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On the Commutators of Singular Integral Operators with Rough Convolution Kernels

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Abstract. Let T_Ω be the singular integral operator with kernel $(\Omega(x))/|x|^n$, where Ω is homogeneous of degree zero, has mean value zero, and belongs to $L^q(S^{n-1})$ for some $q \in (1,\infty]$. In this paper, the authors establish the compactness on weighted L^p spaces and the Morrey spaces, for the commutator generated by $\mathrm{CMO}(\mathbb{R}^n)$ function and T_Ω. The associated maximal operator and the discrete maximal operator are also considered.

1 Introduction

In the last sixty years, considerable attention has been paid to the mapping properties of singular integral operators with homogeneous kernels. Let Ω be homogeneous of degree zero in $\mathbb{R}^n$, integrable, and have mean value zero on the unit sphere S^{n-1}. Define the singular integral operator T_Ω by

$$T_\Omega f(x) = \mathrm{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy. \tag{1.1}$$

The maximal operator associated with T_Ω is defined by

$$T_\Omega^\star f(x) = \sup_{\epsilon>0} \Big| \int_{|x-y|>\epsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy \Big|.$$

These operators were introduced by Calderón and Zygmund [5] and were subsequently studied by many authors. Calderón and Zygmund [6] proved that if $\Omega \in L\ln L(S^{n-1})$, then T_Ω and $T_\Omega^\star$ are bounded on $L^p(\mathbb{R}^n)$ for $p \in (1,\infty)$. Connett [13], and Ricci and Weiss [24] improved the result of Calderón and Zygmund and showed that $\Omega \in H^1(S^{n-1})$ guarantees the $L^p(\mathbb{R}^n)$ boundedness on $L^p(\mathbb{R}^n)$ for $p \in (1,\infty)$. Seeger [26] showed that $\Omega \in L\ln L(S^{n-1})$ is a sufficient condition for T_Ω to be bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. Duoandikoetxea and Rubio de Francia [16], Duoandikoetxea [15], and Watson [30] considered the weighted estimates for T_Ω and $T_\Omega^\star$ when $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$. For other works on T_Ω and $T_\Omega^\star$, see [14, 18] and the references therein.

Let $\mathrm{BMO}(\mathbb{R}^n)$ be the space of functions of bounded mean oscillation introduced by John and Nirenberg, and let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Define the commutator of T_Ω and b

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by

$$T_{\Omega,b}f(x) = b(x)T_\Omega f(x) - T_\Omega(bf)(x)$$

initially for $f \in \mathcal{S}(\mathbb{R}^n)$. As usual, the maximal operator associated with $T_{\Omega,b}$ is defined as

$$T^\star_{\Omega,b}f(x) = \sup_{\epsilon>0}\Big|\int_{|x-y|>\epsilon}\big(b(x)-b(y)\big)\frac{\Omega(x-y)}{|x-y|^n}f(y)\,\mathrm{d}y\Big|. \tag{1.2}$$

Coifman, Rochberg, and Weiss [11] proved that if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ $(\alpha \in (0,1))$, then $T_{\Omega,b}$ is bounded on $L^p(\mathbb{R}^n)$ $(p \in (1,\infty))$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. For $p \in [1,\infty)$, let $A_p(\mathbb{R}^n)$ be the weight functions class of Muckenhoupt (see [17, Chap. 9] for definitions and properties of $A_p(\mathbb{R}^n)$). Using the weighted estimates with $A_p(\mathbb{R}^n)$ weights of T_Ω, and the relation of A_p weights and $\mathrm{BMO}(\mathbb{R}^n)$ functions, Alvarez et al. [2] proved that $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$ guarantees the boundedness on $L^p(\mathbb{R}^n,w)$ for $T_{\Omega,b}$ when $p \in (q',\infty)$ and $w \in A_{p/q'}(\mathbb{R}^n)$, which, via duality, shows that $T_{\Omega,b}$ is bounded on $L^p(\mathbb{R}^n,w)$ if $p \in (1,q)$ and $w^{-1/(p-1)} \in A_{p'/q'}(\mathbb{R}^n)$, where and in the following, for $p \in (1,\infty)$, p' denotes the dual exponent of p, that is, $p' = p/(p-1)$. Hu [19] showed that the maximal commutator $T^\star_{\Omega,b}$ is also bounded on $L^p(\mathbb{R}^n,w)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$, provided that $p \in (q',\infty)$ and $w \in A_{p/q'}(\mathbb{R}^n)$ or $p \in (1,q)$ and $w^{-1/(p-1)} \in A_{p'/q'}(\mathbb{R}^n)$. Hu [20] proved that $\Omega \in L(\ln L)^2(S^{n-1})$ is a sufficient condition such that $T_{\Omega,b}$ and $T^\star_{\Omega,b}$ are bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$ for all $p \in (1,\infty)$.

The compactness of $T_{\Omega,b}$ on function spaces is of interest and has been considered by many authors. Let $\mathrm{CMO}(\mathbb{R}^n)$ be the closure of $C_0^\infty(\mathbb{R}^n)$ in the $\mathrm{BMO}(\mathbb{R}^n)$ topology, which coincide with the space of functions of vanishing mean oscillation; see [4, 12]. For the case of $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ $(\alpha \in (0,1))$, Uchiyama [29] proved that $T_{\Omega,b}$ is compact on $L^p(\mathbb{R}^n)$ if and only if $b \in \mathrm{CMO}(\mathbb{R}^n)$. Fairly recently, Chen and Hu [8] considered the compactness on $L^p(\mathbb{R}^n)$ for $T_{\Omega,b}$ when Ω satisfies a certain minimum size condition. Our first purpose in this paper is to consider the compactness on weighted L^p spaces for $T_{\Omega,b}$ and its discrete maximal operator (see (1.3)) when $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$. To formulate our result, we first recall some notation and definitions.

For a weight w, let $L^p(\mathbb{R}^n,w)$ be the weighted $L^p(\mathbb{R}^n)$ spaces with weight w, defined by

$$L^p(\mathbb{R}^n,w) = \{f : \|f\|_{L^p(\mathbb{R}^n,w)} < \infty\},$$

with

$$\|f\|_{L^p(\mathbb{R}^n,w)} = \Big(\int_{\mathbb{R}^n}|f(x)|^p w(x)\,\mathrm{d}x\Big)^{1/p}.$$

Definition 1.1 Let $\mathcal{X}$ be a normed linear spaces and let $\mathcal{X}^*$ be its dual space, $\{x_k\} \subset \mathcal{X}$ and $x \in \mathcal{X}$. If for all $f \in \mathcal{X}^*$,

$$\lim_{k\to\infty}|f(x_k) - f(x)| = 0,$$

then $\{x_k\}$ is said to converge to x weakly, or $x_k \rightharpoonup x$.

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Definition 1.2 Let $\mathcal{X}$, $\mathcal{Y}$ be two Banach spaces and let S be a bounded operator from $\mathcal{X}$ to $\mathcal{Y}$.

(i) If for each bounded set $\mathcal{G} \subset \mathcal{X}$, $S\mathcal{G} = \{Sx : x \in \mathcal{G}\}$ is a strongly pre-compact set in $\mathcal{Y}$, then S is called a *compact operator* from $\mathcal{X}$ to $\mathcal{Y}$.

(ii) If for $\{x_k\} \subset \mathcal{X}$ and $x \in \mathcal{X}$,

$$x_k \rightharpoonup x \text{ in } \mathcal{X} \Rightarrow \|Sx_k - Sx\|_{\mathcal{Y}} \to 0,$$

then S is said to be a *completely continuous operator*.

It is well known that if $\mathcal{X}$ is a reflexive space and S is completely continuous from $\mathcal{X}$ to $\mathcal{Y}$, then S is also compact from $\mathcal{X}$ to $\mathcal{Y}$. On the other hand, if S is a linear compact operator from $\mathcal{X}$ to $\mathcal{Y}$, then S is also a completely continuous operator. However, if S is not linear, then S being compact operator does not imply that S is completely continuous. For example, the operator $Sx = \|x\|_{l^2}$ is compact from l^2 to $\mathbb{R}$, but not completely continuous.

Our first result can be stated as follows.

Theorem 1.3 *Let Ω be homogeneous of degree zero, $\Omega \in L^q(S^{n-1})$ for some $q \in (1, \infty]$ and have mean value zero on S^{n-1}. Let p and w satisfy one of the following conditions:*

(i) *$p \in (q', \infty)$ and $w \in A_{p/q'}(\mathbb{R}^n)$;*

(ii) *$p \in (1, q)$ and $w^{-1/(p-1)} \in A_{p'/q'}(\mathbb{R}^n)$.*

Then for $b \in \mathrm{CMO}(\mathbb{R}^n)$, $T_{\Omega,b}$ and the discrete maximal operator $T^{\star\star}_{\Omega,b}$ defined by

$$T^{\star\star}_{\Omega,b} f(x) = \sup_{k\in\mathbb{Z}} \left| \int_{|x-y|>2^k} \big(b(x) - b(y)\big) \frac{\Omega(x-y)}{|x-y|^n} f(y)\,\mathrm{d}y \right| \tag{1.3}$$

are completely continuous (and compact) on $L^p(\mathbb{R}^n, w)$.

Remark 1.4 Let $\beta > 1$. The conclusions of Theorem 1.3 are still true for the discrete maximal operator defined by

$$T^{\star\star,\beta}_{\Omega,b} f(x) = \sup_{k\in\mathbb{Z}} \left| \int_{|x-y|>\beta^k} \big(b(x) - b(y)\big) \frac{\Omega(x-y)}{|x-y|^n} f(y)\,\mathrm{d}y \right|.$$

To prove Theorem 1.3, we will approximate the operators T_Ω and the maximal operator

$$T^{\star\star}_{\Omega} f(x) = \sup_{k\in\mathbb{Z}} \left| \int_{|x-y|>2^k} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,\mathrm{d}y \right|$$

by convolution operators whose kernels enjoy appropriate regularity. This idea was developed by Watson [30] and was used to prove the compactness on $L^p(\mathbb{R}^n)$ for the commutators of rough operators by Chen and Hu [8]. We do not know if $T^{\star}_{\Omega}$ can be approximated by convolution operators whose kernels are smooth, or if the conclusion in Theorem 1.3 holds true for the maximal commutator $T^{\star}_{\Omega,b}$ defined by (1.2). As a substitution, we can prove the following theorem.

Theorem 1.5 *Let Ω be homogeneous of degree zero and have mean value zero on S^{n-1}. Suppose that $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$, p and w satisfy one of the conditions in Theorem 1.3. Then for $\{f_k\} \subset L^p(\mathbb{R}^n, w)$ and $f \in L^p(\mathbb{R}^n, w)$,*

$$|f_k - f| \rightharpoonup 0 \text{ in } L^p(\mathbb{R}^n, w) \Rightarrow \|T^{\star}_{\Omega,b} f_k - T^{\star}_{\Omega,b} f\|_{L^p(\mathbb{R}^n,w)} \to 0.$$

Remark 1.6 Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Define the operator $M_{\Omega,b}$ by

$$M_{\Omega,b} f(x) = \sup_{l\in\mathbb{Z}} \Big| \int_{2^l \le |x-y| < 2^{l+1}} \frac{\Omega(x-y)}{|x-y|^n} |b(x) - b(y)|^2 f(y) \mathrm{d}y \Big|. \tag{1.4}$$

We can verify that

$$\begin{aligned}\big\| T^{\star}_{\Omega,b} f_k - T^{\star}_{\Omega,b} f \big\|_{L^p(\mathbb{R}^n,w)} \le \big\| M_{\Omega,b}(|f_k - f|) \big\|^{\frac{1}{2}}_{L^p(\mathbb{R}^n,w)} \|f_k - f\|^{\frac{1}{2}}_{L^p(\mathbb{R}^n,w)} \\ + \big\| T^{\star\star}_{\Omega,b}(f_k - f) \big\|_{L^p(\mathbb{R}^n,w)}.\end{aligned}$$

Under the hypothesis of Theorem 1.3, for $b \in C_0^\infty(\mathbb{R}^n)$, we can prove that

$$|f_k - f| \rightharpoonup 0 \text{ in } L^p(\mathbb{R}^n, w) \Rightarrow \|M_{\Omega,b}(|f_k - f|)\|_{L^p(\mathbb{R}^n)} \to 0;$$

see the proof of Theorem 1.5 for details. However,

$$f_k - f \rightharpoonup 0 \text{ in } L^p(\mathbb{R}^n, w) \not\Rightarrow \|M_{\Omega,b}(|f_k - f|)\|_{L^p(\mathbb{R}^n)} \to 0.$$

To see this, let $g(x) = \chi_{[0,1]^n}(x)$ and $g_m(x) = \exp(2\pi i m x) g(x)$ for $m \in \mathbb{Z}^n$. It is easy to verify that $\{g_m\}_{m\in\mathbb{Z}^n}$ is an orthogonal system of $L^2(\mathbb{R}^n)$. Thus, in $L^2(\mathbb{R}^n)$, $g_m \rightharpoonup 0$ ($|m| \to \infty$), but $\big\| M_{\Omega,b}(|g_m|) \big\|_{L^2(\mathbb{R}^n)} = \|M_{\Omega,b} g\|_{L^2(\mathbb{R}^n)}$. So, our argument in the proof of Theorem 1.5 does not lead to $T^{\star}_{\Omega,b}$ being completely continuous.

