z \in D.$$

The theorem is proved.

THEOREM 7. *For a real-valued function h harmonic in D the following five conditions (the constants M may be different) are equivalent:*

(i) h is normal;

(ii) there exists a positive number M and a real value a such that

$$(1-|z|^2)|\operatorname{grad} h(z)| \le M$$

for $z \in D$ with $h(z) = a$;

(iii) there exists a positive number M such that

$$(1-|z|^2)|\operatorname{grad} h(z)| \le M + 2|h(z)| \text{ for } z \in D;$$

(iv) there exists a positive number M such that

$$(1-|z|^2)|\operatorname{grad} h(z)| \le M\big(1+h^2(z)\big) \text{ for } z \in D;$$

(v) there exists a constant M such that

$$(1-|z|^2)|\operatorname{grad} h(z)| \le M\Big(\exp\big(-h(z)\big) + \exp\big(h(z)\big)\Big) \text{ for } z \in D.$$

PROOF. It is obvious that (iii) implies (iv), (iv) implies (v), and (v) implies (ii). Theorem 6 asserts that (ii) implies (iii). It is known that (i) is equivalent to (iv). This proves Theorem 7.

EXAMPLE 4. Consider the normal harmonic function $h(z) = y$ in the upper half-plane. We have $|\operatorname{grad} h(z)| = 1$. Recall that the Poincaré metric of the upper half-plane is

$$\lambda(z)|dz| = (2y)^{-1}|dz|.$$

Then,

$$\lambda(z)^{-1}|\operatorname{grad} h(z)| = 2h(z).$$

This shows that, for a normal harmonic function $h(z)$, (iii) is the best upper bound for $\lambda(z)^{-1}|\operatorname{grad} h(z)|$ in terms of $|h(z)|$.

To conclude this paper, we give a bound for $(1-|z|^2)|\operatorname{grad} h(z)|/\big(1+h^2(z)\big)$ in terms of the integrals in (17) and (18).

THEOREM 8. *Let h be a function harmonic in D and let*

$$I = \iint_D \frac{|\operatorname{grad} h(z)|^2}{\Big(\exp\big(-h(z)\big) + \exp\big(h(z)\big)\Big)^2}\, dx\, dy,$$

$$I' = \iint_D \left\{\frac{|\operatorname{grad} h(z)|}{1+h^2(z)}\right\}^2 dx\, dy,$$

$$M' = \sup_{z\in D}(1-|z|^2)\frac{|\operatorname{grad} h(z)|}{1+h^2(z)}.$$

If $I < \infty$, then

$$M' \leq \max(A_M, 2M), \tag{19}$$

where, $M = C\max(I^{1/2}, I)$ with the absolute constant C defined in Theorem 2,

$$A_M = M \cdot \frac{e^{x_M} + e^{-x_M}}{1 + x_M^2} = \frac{2x_M + 2M}{1 + x_M^2},$$

and x_M is the unique positive solution of the equation $M \operatorname{ch} x = x + M$. If $I' < \infty$, then

$$M' \leq \max\bigl((2/\pi)^{1/2} I'^{1/2}, \quad 3CI'\bigr), \tag{20}$$

where C is also the absolute constant defined in Theorem 2.

PROOF. If $I < \infty$, by Theorem 4 we know that

$$\sup_{z\in D}(1 - |z|^2)\frac{|\operatorname{grad} h(z)|}{\exp\bigl(-h(z)\bigr) + \exp\bigl(h(z)\bigr)} \leq M = C\max(I^{1/2}, I). \tag{21}$$

In particular,

$$(1 - |z|^2)|\operatorname{grad} h(z)| \leq 2M$$

for $z \in D$ with $h(z) = 0$. Then, by Theorem 6,

$$(1 - |z|^2)|\operatorname{grad} h(z)| \leq 2|h(z)| + 2M$$

for every $z \in D$. Consequently,

$$(1 - |z|^2)\frac{|\operatorname{grad} h(z)|}{1 + h^2(z)} \leq \frac{2|h(z)| + 2M}{1 + h^2(z)} \tag{22}$$

for every $z \in D$. From (21), we have

$$(1 - |z|^2)\frac{|\operatorname{grad} h(z)|}{1 + h^2(z)} \leq M \cdot \frac{\exp\bigl(-h(z)\bigr) + \exp\bigl(h(z)\bigr)}{1 + h^2(z)} \tag{23}$$

for every $z \in D$.