It should be pointed out that the estimates used in the proof of Theorem 1.3 also lead to the compactness on weighted Morrey spaces for $T_{\Omega,b}$ and $T^{\star\star}_{\Omega,b}$.

Definition 1.7 Let $p \in (0,\infty)$ and $\lambda \in (0,n)$. The Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is defined as

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} < \infty \right\},$$

with

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{y\in\mathbb{R}^n, r>0} \Big(\frac{1}{r^\lambda} \int_{B(y,r)} |f(x)|^p \,\mathrm{d}x \Big)^{1/p},$$

where $B(y,r)$ denotes the ball in $\mathbb{R}^n$ centered at y and having radius r.

The space $L^{p,\lambda}(\mathbb{R}^n)$ was introduced by Morrey [22]. It is well known that this space is closely related to some problems in PDE (see [25, 27]), and has interest in harmonic analysis (see [1] and the references therein). Chen et al. [9] considered the compactness of $T_{\Omega,b}$ on Morrey spaces. They proved that if $\lambda \in (0,n)$, $\Omega \in L^q(S^{n-1})$ for $q \in (n/(n-\lambda), \infty]$ and satisfies the regularity condition that

$$\int_0^1 \omega_q(\delta)(1 + |\ln\delta|) \frac{\mathrm{d}\delta}{\delta} < \infty, \tag{1.5}$$

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then for $b \in \mathrm{CMO}(\mathbb{R}^n)$, $T_{\Omega,b}$ is bounded on $L^{p,\lambda}(\mathbb{R}^n)$. Here ω_q denotes the L^q-integral modulus of continuity of Ω defined by

$$\omega_q(\delta) = \Big(\sup_{\|\rho\|<\delta} \int_{S^{n-1}} |\Omega(\rho x') - \Omega(x')|^q \mathrm{d}x' \Big)^{1/q}$$

and sup is taken over all rotations on S^{n-1}, $\|\rho\| = \sup_{x' \in S^{n-1}} |\rho x' - x'|$. Applying the estimates used in the proof of Theorem 1.2, we will show that to guarantee the compactness of $T_{\Omega,b}$ on Morrey space, assumption (1.5) is superfluous. More precisely, we will prove the following theorem.

Theorem 1.8 *Let Ω be homogeneous of degree zero and have mean value zero on S^{n-1}. Suppose that $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$, $p \in (q',\infty)$ and $\lambda \in (0,n)$, or $p \in (1,q']$ and $\lambda \in (0,n/q')$. Then for $b \in \mathrm{CMO}(\mathbb{R}^n)$,*

(i) *the operators $T_{\Omega,b}$ and $T^{\star\star}_{\Omega,b}$ are completely continuous and compact on $L^{p,\lambda}(\mathbb{R}^n)$;*
(ii) *for $\{f_k\} \subset L^{p,\lambda}(\mathbb{R}^n)$ and $f \in L^{p,\lambda}(\mathbb{R}^n)$,*

$$|f_k - f| \rightharpoonup 0 \, in \, L^{p,\lambda}(\mathbb{R}^n) \Rightarrow \|T^{\star}_{\Omega,b} f_k - T^{\star}_{\Omega,b} f\|_{L^{p,\lambda}(\mathbb{R}^n)} \to 0.$$

Remark 1.9 We do not know if the conclusion in Theorem 1.8 holds true for the weighted case.

In what follows, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \le CB$. For a set $E \subset \mathbb{R}^n$, χ_E denotes its characteristic function. Let M be the Hardy–Littlewood maximal operator. For $r \in (0,\infty)$, we use M_r to denote the operator $M_r f(x) = (M(|f|^r)(x))^{1/r}$.

2 Approximations

Let Ω be the same as in Theorem 1.3. Set $K(y) = (\Omega(y))/|y|^n$. For each $l \in \mathbb{Z}$, let

$$K^l_\Omega(y) = \frac{\Omega(y)}{|y|^n} \chi_{\{2^l<|y|\le 2^{l+1}\}}(y).$$

By the vanishing moment of Ω, it is easy to verify that if $\Omega \in L^q(S^{n-1})$, then there exists a constant $\alpha \in (0,1)$ such that for $\xi \in \mathbb{R}^n \backslash \{0\}$,

$$|\widehat{K^l_\Omega}(\xi)| \lesssim \min\big\{ |2^l \xi|, |2^l \xi|^{-\alpha} \big\}. \tag{2.1}$$

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a nonnegative function such that

$$\int_{\mathbb{R}^n} \phi(x) \mathrm{d}x = 1, \quad \operatorname{supp} \phi \subset \{x : |x| \le 1/4\}.$$

For $l \in \mathbb{Z}$, let $\phi_l(y) = 2^{-nl}\phi(2^{-l}y)$. We then have that for all $\gamma \in (0,1)$ and $\xi \in \mathbb{R}^n$,

$$|\widehat{\phi}_l(\xi) - 1| = |\widehat{\phi}(2^l \xi) - 1| \lesssim \min\{1, |2^l \xi|^\gamma\}. \tag{2.2}$$

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As in [30], for a positive integer j, let

$$K^j(y) = \sum_{l=-\infty}^{\infty} K_\Omega^l * \phi_{l-j}(y), \tag{2.3}$$

and T_Ω^j be the convolution operator to be given by

$$T_\Omega^j f(x) = \text{p. v.} \int_{\mathbb{R}^n} K^j(x-y) f(y) \mathrm{d}y. \tag{2.4}$$

As usual, the maximal operator corresponding to T_Ω^j is given by

$$T_\Omega^{j,\star} f(x) = \sup_{\epsilon>0} \Big| \int_{|x-y|>\epsilon} K^j(x-y) f(y) \mathrm{d}y \Big|.$$

Lemma 2.1 *Let $s \in (1,\infty]$, let Ω be homogeneous of degree zero and integrable on S^{n-1}, and let K^j be the function defined as in (2.3). Then for any $y \in \mathbb{R}^n$ and $R > 0$ with $R > 4|y|$,*

$$\sum_{l\in\mathbb{Z}} \sum_{m=1}^{\infty} (2^m R)^{\frac{n}{q'}} \Big(\int_{2^{m-1}R<|x|\le 2^m R} \big| K_\Omega^l * \phi_{l-j}(x+y) - K_\Omega^l * \phi_{l-j}(x) \big|^q \mathrm{d}x \Big)^{\frac{1}{q}} \lesssim j \|\Omega\|_{L^q(S^{n-1})}, \tag{2.5}$$

$$\sum_{l\in\mathbb{Z}} \sum_{m=1}^{\infty} (2^m R)^{\frac{n}{s}} \Big(\int_{2^m R<|x|\le 2^{m+1}R} \big| K_\Omega^l * \phi_{l-j}(x+y) - K_\Omega^l * \phi_{l-j}(x) \big|^{s'} \mathrm{d}x \Big)^{\frac{1}{s'}} \lesssim 2^{j(n+1)} \|\Omega\|_{L^1(S^{n-1})} \frac{|y|}{R}. \tag{2.6}$$

Proof Estimate (2.5) was proved in [30]. To prove (2.6), observing that

$$\|\phi_{l-j}(\cdot+y) - \phi_{l-j}(\cdot)\|_{L^{s'}(\mathbb{R}^n)} \lesssim 2^{(j-l)n/s} \min\{1, 2^{j-l}|y|\},$$

we know that for all $k \in \mathbb{N}$,

$$\begin{aligned}(2^k R)^{n/s} \sum_{l\in\mathbb{Z}} &\Big(\int_{2^k R<|x|\le 2^{k+1}R} |K_\Omega^l * \phi_{l-j}(x+y) - K_\Omega^l * \phi_{l-j}(x)|^{s'} \mathrm{d}x \Big)^{1/s'} \\ &\lesssim (2^k R)^{n/s} \sum_{l\in\mathbb{Z}:2^l\approx 2^k R} \|K_\Omega^l\|_{L^1(\mathbb{R}^n)} \|\phi_{l-j}(\cdot+y) - \phi_{l-j}(\cdot)\|_{L^{s'}(\mathbb{R}^n)} \\ &\lesssim 2^{jn/s} \min\{1, 2^j \frac{|y|}{2^k R}\}.\end{aligned}$$

This in turn leads to

$$\sum_{l\in\mathbb{Z}} \sum_{k=1}^{\infty} (2^k R)^{n/s} \Big(\int_{2^k R<|x|\le 2^{k+1}R} \big| K_\Omega^l * \phi_{l-j}(x+y) - K_\Omega^l * \phi_{l-j}(x) \big|^{s'} \mathrm{d}x \Big)^{1/s'} \lesssim 2^{jn/s} 2^j |y| \sum_{k=1}^{\infty} (2^k R)^{-1} \lesssim 2^{jn/s} 2^j \frac{|y|}{R},$$

which completes the proof of Lemma 2.1. ∎

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Lemma 2.2 *Let Ω be homogeneous of degree zero, have mean value zero, let $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$, and let p and w be the same as in Theorem 1.3. Then the operators T_Ω^j and $T_\Omega^{j,\star}$ are bounded on $L^p(\mathbb{R}^n,w)$ with bound Cj.*

Proof Applying the estimate (2.1) and the fact that $|\widehat{\phi_l}(\xi)| \lesssim 1$, we can verify that for $j \in \mathbb{N}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$\sum_{l\in\mathbb{Z}} |\widehat{K_\Omega^l}(\xi)||\widehat{\phi_{l-j}}(\xi)| \lesssim 1.$$

It then follows from the Plancherel theorem that T_Ω^j is bounded on $L^2(\mathbb{R}^n)$ with bounded C. This, along with (2.4) in Lemma 2.1 and the result of Kurtz and Wheeden in [21], yields the desired conclusion for T_Ω^j.

To consider the operator $T_\Omega^{j,\star}$, we will use the ideas from [7]. As in [7, Lemma 3], by Lemma 2.1, we can verify that for bounded function f with compact support,

$$T_\Omega^{j,\star} f(x) \lesssim M(T_\Omega^j f)(x) + jM_{q'}f(x),$$

which, together with the weighted L^p estimates for T_Ω^j and M, shows that $T_\Omega^{j,\star}$ is bounded on $L^p(\mathbb{R}^n,w)$ with bound Cj provided that $p > q'$ and $w \in A_{p/q'}(\mathbb{R}^n)$. Let M_Ω be the maximal operator defined by

$$M_\Omega f(x) = \sup_{r>0} \int_{|y-x|<r} |\Omega(x-y)||f(y)|\,\mathrm{d}y.$$

It was proved by Duoandikoetxea in [15] that, if $\Omega \in L^q(S^{n-1})$ for $q \in (1,\infty]$, then M_Ω is bounded on $L^p(\mathbb{R}^n,w)$ with p and w as in Theorem 1.3(ii). Observe that for each fixed $R > 0$,

$$\int_{R<|x-y|\le 2R} \left|K^j(x-y)f(y)\right| \mathrm{d}y \lesssim M_\Omega Mf(x).$$

As in the proof of [7, Lemma 4], we can verify that if $p \in (1,q)$ and $w^{-1/(p-1)} \in A_{p'/q'}(\mathbb{R}^n)$, then $T_\Omega^{j,\star}$ is bounded from $L^p(\mathbb{R}^n,w)$ to $L^{p,\infty}(\mathbb{R}^n,w)$ with bound Cj. This, together with the inverse Hölder inequality of $A_{p'/q'}(\mathbb{R}^n)$, leads to that $T_\Omega^{j,\star}$ is bounded on $L^p(\mathbb{R}^n,w)$ with bound Cj. ∎

We now formulate the main theorem in this section.

Theorem 2.3 *Let Ω be homogeneous of degree zero and have mean value zero, let T_Ω and T_Ω^j be the operators defined by (1.1) and (2.4), respectively. Suppose that $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$, and let p and w be the same as in Theorem 1.3. Then there exists a constant $\beta \in (0,1)$ such that*

$$\|T_\Omega f - T_\Omega^j f\|_{L^p(\mathbb{R}^n,w)} \lesssim 2^{-\beta j}\|f\|_{L^p(\mathbb{R}^n,w)}, \tag{2.7}$$

$$\Big\| \sup_{k\in\mathbb{Z}} \Big| \sum_{l=k}^{\infty} S_l^j * f \Big| \Big\|_{L^p(\mathbb{R}^n,w)} \lesssim 2^{-\beta j}\|f\|_{L^p(\mathbb{R}^n,w)}, \tag{2.8}$$

$$\Big\| \sup_{l\in\mathbb{Z}} \Big| \widetilde{S}_l^j * f \Big| \Big\|_{L^p(\mathbb{R}^n,w)} \lesssim 2^{-\beta j}\|f\|_{L^p(\mathbb{R}^n,w)}. \tag{2.9}$$

Here and in the following, for $l \in \mathbb{Z}$ and $j \in \mathbb{N}$, we set

$$S_l^j(y) = K_\Omega^l(y) - K_\Omega^l * \phi_{l-j}(y), \quad \widetilde{S}_l^j(y) = |K_\Omega^l(y)| - |K_\Omega^l| * \phi_{l-j}(y).$$

Proof Estimate (2.7) was established by Watson [30]. To prove (2.8), we will use an idea from [16], with appropriate modifications. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp}\psi \subset \{x \in \mathbb{R}^n : |x| \le 2\}, \quad \psi(x) \equiv 1, \text{ if } |x| \le 1.$$

For each integer k, let $\Psi_k \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\Psi}_k(\xi) = \psi(2^k\xi)$. For each fixed $k \in \mathbb{Z}$, write

$$\begin{aligned}\sum_{l=k}^{\infty} S_l^j * f(x) &= \Psi_k * \big(T_\Omega f - T_\Omega^j f\big)(x) - \Psi_k * \Big(\sum_{l=-\infty}^{k-1} S_l^j * f\Big)(x) \\ &\quad + \sum_{l=k}^{\infty}(\delta - \Psi_k) * S_l^j * f(x) \\ &= \mathrm{I}_k^j f(x) + \mathrm{II}_k^j f(x) + \mathrm{III}_k^j f(x),\end{aligned}$$

with δ the Dirac distribution. It is obvious that

$$\big|\mathrm{I}_k^j f(x)\big| \lesssim M\big(T_\Omega f - T_\Omega^j f\big)(x),$$

and so for $\beta_1 \in (0,1)$,

$$\Big\| \sup_{k\in\mathbb{Z}} |\mathrm{I}_k^j f| \Big\|_{L^2(\mathbb{R}^n)} \lesssim \|T_\Omega f - T_\Omega^j f\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-\beta_1 j}\|f\|_{L^2(\mathbb{R}^n)}.$$

To give the desired estimate for $\sup_{k\in\mathbb{Z}}\big|\mathrm{II}_k^j f\big|$, write

$$\sup_{k\in\mathbb{Z}}\big|\mathrm{II}_k^j f(x)\big| \lesssim \Big(\sum_{u=-\infty}^{\infty}\Big|\Psi_u * \sum_{l=-\infty}^{u-1} S_l^j * f(x)\Big|^2\Big)^{1/2}.$$

Noticing that for any $\xi \in \mathbb{R}^n$,

$$\begin{aligned}\Big|\psi(2^u\xi)\sum_{l=-\infty}^{u-1}\widehat{K}_\Omega^l(\xi)\big(\widehat{\phi}(2^{l-j}\xi) - 1\big)\Big| &\lesssim \big|\psi(2^u\xi)\big|\sum_{l=-\infty}^{u-1}|2^{l-j}\xi| \\ &\lesssim 2^{-j}\big|\psi(2^u\xi)\big||2^u\xi|,\end{aligned}$$

we have, by the Plancherel theorem, that

$$\begin{aligned}\Big\|\sup_{k\in\mathbb{Z}}\big|\mathrm{II}_k^j f\big|\Big\|_{L^2(\mathbb{R}^n)}^2 &\le \sum_{u=-\infty}^{\infty}\Big\|\Psi_u * \sum_{l=-\infty}^{u-1} S_l^j * f\Big\|_{L^2(\mathbb{R}^n)}^2 \\ &= \sum_{u=-\infty}^{\infty}\int_{\mathbb{R}^n}\Big|\sum_{l=-\infty}^{u-1}\widehat{K_\Omega^l}(\xi)\big(\widehat{\phi}(2^{l-j}\xi) - 1\big)\Big|^2\big|\psi(2^u\xi)\widehat{f}(\xi)\big|^2 d\xi \\ &\lesssim 2^{-2j}\int_{\mathbb{R}^n}\sum_{u=-\infty}^{\infty}\big|\psi(2^u\xi)\big|^2|2^u\xi|^2\big|\widehat{f}(\xi)\big|^2 d\xi.\end{aligned}$$

This, together with the fact that $\operatorname{supp}\psi \subset \{x : |x| \le 2\}$, implies

$$\Big\|\sup_{k\in\mathbb{Z}}\big|\mathrm{II}_k^j f\big|\Big\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-j}\|f\|_{L^2(\mathbb{R}^n)}.$$

As for the term $\sup_{k\in\mathbb{Z}} \left|\mathrm{III}_k^j f\right|$, write

$$\begin{aligned}\sup_{k\in\mathbb{Z}} |\mathrm{III}_k^j f(x)| &\le \sum_{l=0}^{\infty} \sup_{k\in\mathbb{Z}} \left|(\delta-\Psi_k)*S_{l+k}^j * f(x)\right| \\ &\lesssim \sum_{l=0}^{\infty}\Big(\sum_{u=-\infty}^{\infty}\left|(\delta-\Psi_{u-l})*S_u^j*f(x)\right|^2\Big)^{1/2}.\end{aligned}$$

An application of (2.1) and (2.2) tells us that

$$\begin{aligned}&\Big\|\Big(\sum_{u=-\infty}^{\infty}\left|(\delta-\Psi_{u-l})*S_u^j*f(x)\right|^2\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^n)}^2 \\ &= \sum_{u=-\infty}^{\infty}\int_{\mathbb{R}^n}\left|1-\psi(2^{u-l}\xi)\right|^2\left|\widehat{K}_\Omega^u(\xi)\big(\widehat{\phi}(2^{u-j}\xi)-1\big)\right|^2|\widehat{f}(\xi)|^2 d\xi \\ &\lesssim \int_{\mathbb{R}^n}\sum_{u=-\infty}^{\infty}|1-\psi(2^{u-l}\xi)|^2|2^u\xi|^{-2\alpha}|2^{u-j}\xi|^{\alpha}\left|\widehat{f}(\xi)\right|^2 d\xi \\ &\lesssim 2^{-\alpha l}2^{-j\alpha}\|f\|_{L^2(\mathbb{R}^n)}^2,\end{aligned}$$

where we have invoked the Fourier transform (2.2) with $\gamma = \alpha/2$. Combining the estimates for $\sup_{k\in\mathbb{Z}} |\mathrm{I}_k^j f|$, $\sup_{k\in\mathbb{Z}} |\mathrm{II}_k^j f|$ and $\sup_{k\in\mathbb{Z}} |\mathrm{III}_k^j f|$ leads to

$$\Big\|\sup_{k\in\mathbb{Z}}\Big|\sum_{l=k}^{\infty}S_l^j*f\Big|\Big\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-\beta_2 j}\|f\|_{L^2(\mathbb{R}^n)} \tag{2.10}$$

with β_2 a positive constant. On the other hand, applying Lemma 2.2, we then obtain that for the same p and w as in Theorem 1.3,

$$\begin{aligned}\Big\|\sup_{k\in\mathbb{Z}}\Big|\sum_{l=k}^{\infty}S_l^j*f\Big|\Big\|_{L^p(\mathbb{R}^n,w)} &\lesssim \|T_\Omega^{j,\star}f\|_{L^p(\mathbb{R}^n,w)} + \|T_\Omega^{\star}f\|_{L^p(\mathbb{R}^n,w)} \\ &\lesssim j\|f\|_{L^p(\mathbb{R}^n,w)}.\end{aligned} \tag{2.11}$$

Recall that if $p \in (q',\infty)$ and $w \in A_{p/q'}(\mathbb{R}^n)$ (or $p \in (1,q)$ and $w^{-1/(p-1)} \in A_{p'/q'}(\mathbb{R}^n)$), then there exists a constant $\theta > 1$, such that $w^\theta \in A_{p/q'}(\mathbb{R}^n)$ (or $p \in (1,q)$ and $w^{-\theta/(p-1)} \in A_{p'/q'}(\mathbb{R}^n)$). The inequalities (2.10) and (2.11), via the interpolation with change of measures (see [28]), yield (2.8).

It remains to prove (2.9). Note that

$$\left|\widetilde{S}_l^j * f(x)\right| \lesssim M_\Omega f(x) + M_\Omega M f(x).$$

Thus, it suffices to prove that for some $\alpha \in (0,1)$ in (2.1),

$$\|\widetilde{S}_l^j * f\|_{L^2(\mathbb{R}^n)} \lesssim 2^{-\alpha j}\|f\|_{L^2(\mathbb{R}^n)}. \tag{2.12}$$

On the other hand, by the Plancherel theorem,

$$\Big\|\sup_{l\in\mathbb{Z}}\left|\widetilde{S}_l^j*f\right|\Big\|_{L^2(\mathbb{R}^n)}^2 = \sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}\left|\widehat{\widetilde{S}_l^j}(\xi)\right|^2|\widehat{f}(\xi)|^2 d\xi.$$

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Since $\widetilde{K}^l_\Omega(x) = |K^l_\Omega(x)|$ also satisfies $|\widehat{\widetilde{K}^l_\Omega}(\xi)| \lesssim |2^l \xi|^{-\alpha}$, we then get that

$$\sum_{l\in\mathbb{Z}} \Big|\widehat{S^j_l}(\xi)\Big|^2 \lesssim \sum_{l\in\mathbb{Z}} |1-\phi(2^{l-j}\xi)|^2 |2^l\xi|^{-2\alpha}$$
$$= \sum_{l\in\mathbb{Z}:|2^l\xi|>2^j} |2^l\xi|^{-2\alpha} + \sum_{l\in\mathbb{Z}:|2^l\xi|\le 2^j} |2^{l-j}\xi|^2|2^l\xi|^{-2\alpha} \lesssim 2^{-2\alpha j}.$$

This implies (2.12), which completes the proof of Theorem 2.3. ∎

3 Proofs of Theorems 1.3 and 1.5

Let $p \in [1,\infty)$, let w be a weight, and let $L^p(l^\infty;\mathbb{R}^n,w)$ be the space of sequences of functions defined by

$$L^p(l^\infty;\mathbb{R}^n,w) = \Big\{ \{f_k\}_{k\in\mathbb{Z}} : \|\{f_k\}\|_{L^p(l^\infty;\mathbb{R}^n,w)} < \infty \Big\},$$

with

$$\|\{f_k\}\|_{L^p(l^\infty;\mathbb{R}^n,w)} = \Big\| \sup_{k\in\mathbb{Z}} |f_k| \Big\|_{L^p(\mathbb{R}^n,w)}.$$

With usual addition and scalar multiplication, $L^p(l^\infty;\mathbb{R}^n,w)$ is a Banach space.

Lemma 3.1 *Let $p\in(1,\infty)$ and $w\in A_p(\mathbb{R}^n)$, $\mathcal{G}\subset L^p(l^\infty;\mathbb{R}^n,w)$. Suppose that $\mathcal{G}$ satisfies the following four conditions:*

(i) *$\mathcal{G}$ is bounded, that is, there exists a constant C such that for all $\vec{f} = \{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}$, $\|\vec{f}\|_{L^p(l^\infty;\mathbb{R}^n,w)} \le C$;*

(ii) *for each fixed $\epsilon>0$, there exists a constant $A>0$ such that for all $\{f_k\}_{k\in\mathbb{Z}}\in\mathcal{G}$,*

$$\Big\| \sup_{k\in\mathbb{Z}} |f_k|\chi_{\{|\cdot|>A\}}(\cdot)\Big\|_{L^p(\mathbb{R}^n,w)} < \epsilon;$$

(iii) *for each fixed $\epsilon>0$, there exists a constant $\rho>0$ such that for all $t\in\mathbb{R}^n$ with $|t|<\rho$ and $\vec{f}=\{f_k\}_{k\in\mathbb{Z}}\in\mathcal{G}$,*

$$\|\vec{f}(\cdot+t)-\vec{f}(\cdot)\|_{L^p(l^\infty;\mathbb{R}^n,w)} < \epsilon;$$

(iv) *for each fixed $D>0$ and $\epsilon>0$, there exists $N\in\mathbb{N}$ such that for all $\{f_k\}_{k\in\mathbb{Z}}\in\mathcal{G}$,*

$$\Big\| \sup_{k>N} |f_k|\chi_{B(0,D)}\Big\|_{L^p(\mathbb{R}^n,w)} < \epsilon \quad \text{and} \quad \Big\| \sup_{k<-N} |f_k - f_{-N}|\Big\|_{L^p(\mathbb{R}^n,w)} < \epsilon.$$

Then $\mathcal{G}$ is a strongly pre-compact set in $L^p(l^\infty;\mathbb{R}^n,w)$.