Consider the functions

$$f_M(x) = M \cdot \frac{e^x + e^{-x}}{1 + x^2}, \quad g_M(x) = \frac{2x + 2M}{1 + x^2},$$

for $x \geq 0$. We have $f_M(0) = g_M(0)$. The other point x such that $f_M(x) = g_M(x)$ is the unique positive solution x_M of the equation

$$M \operatorname{ch} x = x + M. \tag{24}$$

It is obvious that x_M increases with $1/M$ and that $x_M \longrightarrow 0$ as $M \longrightarrow \infty$ and $x_M \longrightarrow \infty$ as $M \longrightarrow 0$. There exists an absolute constant x_1 such that $f_M(x)$ decreases as $0 \leq x \leq x_1$ and increases as $x \geq x_1$, while $g_M(x)$ increases as $0 \leq x \leq x_2$ and decreases as $x \geq x_2$, where $x_2 = \big((M+1)^{1/2} + M\big)^{-1} < 1$. We can show that $x_1 \approx 1.5434$, however the easier estimate $1 < x_1 < 2$ is sufficient for our purposes. From the above facts, $x_M > x_2$, $g_M(x)$ decreases as $x \ge x_M$, and

$$\min\big(f_M(x), g_M(x)\big) = f_M(x)$$

(25) $$\le \max(f_M(x_M), 2M) = \max(A_M, 2M), \text{ for } 0 \le x \le x_M,$$

(26) $$\min\big(f_M(x), g_M(x)\big) = g_M(x) \le g_M(x_M) = A_M, \text{ for } x \ge x_M.$$

It follows from (22), (23), (25) and (26) that

$$\sup_{z\in D}(1-|z|^2)\frac{|\operatorname{grad} h(z)|}{1+h^2(z)} \le \max(A_M, 2M).$$

This proves (19).

Now, assume that $I' < \infty$. It is obvious that $I \le I'$. If $I' \le \pi/2$, it follows from Lemma 7 that

$$\frac{|\operatorname{grad} h(0)|}{1+h^2(0)} \le \left(\frac{I'/\pi}{1-I'/\pi}\right)^{1/2} \le (2/\pi)^{1/2} I'^{1/2}.$$

For any point $z' \in D$, let $\gamma \in \operatorname{Aut}(D)$ be such that $\gamma(0) = z'$ and let $\phi(z) = h\big(\gamma(z)\big)$. Then,

$$\iint_D \left\{\frac{|\operatorname{grad}\phi(z)|}{1+\phi^2(z)}\right\}^2 dx\,dy = \iint_D \left\{\frac{|\operatorname{grad} h(z)|}{1+h^2(z)}\right\}^2 dx\,dy = I' \le \pi/2.$$

Thus,

$$(1-|z'|^2)\frac{|\operatorname{grad} h(z')|}{1+h^2(z')} = \frac{|\operatorname{grad}\phi(0)|}{1+\phi^2(0)} \le (2/\pi)^{1/2} I'^{1/2}.$$

This proves that $M' \le (2/\pi)^{1/2} I'^{1/2}$ when $I' \le \pi/2$.

If $I' > \pi/2$, it follows from (22), since C is quite large and

$$M = C\max(I^{1/2}, I) \le C\max(I'^{1/2}, I') = CI',$$

that

$$(1-|z|^2)\frac{|\operatorname{grad} h(z)|}{1+h^2(z)} \le \frac{2|h(z)| + 2M}{1+h^2(z)} \le 1 + 2M \le 1 + 2CI' < 3CI'.$$

This proves (20) for $I' > \pi/2$ and the proof of the theorem is complete.

Let us investigate estimate (19) in Theorem 8 further. Assume that M_0 is a constant such that

$$\frac{e^{x_{M_0}} + e^{-x_{M_0}}}{1 + x_{M_0}^2} = 2,$$

i.e., $A_{M_0} = f_{M_0}(x_{M_0}) = 2M_0$. Then, $\max(A_M, 2M) = 2M$ for $M \ge M_0$, and $\max(A_M, 2M) = A_M$ for $M \le M_0$. A numerical calculation gives $x_{M_0} \approx 2.9829$ and $M_0 = x_{M_0}^{-1} \approx 0.3352$. Thus, for $I \ge 1$, $M = CI$ and

(27) $$M' \le 2M = 2CI.$$

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Letting $M \longrightarrow 0$, we have

$$x_M \longrightarrow \infty, \quad A_M = \frac{2x_M + 2M}{1 + x_M^2} \longrightarrow 0,$$
$$A_M = \frac{2}{x_M} \cdot \frac{1 + M/x_M}{1 + (x_M)^{-2}} \approx \frac{2}{x_M}.$$