Proof We employ the argument used in the proof of [10, Theorem 5] with some suitable modifications. We claim that for each fixed $\epsilon>0$, there exists a $\delta=\delta_\epsilon>0$ and a mapping Φ_ϵ on $L^p(l^\infty;\mathbb{R}^n,w)$ such that $\Phi_\epsilon(\mathcal{G}) = \{\Phi_\epsilon(\vec{f}) : \vec{f}\in\mathcal{G}\}$ is a strong pre-compact set in $L^p(l^\infty;\mathbb{R}^n,w)$, and for all $\vec{f},\vec{g}\in\mathcal{G}$,

$$\|\Phi_\epsilon(\vec{f})-\Phi_\epsilon(\vec{g})\|_{L^p(l^\infty;\mathbb{R}^n,w)} < \delta \Rightarrow \|\vec{f}-\vec{g}\|_{L^p(l^\infty;\mathbb{R}^n,w)} < 9\epsilon.$$

If we can prove this, then by [10, Lemma 6], we see that $\mathcal{G}$ is a strongly pre-compact set in $L^p(l^\infty;\mathbb{R}^n,w)$.

Now let $\epsilon>0$. We choose $A>1$ large enough as in assumption (ii), and $\rho<1$ small enough as in assumption (iii). Let Q be the largest cube centered at the origin such that

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$2Q \subset B(0,\rho)$, and let $Q_1,\dots,Q_J$ be J copies of Q such that they are non-overlapping and $\overline{B(0,A)} \subset \overline{\cup_{j=1}^J Q_j} \subset B(0,2A)$. Let $N \in \mathbb{N}$ such that for all $\{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}$,

$$\Big\| \sup_{k>N} |f_k| \chi_{B(0,2A)} \Big\|_{L^p(\mathbb{R}^n,w)} < \epsilon/2, \quad \Big\| \sup_{k<-N} |f_k - f_{-N}| \Big\|_{L^p(\mathbb{R}^n,w)} < \epsilon/2.$$

Define the mapping $\Phi_\epsilon: L^p(l^\infty;\mathbb{R}^n,w) \to L^p(l^\infty;\mathbb{R}^n,w)$ by

$$\Phi_\epsilon(\vec{f})(x) = \Big\{ \dots, \sum_{i=1}^J m_{Q_i}(f_{-N})\chi_{Q_i}(x), \dots, \sum_{i=1}^J m_{Q_i}(f_{-N})\chi_{Q_i}(x), \sum_{i=1}^J m_{Q_i}(f_{-N+1})\chi_{Q_i}(x), \dots, \sum_{i=1}^J m_{Q_i}(f_N)\chi_{Q_i}(x), 0, 0, \dots \Big\}, \tag{3.1}$$

where, and in the following, $m_{Q_i}(f)$ denotes the mean value of f on Q_i. Note that

$$|m_{Q_i}(f_k)| \le \Big(\frac{1}{|Q_i|} \int_{Q_i} |f_k(x)|^p w(x) \mathrm{d}x \Big)^{1/p} \Big(\frac{1}{|Q_i|} \int_{Q_i} w^{-1/(p-1)}(x)\, \mathrm{d}x \Big)^{1/p'}.$$

For $\vec{f} = \{f_k\}_{k\in\mathbb{Z}}$, we see that

$$\|\Phi_\epsilon(\vec{f})\|^p_{L^p(l^\infty;\mathbb{R}^n,w)} = \sum_{i=1}^J \int_{Q_i} \sup_{k\in\mathbb{Z}} |m_{Q_i}(f_k)|^p w(x) \mathrm{d}x \le \|\vec{f}\|^p_{L^p(l^\infty;\mathbb{R}^n,w)}.$$

Thus, Φ_ϵ is bounded on $L^p(l^\infty;\mathbb{R}^n,w)$, and $\Phi_\epsilon(\mathcal{G}) = \{\Phi_\epsilon(\vec{f}) : \vec{f} \in \mathcal{G}\}$ is a strongly pre-compact set in $L^p(l^\infty;\mathbb{R}^n,w)$. Denote $\mathcal{D} = \cup_{i=1}^J Q_i$. Write

$$\begin{aligned}
&\|\vec{f}\chi_{\mathcal{D}} - \Phi_\epsilon(\vec{f})\|_{L^p(l^\infty;\mathbb{R}^n,w)} \\
&\le \Big\| \sup_{|k|\le N} \Big| f_k\chi_{\mathcal{D}} - \sum_{i=1}^J m_{Q_i}(f_k)\chi_{Q_i} \Big| \Big\|_{L^p(\mathbb{R}^n,w)} \\
&\quad + \Big\| \sup_{k<-N} \Big| f_k\chi_{\mathcal{D}} - \sum_{i=1}^J m_{Q_i}(f_{-N})\chi_{Q_i} \Big| \Big\|_{L^p(\mathbb{R}^n,w)} + \Big\| \sup_{k>N} |f_k| \chi_{B(0,2A)} \Big\|_{L^p(\mathbb{R}^n,w)} \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

A straightforward computation leads to

$$\begin{aligned}
\mathrm{I}^p &\le \sum_{i=1}^J \int_{Q_i} \Big\{ \sup_{|k|\le N} \Big| f_k(x) - \sum_{l=1}^J m_{Q_l}(f_k)\chi_{Q_l}(x) \Big| \Big\}^p w(x) \mathrm{d}x \\
&\lesssim \sum_{i=1}^J \frac{1}{|Q_i|} \int_{Q_i} \sup_{|k|\le N} \int_{Q_i} |f_k(x) - f_k(y)|^p \mathrm{d}y\, w(x) \mathrm{d}x \\
&\lesssim \sum_{i=1}^J \frac{1}{|Q_i|} \int_{Q_i} \int_{2Q} \sup_{|k|\le N} |f_k(x) - f_k(x+h)|^p \mathrm{d}h\, w(x) \mathrm{d}x \\
&\lesssim \sup_{|h|\le\rho} \|\vec{f}(\cdot) - \vec{f}(\cdot + h)\|^p_{L^p(l^\infty;\mathbb{R}^n,w)}.
\end{aligned}$$

On the other hand, it follows from the Hölder inequality that

$$\big|m_{Q_i}(f_k)-m_{Q_i}(f_{-N})|^p \le \frac{1}{|Q_i|^p}\int_{Q_i}|f_k(x)-f_{-N}(x)|^p w(x)\mathrm{d}x\Big(\int_{Q_i} w^{-\frac{1}{p-1}}(x)\mathrm{d}x\Big)^{p-1},$$

which, via the fact that $w\in A_p(\mathbb{R}^n)$, implies that

$$\begin{aligned}\mathrm{II}^p &\le \sum_{i=1}^{J}\int_{Q_i}\Big\{\sup_{k<-N}\Big|f_k(x)-\sum_{l=1}^{J}m_{Q_l}(f_k)\chi_{Q_l}(x)\Big|\Big\}^p w(x)\mathrm{d}x\\ &\quad+\sum_{i=1}^{J}\int_{Q_i}\Big\{\sup_{k<-N}\big|m_{Q_i}(f_k)-m_{Q_i}(f_{-N})\big|\Big\}^p w(x)\mathrm{d}x\\ &\lesssim \sup_{|h|\le\rho}\|\vec{f}(\cdot)-\vec{f}(\cdot+h)\|_{L^p(l^\infty;\mathbb{R}^n,w)}+\Big\|\sup_{k<-N}|f_k-f_{-N}|\Big\|^p_{L^p(\mathbb{R}^n,w)}.\end{aligned}$$

The estimates for I, II, together with assumption (iv), prove that

$$\|\vec{f}\chi_{\mathcal{D}}-\Phi_\epsilon(\vec{f})\|_{L^p(l^\infty;\mathbb{R}^n,w)}<3\epsilon,$$

which via assumption (ii) tells us that for all $\vec{f}\in\mathcal{G}$,

$$\|\vec{f}-\Phi_\epsilon(\vec{f})\|_{L^p(l^\infty;\mathbb{R}^n,w)}<4\epsilon.$$

Note that

$$\begin{aligned}\|\vec{f}-\vec{g}\|_{L^p(l^\infty;\mathbb{R}^n,w)} \le \|\vec{f}-\Phi_\epsilon(\vec{f})\|_{L^p(l^\infty;\mathbb{R}^n,w)}+\|\Phi_\epsilon(\vec{f})-\Phi_\epsilon(\vec{g})\|_{L^p(l^\infty;\mathbb{R}^n,w)}\\ +\|\vec{g}-\Phi_\epsilon(\vec{g})\|_{L^p(l^\infty;\mathbb{R}^n,w)}.\end{aligned}$$

Our claim then follows directly. This completes the proof of Lemma 3.1. ■

For $b\in\mathrm{BMO}(\mathbb{R}^n)$, let $T^j_{\Omega,b}$ be the commutator of T^j_Ω, and let

$$T^{j,\star\star}_{\Omega,b}f=\sup_{k\in\mathbb{Z}}\big|T^{j,k}_{\Omega,b}f(x)\big|$$

with

$$T^{j,k}_{\Omega,b}f(x)=\sum_{l=k}^{\infty}\int_{\mathbb{R}^n}\big(b(x)-b(y)\big)K^l_\Omega*\phi_{l-j}(x-y)f(y)\mathrm{d}y.$$

As in [3], let φ be a non-negative function in $C^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp}\varphi\subset\{x\in\mathbb{R}^n:|x|\ge1\}\quad\text{and}\quad\varphi(x)\equiv1$$

when $|x|\ge2$. For $\delta>0$, let $K^{j,\delta}(x)=K^j(x)\varphi(\delta^{-1}x)$, $T^{j,\delta}_\Omega$ be the convolution operator with kernel $K^{j,\delta}$. For $b\in\mathrm{BMO}(\mathbb{R}^n)$, let $T^{j,\delta}_{\Omega,b}$ be the commutator of $T^{j,\delta}_\Omega$ and $T^{j,\delta,\star\star}_{\Omega,b}$ the maximal operator defined by

$$T^{j,\delta,\star\star}_{\Omega,b}f(x)=\sup_{v\in\mathbb{Z}}|T^{j,\delta,v}_{\Omega,b}f(x)|,$$

with

$$T^{j,\delta,v}_{\Omega,b}f(x)=\sum_{l=v}^{\infty}\int_{\mathbb{R}^n}\big(b(x)-b(y)\big)K^l_\Omega*\phi_{l-j}(x-y)\varphi\big(\delta^{-1}(x-y)\big)f(y)\mathrm{d}y.$$

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Lemma 3.2 *Let $b \in C_0^\infty(\mathbb{R}^n)$, $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$, and let p and w be the same as in Theorem 1.3. Then for $j \in \mathbb{N}$,*

$$\|T_{\Omega,b}^{j,\delta}f - T_{\Omega,b}^{j}f\|_{L^p(\mathbb{R}^n,w)} + \Big\|\sup_{\nu\in\mathbb{Z}}|T_{\Omega,b}^{j,\delta,\nu}f - T_{\Omega,b}^{j,\nu}f|\Big\|_{L^p(\mathbb{R}^n,w)} \lesssim \delta\|f\|_{L^p(\mathbb{R}^n,w)}.$$

Proof Let $b \in C_0^\infty(\mathbb{R}^n)$. For each fixed $\delta > 0$, it is easy to verify that

$$\begin{aligned}
&\Big|T_{\Omega,b}^{j,\delta}f(x) - T_{\Omega,b}^{j}f(x)\Big| + \sup_{\nu\in\mathbb{Z}}\Big|T_{\Omega,b}^{j,\delta,\nu}f(x) - T_{\Omega,b}^{j,\nu}f(x)\Big|\\
&\lesssim \delta\|\nabla b\|_{L^\infty(\mathbb{R}^n)}\sum_{k=-\infty}^{0}2^k\sum_{l\in\mathbb{Z}}\int_{2^k\delta<|x-y|\le 2^{k+1}\delta}\big|K_\Omega^l * \phi_{l-j}(x-y)\big||f(y)|\,dy\\
&\lesssim \delta\|\nabla b\|_{L^\infty(\mathbb{R}^n)}M_\Omega Mf(x).
\end{aligned}$$

Our desired conclusion now follows from the weighted estimates for M_Ω and M immediately. ∎

Lemma 3.3 *Let Ω be homogeneous of degree zero and have mean value zero, and let p and w be the same as in Theorem 1.3. Suppose that $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$. Then for $b \in C_0^\infty(\mathbb{R}^n)$ and $\delta \in (0,1/2)$,*

(i) *the operator $T_{\Omega,b}^{j,\delta}$ is compact on $L^p(\mathbb{R}^n,w)$;*
(ii) *the operator $\Gamma_{j,\delta}$ defined by*

$$\Gamma_{j,\delta}f(x) = \{T_{\Omega,b}^{j,\delta,\nu}f(x)\}_{\nu\in\mathbb{Z}} \tag{3.2}$$

is compact from $L^p(\mathbb{R}^n,w)$ to $L^p(l^\infty;\mathbb{R}^n,w)$.

Proof We only prove (ii). By Lemmas 3.2 and 2.2, it is obvious that $\Gamma_{j,\delta}$ is bounded from $L^p(\mathbb{R}^n,w)$ to $L^p(l^\infty;\mathbb{R}^n,w)$. Let p and w be as in Theorem 1.3. We choose $s \in (1,p)$ such that p/s and w satisfies the condition as p and w. For each fixed $\delta \in (0,1/2)$, we claim that if $b \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp} b \subset B(0,R)$, then

(i) for all $x \in \mathbb{R}^n$ with $|x| > 4R$,

$$T_{\Omega,b}^{j,\delta,\star\star}f(x) \lesssim \Big(M_\Omega M(|f|^s)(x)\Big)^{1/s}R^{\frac{n}{s'q'}}|x|^{-\frac{n}{s'q'}}; \tag{3.3}$$

(ii) for each $t \in \mathbb{R}^n$ with $|t| < \min\{1,\delta/4\}$,

$$\Big\|\sup_{\nu\in\mathbb{Z}}|T_{\Omega,b}^{j,\delta,\nu}f(x) - T_{\Omega,b}^{j,\delta,\nu}f(x+t)|\Big\|_{L^p(\mathbb{R}^n,w)} \lesssim \frac{|t|}{\delta}2^{j(n+1)}\|f\|_{L^p(\mathbb{R}^n,w)}. \tag{3.4}$$

(iii) for each fixed $D > 0$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|\sup_{\nu>N}|T_{\Omega,b}^{j,\delta,\nu}f|\chi_{B(0,D)}\|_{L^p(\mathbb{R}^n,w)} < \epsilon\|f\|_{L^p(\mathbb{R}^n,w)}, \tag{3.5}$$

$$\Big\|\sup_{\nu<-N}|T_{\Omega,b}^{j,\delta,\nu}f - T_{\Omega,b}^{j,\delta,-N}f|\Big\|_{L^p(\mathbb{R}^n,w)} < \epsilon\|f\|_{L^p(\mathbb{R}^n,w)}. \tag{3.6}$$

If we can prove this, we then know from Lemma 3.1 that $\Gamma_{j,\delta}$ is compact from $L^p(\mathbb{R}^n,w)$ to $L^p(l^\infty;\mathbb{R}^n,w)$.

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We first prove (3.3). For $x \in \mathbb{R}^n$ with $|x| > 4R$, by applying the Hölder inequality, we deduce that

$$\begin{aligned}\int_{|z|<R} \big|K_\Omega^l * \phi_{l-j}(x-z)\big| \mathrm{d}z &\lesssim \Big(\int_{|z|<R} \big|K_\Omega^l * \phi_{l-j}(x-z)\big|^q \mathrm{d}z\Big)^{1/q} R^{\frac{n}{q'}}\\ &\lesssim \Big(\int_{\frac{|x|}{2}\le|z|<2|x|} \big|K_\Omega^l * \phi_{l-j}(z)\big|^q \mathrm{d}z\Big)^{\frac{1}{q}} R^{\frac{n}{q'}}\\ &\lesssim |x|^{-\frac{n}{q'}} R^{\frac{n}{q'}}.\end{aligned}$$

Another application of the Hölder inequality then gives

$$\begin{aligned}\big|T_{\Omega,b}^{j,\delta,\star\star} f(x)\big| &\lesssim \|b\|_{L^\infty(\mathbb{R}^n)} \sum_{l\in\mathbb{Z}} \int_{|z|<R} \big|K_\Omega^l * \phi_{l-j}(x-z)\big| |f(z)| \mathrm{d}z\\ &\lesssim \|b\|_{L^\infty(\mathbb{R}^n)} \sum_{l\in\mathbb{Z}} \Big(\int_{|x|/2\le|x-z|\le2|x|} \big|K_\Omega^l * \phi_{l-j}(x-z)\big| |f(z)|^s \mathrm{d}z\Big)^{\frac{1}{s}}\\ &\quad\times \Big(\int_{|z|<R} \big|K_\Omega^l * \phi_{l-j}(x-z)\big| \mathrm{d}z\Big)^{1/s'}\\ &\lesssim \|b\|_{L^\infty(\mathbb{R}^n)} \Big(M_\Omega M(|f|^s)(x)\Big)^{1/s} R^{\frac{n}{s'q'}} |x|^{-\frac{n}{s'q'}},\end{aligned}$$

which gives (3.3).

We turn our attention to (3.4). Let $b \in C_0^\infty(\mathbb{R}^n)$. Without loss of generality, we may assume that $\|b\|_{L^\infty(\mathbb{R}^n)} + \|\nabla b\|_{L^\infty(\mathbb{R}^n)} = 1$. For each fixed $t \in \mathbb{R}^n$ with $|t| < \delta/4$, write

$$\begin{aligned}&\sup_{\nu\in\mathbb{Z}} \big|T_{\Omega,b}^{j,\delta,\nu} f(x) - T_{\Omega,b}^{j,\delta,\nu} f(x+t)\big|\\ &\quad\lesssim \big|b(x+t) - b(x)\big| \sup_{k\in\mathbb{Z}} \Big|\sum_{l=k}^{\infty} \int_{\mathbb{R}^n} K_\Omega^l * \phi_{l-j}(x-y)\varphi\big(\delta^{-1}(x-y)\big) f(y) \mathrm{d}y\Big|\\ &\qquad+ \sup_{k\in\mathbb{Z}} \Big|\int_{\mathbb{R}^n} U_{j,\delta;k}(x,y;t)\big(b(y) - b(x+t)\big) f(y) \mathrm{d}y\Big|\\ &\quad= \mathrm{J}_1^j f(x,t) + \mathrm{J}_2^j f(x,t),\end{aligned}$$

with

$$\begin{aligned}U_{j,\delta;k}(x,y;t) = \sum_{l=k}^{\infty} \Big(&K_\Omega^l * \phi_{l-j}(x-y)\varphi\big(\delta^{-1}(x-y)\big)\\ &- K_\Omega^l * \phi_{l-j}(x+t-y)\varphi\big(\delta^{-1}(x+t-y)\big)\Big).\end{aligned}$$

To estimate J_1^j, let

$$\begin{aligned}\mathrm{J}_{11}^j f(x,t) = \sum_{l\in\mathbb{Z}} \int_{\mathbb{R}^n} \Big|&K_\Omega^l * \phi_{l-j}(x-y)\varphi\big(\delta^{-1}(x-y)\big)\\ &- K_\Omega^l * \phi_{l-j}(x-y)\chi_{\{|x-y|>2\delta\}}(x-y)\Big| |f(y)| \mathrm{d}y,\end{aligned}$$

and

$$\mathrm{J}_{12}^j f(x,t) = \sup_{k\in\mathbb{Z}} \Big|\sum_{l=k}^{\infty} \int_{\mathbb{R}^n} K_\Omega^l * \phi_{l-j}(x-y)\chi_{\{|x-y|>2\delta\}}(x-y) f(y) \mathrm{d}y\Big|.$$

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A trivial computation gives us

$$\mathrm{J}_{11}^{j} f(x,t) \lesssim \sum_{l\in\mathbb{Z}} \int_{\delta/2\le|x-y|\le2\delta} \big|K_{\Omega}^{l} * \phi_{l-j}(x-y)\big|\,|f(y)|\mathrm{d}y \lesssim M_{\Omega}Mf(x).$$

On the other hand, we have

$$\begin{aligned}\mathrm{J}_{12}^{j} f(x,t) &\lesssim \int_{\mathbb{R}^n} \Big|\sum_{l=k}^{\infty} K_{\Omega}^{l} * \phi_{l-j}(x-y) - K^{j}(x-y)\chi_{\{|x-y|>2^k\}}(x-y)\Big|\,|f(y)|\mathrm{d}y\\ &\quad+ \Big|\int_{\mathbb{R}^n} K^{j}(x-y)\chi_{\{|x-y|>\max\{2\delta,2^k\}\}}(x-y)f(y)\mathrm{d}y\Big|\\ &\lesssim M_{\Omega}Mf(x) + T_{\Omega}^{j,\star} f(x).\end{aligned}$$

Combining the estimates for J_{11}^{j} and $\mathrm{J}_{12}^{j} f(x,t)$ leads to

$$\mathrm{J}_{1}^{j} f(x,t) \lesssim |t|\big(\mathrm{J}_{11}^{j} f(x,t) + \mathrm{J}_{12}^{j} f(x,t)\big) \lesssim |t|\big(M_{\Omega}Mf(x) + T_{\Omega}^{j,\star} f(x)\big).$$

To consider the term $\mathrm{J}_{2}^{j} f(x,t)$, set

$$\mathrm{J}_{21}^{j} f(x,t) = \sum_{l\in\mathbb{Z}} \int_{|x-y|>\delta} \big|K_{\Omega}^{l} * \phi_{l-j}(x-y) - K_{\Omega}^{l} * \phi_{l-j}(x+t-y)\big|\,|f(y)|\mathrm{d}y,$$

$$\mathrm{J}_{22}^{j} f(x,t) = \sum_{l\in\mathbb{Z}} \int_{\mathbb{R}^n} \big|K_{\Omega}^{l} * \phi_{l-j}(x-y)\big|\,\Big|\varphi\Big(\frac{x-y}{\delta}\Big) - \varphi\Big(\frac{x+t-y}{\delta}\Big)\Big|\,|f(y)|\mathrm{d}y.$$

It then follows that

$$\mathrm{J}_{2}^{j} f(x,t) \lesssim \mathrm{J}_{21}^{j} f(x,t) + \mathrm{J}_{22}^{j} f(x,t).$$

We know from (2.6) in Lemma 2.1 that for $s\in(1,\infty)$,

$$\mathrm{J}_{21}^{j} f(x,t) \lesssim \frac{|t|}{\delta} 2^{jn/s} 2^{j} M_s f(x).$$

On the other hand, when $|t|<\delta/4$, it is obvious that $\varphi\big(\frac{x-y}{\delta}\big) - \varphi\big(\frac{x+t-y}{\delta}\big) \neq 0$ only if $|x-y|>\delta/2$; we then deduce that

$$\mathrm{J}_{22}^{j} f(x,t) \lesssim \frac{|t|}{\delta} \sum_{l\in\mathbb{Z}} \int_{\delta/2<|x-y|\le3\delta} \big|K_{\Omega}^{l} * \phi_{l-j}(x-y)\big|\,\big|f(y)\big|\mathrm{d}y \lesssim \frac{|t|}{\delta} M_{\Omega}Mf(x).$$

Therefore,

$$\mathrm{J}_{2}^{j} f(x,t) \lesssim \frac{|t|}{\delta} 2^{j(n+1)} M_s f(x) + \frac{|t|}{\delta} M_{\Omega}Mf(x).$$

The estimate (3.4) follows from the estimates for J_{1}^{j}, J_{2}^{j}, Lemma 2.2, and the weighted estimate for M_{Ω}.