Let $x = \log M^{-1}$ in equation (24). Then the right side will be smaller than the left side provided that M is sufficiently small. This shows $x_M > \log M^{-1}$ and $A_M < \big(2 + o(1)\big)(\log M^{-1})^{-1}$ as $M \longrightarrow 0$. Thus, since $M = CI^{1/2}$ for $I \leq 1$, (19) becomes

$$M' \leq A_M < \big(4 + o(1)\big)(\log I^{-1})^{-1} \tag{28}$$

for sufficiently small I. We do not know if the coefficient $4+o(1)$ in (28) is best. However, the following example indicates that $4 + o(1)$ cannot be replaced by a constant $k < 1$.

EXAMPLE 5. Let $h_{m,n}(z) = n(m+x)$ for $z = x + iy \in D$, $0 < m < 1$ and $n = 1, 2, \cdots$. We have

$$M'_{m,n} = \sup_{z\in D}(1 - |z|^2)\frac{|\operatorname{grad} h_{m,n}(z)|}{1 + h_{m,n}^2(z)}$$
$$= \sup_{z\in D} \frac{n(1 - |z|^2)}{1 + n^2(m + x)^2} = \sup_{-1<x<1} \frac{n(1 - x^2)}{1 + n^2(m + x)^2}.$$

The function $n(1-x^2)/(1+n^2(m+x)^2)$ attains its maximum at $x_{m,n} \in (-1, 1)$, where $x_{m,n}$ is the solution of the equation $x + n^2(m + x)(mx + 1) = 0$. It is obvious that $x_{m,n} \longrightarrow -1/m$ as $n \longrightarrow \infty$. Thus, for a given m,

$$nM'_{m,n} = \frac{1 - x_{m,n}^2}{n^{-2} + (m + x_{m,n})^2} \longrightarrow \frac{1 - m^{-2}}{(m - m^{-1})^2} = \frac{1}{m^2 - 1}, \tag{29}$$
$$M'_{m,n} \approx \frac{1}{m^2 - 1} \cdot \frac{1}{n} \text{ as } n \longrightarrow \infty.$$

On the other hand,

$$I_{m,n} = \iint_D \frac{|\operatorname{grad} h_{m,n}(z)|^2}{\Big(\exp\big(-h_{m,n}(z)\big) + \exp\big(h_{m,n}(z)\big)\Big)^2}\, dx\, dy$$
$$= \iint_D \frac{n^2}{(e^{-n(m+x)} + e^{n(m+x)})^2}\, dx\, dy$$
$$\leq \iint_D n^2 e^{-2n(m+x)}\, dx\, dy$$
$$\leq 2n^2 e^{-2nm} \int_{-1}^{1} e^{-2nx}\, dx$$
$$\leq ne^{-2n(m-1)}.$$

For a given m, we have

$$(\log I_{m,n}^{-1})^{-1} \leq \big(\log n^{-1} + 2n(m - 1)\big)^{-1}$$
$$\leq \left(\frac{1}{2(m - 1)} + o(1)\right) \cdot \frac{1}{n} \text{ as } n \longrightarrow \infty. \tag{30}$$

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Combining (29) and (30), we see that the coefficient $4 + o(1)$ in (28) cannot be replaced by a constant $k < 1$, since

$$\frac{1}{m^2-1} : \frac{1}{2(m-1)} \longrightarrow 1 \text{ as } m \longrightarrow 1.$$

Theorem 8 asserts that

$$M' = \sup_{z\in D}(1-|z|^2)\frac{|\operatorname{grad} h(z)|}{1+h^2(z)} \le 3CI' = 3C\iint_D \frac{|\operatorname{grad} h(z)|^2}{\big(1+h^2(z)\big)^2}\,dx\,dy$$

for large I', where C is the absolute constant defined in Theorem 2. The following example shows that in the above estimate I' cannot be replaced by I'^{α} with $\alpha < 1$.

EXAMPLE 6. Let $h_n(z) = nx$ for $z \in D$ and $n = 1, 2, \cdots$. Then,

$$M' = \sup_{z\in D}\frac{n(1-|z|^2)}{1+n^2x^2} = n,$$

$$I' = \iint_D \frac{n^2}{(1+n^2x^2)^2}\,dx\,dy = \frac{n^2}{2}\int_0^{2\pi}\frac{d\theta}{1+n^2\cos^2\theta} = \frac{n^2\pi}{\sqrt{1+n^2}}.$$

Thus, $M' \approx I'/\pi$ as $n \longrightarrow \infty$.

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