We now verify claim (iii). Let $D>0$ and $N\in\mathbb{N}$ such that $2^{N-2}>D$. Then for $l\ge N$ and $x\in\mathbb{R}^n$ with $|x|\le D$,

$$\int_{\mathbb{R}^n} |K_{\Omega}^{l} * \phi_{l-j}(x-y)||f(y)|\mathrm{d}y = \int_{\mathbb{R}^n} |K_{\Omega}^{l} * \phi_{l-j}(x-y)||f(y)\chi_{\{|y|\le2^{l+3}\}}|\mathrm{d}y.$$

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Therefore, for $v \in \mathbb{Z}$ with $v > N$,

$$\begin{aligned}
|T^{j,\delta,v}_{\Omega,b} f(x)| &\lesssim \sum_{l>N} \int_{\mathbb{R}^n} |K^l_\Omega * \phi_{l-j}(x-y)||f(y)|\mathrm{d}y \\
&\lesssim \sum_{l>N} \int_{|y|\le 2^{l+3}} |f(y)|\mathrm{d}y \|K^l_\Omega\|_{L^1(\mathbb{R}^n)} \|\phi_{l-j}\|_{L^\infty(\mathbb{R}^n)} \\
&\lesssim 2^{nj} \sum_{l>N} 2^{-nl} \int_{|y|\le 2^{l+3}} |f(y)|\mathrm{d}y \\
&\lesssim 2^{nj} \|f\|_{L^p(\mathbb{R}^n,w)} \sum_{l>N} 2^{-nl} \Big(\int_{B(0,2^{l+3})} w^{-\frac{1}{p-1}}(y)\,\mathrm{d}y \Big)^{\frac{1}{p'}}.
\end{aligned}$$

Since $w \in A_\infty(\mathbb{R}^n)$, we can take a positive constant θ such that

$$\int_{B(0,D)} w(y)\,\mathrm{d}y \le \Big(\frac{D}{2^l}\Big)^{n\theta} \int_{B(0,2^{l+3})} w(y)\,\mathrm{d}y;$$

see [17, p. 305]. A straightforward computation now leads to

$$\begin{aligned}
&\Big(\int_{B(0,D)} \sup_{v>N} |T^{j,\delta,v}_{\Omega,b} f(x)|^p\, w(x)\mathrm{d}x \Big)^{1/p} \\
&\lesssim 2^{nj} \|f\|_{L^p(\mathbb{R}^n,w)} \sum_{l>N} 2^{-nl} \Big(\int_{B(0,2^{l+3})} w^{-\frac{1}{p-1}}(y)\,\mathrm{d}y \Big)^{\frac{1}{p'}} \Big(\int_{B(0,D)} w(x)\,\mathrm{d}x \Big)^{\frac{1}{p}} \\
&\lesssim 2^{nj} \|f\|_{L^p(\mathbb{R}^n,w)} \Big(\frac{D}{2^N}\Big)^{\frac{n\theta}{p}}.
\end{aligned}$$

This gives us (3.5) immediately. On the other hand, we have that for $N \in \mathbb{N}$ and $v < -N$,

$$\begin{aligned}
&\Big| T^{j,\delta,v}_{\Omega,b} f(x) - T^{j,\delta,-N}_{\Omega,b} f(x) \Big| \\
&\le \|\nabla b\|_{L^\infty(\mathbb{R}^n)} \sum_{l=-\infty}^{-N} \int_{\mathbb{R}^n} |x-y| \big| K^l_\Omega * \phi_{l-j}(x-y) \big| |f(y)|\mathrm{d}y \\
&\lesssim \|\nabla b\|_{L^\infty(\mathbb{R}^n)} 2^{-N} M_\Omega M f(x),
\end{aligned}$$

which obviously implies that

$$\Big\| \sup_{v<-N} \big| T^{j,\delta,v}_{\Omega,b} f - T^{j,\delta,-N}_{\Omega,b} f \big| \Big\|_{L^p(\mathbb{R}^n,w)} \lesssim 2^{-N} \|\nabla b\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n,w)},$$

and in turn gives (3.6). ∎

Now let $j \in \mathbb{N}$ and $l \in \mathbb{Z}$. Define the operator $W^{j,v}_{\Omega,b}$ by

$$W^{j,l}_{\Omega,b} f(x) = \Big| \int_{\mathbb{R}^n} |K^l_\Omega| * \phi_{l-j}(x-y) |b(x) - b(y)|^2 f(y)\mathrm{d}y \Big|.$$

Lemma 3.4 *Let Ω be homogeneous of degree zero, let $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$, and let p and w be the same as in Theorem 1.3. Then for $b \in C^\infty_0(\mathbb{R}^n)$, the operator Δ_j defined by*

$$\Delta_j f(x) = \big\{ W^{j,l}_{\Omega,b} f(x) \big\}_{l\in\mathbb{Z}} \tag{3.7}$$

is compact from $L^p(\mathbb{R}^n,w)$ to $L^p(l^\infty;\mathbb{R}^n,w)$.

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Proof For $\delta \in (0,1/2)$, let

$$W_{\Omega,b}^{j,\delta,l} f(x) = \Big| \int_{\mathbb{R}^n} |K_\Omega^l| * \phi_{l-j}(x-y)|b(x)-b(y)|^2 \varphi\big(\delta^{-1}(x-y)\big) f(y)\mathrm{d}y\Big|.$$

It is obvious that for $b \in C_0^\infty(\mathbb{R}^n)$,

$$\sup_{l\in\mathbb{Z}} \big| W_{\Omega,b}^{j,\delta,l} f(x)\big| \lesssim M_\Omega M f(x),$$

and so $\sup_{l\in\mathbb{Z}} |W_{\Omega,b}^{j,\delta,l} f(x)|$ define a bounded operator on $L^p(\mathbb{R}^n, w)$. On the other hand, as in the proof of Lemma 3.3, we can verify that for $\delta \in (0,1/2)$, the operator $\Delta_{j,\delta}$ defined by

$$\Delta_{j,\delta} f(x) = \big\{ W_{\Omega,b}^{j,\delta,l} f(x)\big\}_{l\in\mathbb{Z}}$$

is compact from $L^p(\mathbb{R}^n, w)$ to $L^p(l^\infty; \mathbb{R}^n, w)$. Also, as in Lemma 3.2, we deduce that

$$\|\Delta_j f - \Delta_{j,\delta} f\|_{L^p(l^\infty;\mathbb{R}^n,w)} \lesssim \delta \|f\|_{L^p(\mathbb{R}^n,w)}.$$

Thus, Δ_j is compact from $L^p(\mathbb{R}^n, w)$ to $L^p(l^\infty; \mathbb{R}^n, w)$. ∎

Proof of Theorem 1.3 We only consider the compactness of $T_{\Omega,b}^{\star\star}$ on $L^p(\mathbb{R}^n, w)$, since the argument for $T_{\Omega,b}$ is similar and simpler. Let p and w be the same as in Theorem 1.3. For $j \in \mathbb{N}$, let Γ_j be the operator defined by

$$\Gamma_j f(x) = \{T_{\Omega,b}^{j,\nu} f(x)\}_{\nu\in\mathbb{Z}}, \tag{3.8}$$

with

$$T_{\Omega,b}^{j,\nu} f(x) = \sum_{l=\nu}^{\infty} \int_{\mathbb{R}^n} K_\Omega^l * \phi_{l-j}(x-y)\big(b(x)-b(y)\big) f(y)\,\mathrm{d}y.$$

Also, set

$$\Gamma f(x) = \{T_{\Omega,b}^{\nu} f(x)\}_{\nu\in\mathbb{Z}}, \tag{3.9}$$

with

$$T_{\Omega,b}^{\nu} f(x) = \sum_{l=\nu}^{\infty} \int_{\mathbb{R}^n} K_\Omega^l(x-y)\big(b(x)-b(y)\big) f(y)\,\mathrm{d}y.$$

Lemma 3.2 now tells us that for $b \in C_0^\infty(\mathbb{R}^n)$,

$$\|\Gamma_j f - \Gamma_{j,\delta} f\|_{L^p(l^\infty;\mathbb{R}^n,w)} \lesssim \delta \|f\|_{L^p(\mathbb{R}^n,w)}. \tag{3.10}$$

Thus, by Lemma 3.3, Γ_j is compact from $L^p(\mathbb{R}^n, w)$ to $L^p(l^\infty; \mathbb{R}^n, w)$. On the other hand, for $b \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\big\| \Gamma_j f(x) - \Gamma f(x)\big\|_{l^\infty} &\lesssim \sup_{k\in\mathbb{Z}} \Big| \sum_{l=k}^{\infty} \int_{\mathbb{R}^n} \big(b(x)-b(y)\big) S_l^j(x-y) f(y)\mathrm{d}y\Big| \\ &\lesssim \|b\|_{L^\infty(\mathbb{R}^n)} \sup_{k\in\mathbb{Z}} \Big| \sum_{l=k}^{\infty} S_l^j * f(x)\Big| + \sup_{k\in\mathbb{Z}} \Big| \sum_{l=k}^{\infty} S_l^j * (bf)(x)\Big|,\end{aligned}$$

which, via Theorem 2.3, yields

$$\big\| \Gamma_j f - \Gamma f\big\|_{L^p(l^\infty;\mathbb{R}^n,w)} \lesssim 2^{-\beta j} \|b\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n,w)}. \tag{3.11}$$

Therefore, for $b \in C_0^\infty(\mathbb{R}^n)$, Γ is also compact (and completely continuous) from $L^p(\mathbb{R}^n, w)$ to $L^p(l^\infty; \mathbb{R}^n, w)$. Observing that for functions f_1 and f_2,

$$|T^{\star\star}_{\Omega,b} f_1(x) - T^{\star\star}_{\Omega,b} f_2(x)| \le \sup_{v\in\mathbb{Z}} \big| T^{v}_{\Omega,b} f_1(x) - T^{v}_{\Omega,b} f_2(x) \big|,$$

we then know that $T^{\star\star}_{\Omega,b}$ is completely continuous on $L^p(\mathbb{R}^n, w)$ when $b \in C_0^\infty(\mathbb{R}^n)$. It is well known that the limit of a sequence of completely continuous operators is also a completely continuous operator. Recalling that for $b \in \mathrm{BMO}(\mathbb{R}^n)$, $T^{\star\star}_{\Omega,b}$ is bounded on $L^p(\mathbb{R}^n, w)$ with bounded $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$, we finally deduce that $T^{\star\star}_{\Omega,b}$ is completely continuous on $L^p(\mathbb{R}^n, w)$ when $b \in \mathrm{CMO}(\mathbb{R}^n)$. ∎

Proof of Theorem 1.5 Let p and w be as in Theorem 1.5. Recall that $T^{\star}_{\Omega,b}$ is bounded on $L^p(\mathbb{R}^n, w)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. Thus, it suffices to prove that for $b \in C_0^\infty(\mathbb{R}^n)$, $f \in L^p(\mathbb{R}^n, w)$ and $\{f_k\}_{k\in\mathbb{N}} \subset L^p(\mathbb{R}^n, w)$,

$$|f_k - f| \rightharpoonup 0 \text{ in } L^p(\mathbb{R}^n, w) \Rightarrow \|T^{\star}_{\Omega,b} f_k - T^{\star}_{\Omega,b} f\|_{L^p(\mathbb{R}^n,w)} \to 0. \tag{3.12}$$

To prove (3.12), we observe that for $\{f_k\}$ and f,

$$\big|T^{\star}_{\Omega,b} f_k(x) - T^{\star}_{\Omega,b} f(x)\big| \le \Big(M_{\Omega,b}(|f_k - f|)(x)\Big)^{\frac12} \Big(M_{\Omega}(f_k - f)(x)\Big)^{\frac12} + T^{\star\star}_{\Omega,b}(f_k - f)(x), \tag{3.13}$$

with $M_{\Omega,b}$ the operator defined by (1.4). Via the weighted estimate of M_Ω, this yields

$$\big\| T^{\star}_{\Omega,b} f_k - T^{\star}_{\Omega,b} f\big\|_{L^p(\mathbb{R}^n,w)} \lesssim \big\| M_{\Omega,b}(|f_k - f|)\big\|^{\frac12}_{L^p(\mathbb{R}^n,w)} \|f_k - f\|^{\frac12}_{L^p(\mathbb{R}^n,w)} + \big\| T^{\star\star}_{\Omega,b}(f_k - f)\big\|_{L^p(\mathbb{R}^n,w)}. \tag{3.14}$$

In the proof of Theorem 1.3, we have shown that the operator Γ is compact from $L^p(\mathbb{R}^n, w)$ to $L^p(l^\infty; \mathbb{R}^n, w)$; thus, for $f_k \rightharpoonup f$,

$$\|\Gamma(f_k - f)\|_{L^p(l^\infty;\mathbb{R}^n,w)} \to 0 \quad \text{and} \quad \lim_{k\to\infty} \|T^{\star\star}_{\Omega,b}(f_k - f)\|_{L^2(\mathbb{R}^n)} = 0. \tag{3.15}$$

On the other hand, a trivial computation shows that

$$\begin{aligned}
&\Big| M_{\Omega,b} f(x) - \sup_{l\in\mathbb{Z}} W^{j,l}_{\Omega,b} f(x)\Big| \\
&\quad\le \sup_{l\in\mathbb{Z}} \Big| \int_{\mathbb{R}^n} S^j_l(x-y)|b(x) - b(y)|^2 f(y)\,\mathrm{d}y\Big| \\
&\quad\lesssim \|b\|^2_{L^\infty(\mathbb{R}^n)} \sup_{l\in\mathbb{Z}} |\widetilde{S}^j_l * f(x)| + \sup_{l\in\mathbb{Z}} \big|\widetilde{S}^j_l * (|b|^2 f)(x)\big| \\
&\qquad + \|b\|_{L^\infty(\mathbb{R}^n)} \sup_{l\in\mathbb{Z}} \big|\widetilde{S}^j_l * \big(f\mathrm{Re} b\big)(x)\big| + \|b\|_{L^\infty(\mathbb{R}^n)} \sup_{l\in\mathbb{Z}} \big|\widetilde{S}^j_l \big(f\mathrm{Im} b\big)(x)\big|,
\end{aligned}$$

and so by (2.9) in Theorem 2.3,

$$\lim_{j\to\infty} \Big\| M_{\Omega,b} f - \sup_{l\in\mathbb{Z}} W^{j,l}_{\Omega,b} f\Big\|_{L^p(\mathbb{R}^n,w)} \lesssim 2^{-\beta j} \|f\|_{L^p(\mathbb{R}^n,w)}. \tag{3.16}$$

By Lemma 3.4 and the fact that Δ_j is linear, we know that

$$h_k \rightharpoonup 0 \text{ in } L^p(\mathbb{R}^n, w) \Rightarrow \|\Delta_j h_k\|_{L^p(l^\infty;\mathbb{R}^n,w)} \to 0.$$

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Therefore,

$$|f_k - f| \rightharpoonup 0 \text{ in } L^p(\mathbb{R}^n, w) \Rightarrow \|M_{\Omega,b}(|f_k - f|)\|_{L^p(\mathbb{R}^n,w)} \to 0.$$

This together with (3.14) and (3.15) leads to the conclusion of Theorem 1.5. ∎

4 Proof of Theorem 1.8

For $p \in [1,\infty)$ and $\lambda \in (0,n)$, let $L^{p,\lambda}(l^\infty;\mathbb{R}^n)$ be the Banach space of sequences of functions defined by

$$L^{p,\lambda}(l^\infty;\mathbb{R}^n) = \left\{ \{f_k\}_{k\in\mathbb{Z}} : \|\{f_k\}\|_{L^{p,\lambda}(l^\infty;\mathbb{R}^n)} < \infty \right\},$$

with

$$\|\{f_k\}\|_{L^{p,\lambda}(l^\infty;\mathbb{R}^n)} = \Big\| \sup_{k\in\mathbb{Z}} |f_k| \Big\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

Lemma 4.1 *Let $p \in (1,\infty)$ and $\lambda \in (0,n)$, $\mathcal{G} \subset L^{p,\lambda}(l^\infty;\mathbb{R}^n)$. Suppose that $\mathcal{G}$ satisfies the following four conditions:*

(i) *$\mathcal{G}$ is a bounded set in $L^{p,\lambda}(l^\infty;\mathbb{R}^n)$;*

(ii) *for each fixed $\epsilon > 0$, there exists a constant $A > 0$ such that for all $\{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}$,*

$$\Big\| \sup_{k\in\mathbb{Z}} |f_k| \chi_{\{|\cdot|>A\}}(\cdot) \Big\|_{L^{p,\lambda}(\mathbb{R}^n)} < \epsilon;$$

(iii) *for each fixed $\epsilon > 0$, there exists a constant $\rho > 0$ such that for all $t \in \mathbb{R}^n$ with $|t| < \rho$ and $\vec{f} = \{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}$,*

$$\|\vec{f}(\cdot + t) - \vec{f}(\cdot)\|_{L^{p,\lambda}(l^\infty;\mathbb{R}^n)} < \epsilon;$$

(iv) *for each fixed $D > 0$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}$,*

$$\Big\| \sup_{k>N} |f_k| \chi_{B(0,D)} \Big\|_{L^{p,\lambda}(\mathbb{R}^n)} < \epsilon, \ \Big\| \sup_{k<-N} |f_k - f_{-N}| \Big\|_{L^{p,\lambda}(\mathbb{R}^n)} < \epsilon.$$

Then $\mathcal{G}$ is strongly pre-compact in $L^{p,\lambda}(l^\infty;\mathbb{R}^n)$.

Proof As in the proof of Lemma 3.1, it suffices to prove that for each fixed $\epsilon > 0$, there exists a $\delta = \delta_\epsilon > 0$ and a mapping Φ_ϵ on $L^{p,\lambda}(l^\infty;\mathbb{R}^n)$ such that $\Phi_\epsilon(\mathcal{G}) = \{\Phi_\epsilon(\vec{f}) : \vec{f} \in \mathcal{G}\}$ is a strongly pre-compact set in $L^p(l^\infty;\mathbb{R}^n)$, and for any $\vec{f}, \vec{g} \in \mathcal{G}$,

$$\|\Phi_\epsilon(\vec{f}) - \Phi_\epsilon(\vec{g})\|_{L^{p,\lambda}(l^\infty;\mathbb{R}^n)} < \delta \Rightarrow \|\vec{f} - \vec{g}\|_{L^p(l^\infty;\mathbb{R}^n)} < 10\epsilon.$$

Now let $\epsilon > 0$. As in the proof of Lemma 3.1, we choose $A > 1$ large enough, as in assumption (ii), and ρ small enough, as in assumption (iii). Let Q be the largest cube centered at the origin such that $2Q \subset B(0,\rho)$, let $Q_1,\dots,Q_J$ and $\mathcal{D}$ be as in the proof of Lemma 3.1, and let $N \in \mathbb{N}$ be such that for all $\{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}$,

$$\Big\| \sup_{k>N} |f_k| \chi_{B(0,2A)} \Big\|_{L^{p,\lambda}(\mathbb{R}^n,w)} < \epsilon/2, \ \Big\| \sup_{k<-N} |f_k - f_{-N}| \Big\|_{L^{p,\lambda}(\mathbb{R}^n,w)} < \frac{\epsilon}{2J}.$$

Let Φ_ϵ be the operator defined by (3.1). Note that

$$|m_{Q_i}(f_k)| \lesssim \Big\| \sup_{k\in\mathbb{Z}} |f_k| \Big\|_{L^{p,\lambda}(\mathbb{R}^n)} |Q_i|^{\lambda/(np)-1/p}.$$

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For $\vec{f} = \{f_k\}_{k\in\mathbb{Z}}$ and each ball $B(y,r)$, we see that

$$\int_{B(y,r)} |\Phi_\epsilon(\vec{f})(x)|^p \,\mathrm{d}x = \sum_{i=1}^{J} \int_{Q_i\cap B(y,r)} \sup_{k\in\mathbb{Z}} |m_{Q_i}(f_k)|^p \,\mathrm{d}x \lesssim Jr^\lambda \|\vec{f}\|^p_{L^{p,\lambda}(l^\infty;\mathbb{R}^n)},$$

since $|Q_i \cap B(y,r)| \le |Q_i|^{1-\lambda/n}|B(y,r)|^{\lambda/n}$. Thus, $\Phi_\epsilon(\mathcal{G}) = \{\Phi_\epsilon(\vec{f}) : \vec{f} \in \mathcal{G}\}$ is a strongly pre-compact set in $L^{p,\lambda}(l^\infty;\mathbb{R}^n)$. For a ball $B(y,r)$,

$$\begin{aligned}
&\int_{B(y,r)} \|\vec{f}(x)\chi_{\mathcal{D}}(x) - \Phi_\epsilon(\vec{f})(x)\|^p_{l^\infty} \,\mathrm{d}x\\
&\lesssim \int_{B(y,r)} \sup_{|k|\le N} \Big| f_k(x)\chi_{\mathcal{D}}(x) - \sum_{i=1}^{J} m_{Q_i}(f_k)\chi_{Q_i}(x)\Big|^p \,\mathrm{d}x\\
&\quad + \int_{B(y,r)} \sup_{k\le -N} \Big| f_k(x)\chi_{\mathcal{D}}(x) - \sum_{i=1}^{J} m_{Q_i}(f_{-N})\chi_{Q_i}(x)\Big|^p \,\mathrm{d}x\\
&\quad + \int_{B(y,r)} \Big\{ \sup_{k>N} \big| f_k(x)\big| \chi_{B(0,2A)}(x)\Big\}^p \,\mathrm{d}x\\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

A straightforward computation leads to

$$\begin{aligned}
\mathrm{I} &\le \sum_{i=1}^{J} \int_{Q_i\cap B(y,r)} \Big\{ \sup_{|k|\le N} \Big| f_k(x) - \sum_{l=1}^{J} m_{Q_l}(f_k)\chi_{Q_l}(x)\Big|\Big\}^p \,\mathrm{d}x\\
&\lesssim \sum_{i=1}^{J} \frac{1}{|Q_i|} \int_{Q_i\cap B(y,r)} \sup_{|k|\le N} \int_{Q_i} |f_k(x) - f_k(y)|^p \,\mathrm{d}y\,\mathrm{d}x\\
&\lesssim r^\lambda \sup_{|h|\le\rho} \|\vec{f}(\cdot) - \vec{f}(\cdot + h)\|^p_{L^{p,\lambda}(l^\infty;\mathbb{R}^n)}.
\end{aligned}$$

From the Hölder inequality, we obtain that for $k < -N$,

$$\big| m_{Q_i}(f_k) - m_{Q_i}(f_{-N})|^p \lesssim \Big\| \sup_{k<-N} |f_k - f_{-N}|\Big\|^p_{L^{p,\lambda}(\mathbb{R}^n)} |Q_i|^{\lambda/n-1},$$

which implies that

$$\begin{aligned}
\mathrm{II} &\lesssim \sum_{i=1}^{J} \int_{Q_i\cap B(y,r)} \Big\{ \sup_{k<-N} \Big| f_k(x) - \sum_{l=1}^{J} m_{Q_l}(f_k)\chi_{Q_l}(x)\Big|\Big\}^p \,\mathrm{d}x\\
&\quad + \sum_{i=1}^{J} \int_{Q_i\cap B(y,r)} \Big\{ \sup_{k<-N} \big| m_{Q_i}(f_k) - m_{Q_i}(f_{-N})|\Big\}^p \,\mathrm{d}x\\
&\lesssim r^\lambda \sup_{|h|\le\rho} \|\vec{f}(\cdot) - \vec{f}(\cdot + h)\|^p_{L^{p,\lambda}(l^\infty;\mathbb{R}^n)}\\
&\quad + \Big\| \sup_{k<-N} |f_k - f_{-N}|\Big\|^p_{L^{p,\lambda}(\mathbb{R}^n)} \sum_{i=1}^{J} \frac{|Q_i \cap B(y,r)|}{|Q_i|^{1-\lambda/n}}\\
&\lesssim r^\lambda \sup_{|h|\le\rho} \|\vec{f}(\cdot) - \vec{f}(\cdot + h)\|^p_{L^{p,\lambda}(l^\infty;\mathbb{R}^n)}.
\end{aligned}$$

The estimates for I, II, together with assumption (iv), prove that

$$\|\vec{f}\chi_{\mathcal{D}} - \Phi_\epsilon(\vec{f})\|_{L^{p,\lambda}(l^\infty;\mathbb{R}^n,w)} < 3\epsilon,$$

which, via assumption (ii), tells us that for all $\vec{f} \in \mathcal{G}$,

$$\|\vec{f} - \Phi_\epsilon(\vec{f})\|_{L^{p,\lambda}(l^\infty;\mathbb{R}^n,w)} < 4\epsilon.$$

This leads to our claim and completes the proof of Lemma 4.1. ∎

Lemma 4.2 *Let $p, s \in [1,\infty)$, and let $\{T_l\}_{l\in\mathbb{Z}}$ be a sequence of sublinear operators on $L^p(\mathbb{R}^n)$. Suppose that for all measurable sets E and all $r \in (s,\infty)$,*

$$\Big\| \sup_{l\in\mathbb{Z}} |T_l f| \Big\|_{L^p(\mathbb{R}^n,\chi_E)} \lesssim D(r)\|f\|_{L^p(\mathbb{R}^n,M_r\chi_E)},$$

with $D(r)$ a constant depending only on p, n, and r. Then for $\lambda \in (0,n/s)$, $\sigma \in (1,\infty)$ such that $n > \lambda s\sigma$,

$$\Big\| \sup_{l\in\mathbb{Z}} |T_l f| \Big\|_{L^{p,\lambda}(\mathbb{R}^n)} \lesssim D(s\sigma)\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

Proof This lemma was essentially proved in [9]. For the sake of self-containment, we present the proof here. For fixed ball B and $f \in L^{p,\lambda}(\mathbb{R}^n)$, decompose f as

$$f(y) = f(y)\chi_{2B}(y) + \sum_{k=1}^{\infty} f(y)\chi_{2^{k+1}B\backslash 2^k B}(y) = \sum_{k=1}^{\infty} f_k(y).$$

It is obvious that

$$\begin{aligned}\Big(\int_B\Big(\sup_{l\in\mathbb{Z}}|T_l f_0(y)|\Big)^p \mathrm{d}y\Big)^{1/p} &\lesssim D(s\sigma)\Big(\int_{B(x,2r)}|f(y)|^p\mathrm{d}y\Big)^{1/p}\\ &\lesssim D(s\sigma)r^{\lambda/p}\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.\end{aligned}$$

On the other hand, our assumption implies that for each $k \in \mathbb{N}$,

$$\begin{aligned}\Big(\int_B\Big(\sup_{l\in\mathbb{Z}}|T_l f_k(y)|\Big)^p \mathrm{d}y\Big)^{1/p} &\lesssim D(s\sigma)\Big(\int_{\mathbb{R}^n}|f_k(y)|^p\{M\chi_B(y)\}^{\frac{1}{s\sigma}}\mathrm{d}y\Big)^{1/p}\\ &\lesssim D(s\sigma)2^{\frac{-kn}{s\sigma p}}\Big(\int_{2^{k+1}B}|f(y)|^p\mathrm{d}y\Big)^{1/p}\\ &\lesssim D(s\sigma)r^{\lambda/p}2^{-k(\frac{n}{s\sigma p}-\frac{\lambda}{p})}\|f\|_{L^{p,\lambda}(\mathbb{R}^n)},\end{aligned}$$

where in the second inequality, we have invoked the fact that for $y \in 2^{k+1}B\backslash 2^k B$, $M\chi_B(y) \lesssim 2^{-kn}$; see [23] for details. Recall that $n > \lambda s\sigma$. Therefore,

$$\begin{aligned}\Big(\int_B\Big(\sup_{l\in\mathbb{Z}}|T_l f(y)|\Big)^p \mathrm{d}y\Big)^{1/p} &\lesssim \sum_{k=0}^{\infty}\Big(\int_B\Big(\sup_{l\in\mathbb{Z}}|T_l f_k(y)|\Big)^p\mathrm{d}y\Big)^{1/p}\\ &\lesssim D(s\sigma)r^{\lambda/p}\sum_{k=0}^{\infty}2^{-k(\frac{n}{s\sigma p}-\frac{\lambda}{p})}\|f\|^p_{L^{p,\lambda}(\mathbb{R}^n)}\\ &\lesssim D(s\sigma)r^{\lambda/p}\|f\|^p_{L^{p,\lambda}(\mathbb{R}^n)}.\end{aligned}$$

This leads to our desired conclusion directly. ∎

Lemma 4.3 *Let Ω be homogeneous of degree zero and have mean value zero, let $\Omega \in L^q(S^{n-1})$ for some $q \in (1,\infty]$, $p \in (1,\infty)$ and $\lambda \in (0,n)$ or $p \in (1,q']$ and $\lambda \in (0,n/q')$. Then for $\delta \in (0,1/2)$ and $b \in C_0^\infty(\mathbb{R}^n)$,*

(i) *the operator* $T^{j,\delta}_{\Omega,b}$ *is compact on* $L^{p,\lambda}(\mathbb{R}^n)$;

(ii) *the operator* $\Gamma_{j,\delta}$ *defined by* (3.2) *is compact from* $L^{p,\lambda}(\mathbb{R}^n)$ *to* $L^{p,\lambda}(l^\infty;\mathbb{R}^n)$.

Proof We only prove conclusion (ii). From Lemmas 2.2 and 3.2, we know that

$$\|T^{j,\delta,\star\star}_{\Omega,b}f\|_{L^p(\mathbb{R}^n,w)} \lesssim \|f\|_{L^p(\mathbb{R}^n,w)}, \tag{4.1}$$

if p and w are the same as in Theorem 1.3. By repeating the argument used in the proof of [21, Theorem 2], we see that (4.1) still holds if $p \in (1,\infty)$ and $w^{q'} \in A_p(\mathbb{R}^n)$. Note that for all measurable set E and $r \in (1,\infty)$, $M_r\chi_E \in A_1(\mathbb{R}^n)$ (see [17]). Therefore,

$$\|T^{j,\delta,\star\star}_{\Omega,b}f\|_{L^p(\mathbb{R}^n,\chi_E)} \lesssim \|f\|_{L^p(\mathbb{R}^n,M_s\chi_E)},$$

provided that $p \in (q',\infty)$ and $s \in (1,\infty)$ or $p \in (1,\infty)$ and $s \in (q',\infty)$. Via Lemma 4.2, this shows that

$$\|T^{j,\delta,\star\star}_{\Omega,b}f\|_{L^{p,\lambda}(\mathbb{R}^n)} \lesssim \|f\|_{L^{p,\lambda}(\mathbb{R}^n)},$$

provided $p \in (q',\infty)$ and $\lambda \in (0,n)$, or $p \in (1,q']$ and $\lambda \in (0,n/q')$. Similarly, we can deduce from (3.3) and (3.4) that for any fixed ϵ, we can choose A large enough such that

$$\left\| T^{j,\delta,\star\star}_{\Omega,b}f\chi_{\{|\cdot|>A\}}\right\|_{L^{p,\lambda}(\mathbb{R}^n)} \lesssim \epsilon\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}$$

and ς small enough such that for $t \in \mathbb{R}^n$ with $|t| < \varsigma$,

$$\left\|\Gamma_{j,\delta}f(\cdot) - \Gamma_{j,\delta}f(\cdot+t)\right\|_{L^{p,\lambda}(l^\infty;\mathbb{R}^n)} \lesssim \epsilon\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

Also, for fixed $\epsilon > 0$ and $A > 0$, by (3.5), (3.6), and Lemma 4.2, we can take $N \in \mathbb{N}$ such that

$$\Big\|\sup_{\nu>N}|T^{j,\delta,\nu}_{\Omega,b}f|\chi_{B(0,A)}\Big\|_{L^{p,\lambda}(\mathbb{R}^n)} < \epsilon\|f\|_{L^{p,\lambda}(\mathbb{R}^n)},$$

$$\Big\|\sup_{\nu<-N}\big|T^{j,\delta,\nu}_{\Omega,b}f - T^{j,\delta,-N}_{\Omega,b}f\big|\Big\|_{L^{p,\lambda}(\mathbb{R}^n)} < \epsilon\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

Employing Lemma 4.1 then leads to the compactness from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{p,\lambda}(l^\infty;\mathbb{R}^n)$ for $\Gamma_{j,\delta}$. ∎

Lemma 4.4 *Let* Ω *be homogeneous of degree zero and let* $\Omega \in L^q(S^{n-1})$ *for some* $q \in (1,\infty]$. *Let* $p \in (1,\infty)$ *and* $\lambda \in (0,n)$ *or* $p \in (1,q']$ *and* $\lambda \in (0,n/q')$. *Then for* $b \in C^\infty_0(\mathbb{R}^n)$ *and* $j \in \mathbb{N}$, *the operator* Δ_j *defined by* (3.7) *is compact from* $L^{p,\lambda}(\mathbb{R}^n)$ *to* $L^{p,\lambda}(l^\infty;\mathbb{R}^n)$.

Lemma 4.4 can be proved by the argument in the proof of Lemma 4.3, together with the estimates in the proof of Lemma 3.4. We omit the details for brevity.

We are now ready to prove Theorem 1.8.

Proof of Theorem 1.8 By Lemma 4.2 and the weighted norm inequalities for $T_{\Omega,b}$ and $T^\star_{\Omega,b}$, we see that both $T_{\Omega,b}$ and $T^\star_{\Omega,b}$ are bounded on $L^{p,\lambda}(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$, provided that $p \in (q',\infty)$ and $\lambda \in (0,n)$, or $p \in (1,q']$ and $\lambda \in (0,n/q')$. Thus, it suffices to prove the conclusions for the case $b \in C^\infty_0(\mathbb{R}^n)$. For simplicity, we only consider $T^\star_{\Omega,b}$ and $T^{\star\star}_{\Omega,b}$.

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To consider the compactness of $T^{\star\star}_{\Omega,b}$ on $L^{p,\lambda}(\mathbb{R}^n)$, let $p\in(q'\,\infty)$ and $\lambda\in(0,n)$ or $p\in(1,q']$ and $\lambda\in(0,n/q')$. For $j\in\mathbb{N}$, $\delta\in(0,1/2)$, let $\Gamma_{j,\delta}$ and Γ_j be the operators defined by (3.2) and (3.8), respectively. Let $b\in C_0^\infty(\mathbb{R}^n)$. Repeating the argument used in the proof of [21, Theorem 2], we obtain from (3.10) that

$$\|\Gamma_j f-\Gamma_{j,\delta}f\|_{L^r(l^\infty;\mathbb{R}^n,w)}\lesssim\delta\|b\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^r(\mathbb{R}^n,w)}$$

provided that $r\in(q',\infty)$ and $w\in A_{r/q'}(\mathbb{R}^n)$, or $r\in(1,\infty)$ and $w^{q'}\in A_r(\mathbb{R}^n)$. Thus, by Lemma 4.2,

$$\|\Gamma_j f-\Gamma_{j,\delta}f\|_{L^{p,\lambda}(l^\infty;\mathbb{R}^n)}\lesssim\delta\|b\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

This, via Lemma 4.3, shows that Γ_j is compact from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{p,\lambda}(l^\infty;\mathbb{R}^n)$. Similarly, we get from (3.11) and Lemma 4.2 that for some constant $\iota\in(0,1)$,

$$\big\|\Gamma_j f-\Gamma f\big\|_{L^{p,\lambda}(l^\infty;\mathbb{R}^n)}\lesssim 2^{-\iota j}\|b\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

Therefore, the operator Γ defined by (3.9) is also compact from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{p,\lambda}(l^\infty;\mathbb{R}^n)$ when $b\in C_0^\infty(\mathbb{R}^n)$, and so $T^{\star\star}_{\Omega,b}$ is completely continuous on $L^{p,\lambda}(\mathbb{R}^n)$.

It remains to consider the operator $T^\star_{\Omega,b}$. Let $p\in(1,\infty)$ and $\lambda\in(0,n)$ or $p\in(1,q']$ and $\lambda\in(0,n/q')$. For $\{f_k\}\subset L^{p,\lambda}(\mathbb{R}^n)$ and $f\in L^{p,\lambda}(\mathbb{R}^n)$ with $|f_k-f|\rightharpoonup 0$, we get from (3.13) that

$$\begin{aligned}\big\|T^\star_{\Omega,b}f_k-T^\star_{\Omega,b}f\big\|_{L^{p,\lambda}(\mathbb{R}^n)}&\lesssim\big\|T^{\star\star}_{\Omega,b}(f_k-f)\big\|_{L^{p,\lambda}(\mathbb{R}^n)}\\&\quad+\big\|M_{\Omega,b}(|f_k-f|)\big\|^{1/2}_{L^{p,\lambda}(\mathbb{R}^n)}\|f_k-f\|^{1/2}_{L^{p,\lambda}(\mathbb{R}^n)}.\end{aligned}$$

The fact that Γ is completely continuous from $L^p(\mathbb{R}^n)$ to $L^{p,\lambda}(l^\infty;\mathbb{R}^n)$ implies that

$$\lim_{k\to\infty}\big\|T^{\star\star}_{\Omega,b}(f_k-f)\big\|_{L^{p,\lambda}(\mathbb{R}^n)}=0.$$

On the other hand, the estimate (3.16), via Lemma 4.2, tells us that for $b\in C_0^\infty(\mathbb{R}^n)$,

$$\lim_{j\to\infty}\Big\|M_{\Omega,b}h-\sup_{l\in\mathbb{Z}}W^{j,l}_{\Omega,b}h\Big\|_{L^{p,\lambda}(\mathbb{R}^n)}\lesssim 2^{-\iota j}\|h\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

We then deduce from Lemma 4.4 that

$$\lim_{k\to\infty}\big\|M_{\Omega,b}\big(|f_k-f|\big)\big\|_{L^{p,\lambda}(\mathbb{R}^n)}=0.$$

This leads to

$$\lim_{k\to\infty}\|T^\star_{\Omega,b}f_k-T^\star_{\Omega,b}f\|_{L^{p,\lambda}(\mathbb{R}^n)}=0$$

and completes the proof of Theorem 1.8. ∎

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