Can. J. Math.Vol. 43 (4), 1991 pp. 738–747

EQUIVARIANT FIXED POINT INDEX AND THE PERIOD-DOUBLING CASCADES

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0. **Introduction.** Properties of fixed points of equivariant maps have been studied by several authors including A. Dold (cf. [2], 1982), H. Ulrich (cf. [9], 1988), A. Marzantowicz (cf. [7], 1975) and others. Closely related is the work of R. Rubinsztein (cf. [8], 1976) in which he investigated homotopy classes of equivariant maps between spheres. There have been many attempts to introduce and effectively apply these concepts to nonlinear problems. In particular we mention the work of E. Dancer (cf. [1], 1982) in which some applications to nonlinear problems are given.

Recently K. Komiya (cf. [6], 1988) defined for an equivariant map $f: X \longrightarrow X$ a family of integers $\{ a_H(f) \}$. We believe that, taking into account certain natural properties of the family $\{ a_H(f) \}$, it is appropriate to label this family of integers as the equivariant fixed point index at f. We also note that using the approach taken in ([5], 1989) one can define this fixed point index by means of elementary homotopy theory.

In this paper we present a simple geometric interpretation of the equivariant fixed point index for generic $\mathbb{Z}_n$-equivariant maps. We combine those results with the method of A. Dold (cf. [3], 1983) based on the fact that n-periodic orbits of the map f correspond to fixed points of the $\mathbb{Z}_n$-equivariant map defined by $\hat{f}_n(x_1, \ldots, x_n) = \big(f(x_n), f(x_1), \ldots, f(x_{n-1})\big)$. Consequently we obtain a simple proof of a theorem originally proved by J. Franks (cf. [4], 1985), which describes the period-doubling cascades of $f: D^n \longrightarrow \mathrm{Int}(D^n)$.

The proof given by J. Franks uses homology theory and nontrivial properties of the Lefschetz zeta function. We believe that our approach is somewhat simpler and provides a geometrical interpretation of the phenomena. We also remark that the recent paper of Matsuoka (cf. [10], 1989) is closely related to this subject.

1. **Equivariant fixed point index.** Let G be a finite abelian group and assume that V is a linear finite dimensional *representation* of G, i.e., we assume that there is given a homomorphism $\varphi: G \longrightarrow \mathrm{GL}(V)$, where $\mathrm{GL}(V)$ denotes the general linear group of V. We put $gx := \big(\varphi(g)\big)(x)$, $x \in V$, $g \in G$.

The first and third authors' research is supported by grants from NSERC Canada.

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Received by the editors December 14, 1989.

AMS subject classification: Primary: 55M25; secondary: 55P91, 58C30.

https://doi.org/10.4153/CJM-1991-042-4 Published online by Cambridge University Press

Given $x \in V$, $G_x := \{g \in G : gx = x\}$ denotes the *isotropy group* of x and $Gx := \{y \in V : y = gx \text{ for some } g \in G\}$ denotes the *orbit* of x. For a subgroup H of G we put $V^H := \{x \in V : gx = x \text{ for all } g \in H\}$ and for a subset $X \subset V$ we denote $X^H := X \cap V^H$. Let $\mathcal{H} := \mathcal{H}(G)$ denote the family of subgroups of G.

Let Ω be an open bounded invariant subset of V and suppose that $f: \bar{\Omega} \longrightarrow V$ is a continuous equivariant map such that $f(x) \neq x$ for all $x \in \partial\Omega$. Then there is defined a sequence of integers $\{i_H\}$, $H \in \mathcal{H}(G)$, called the *equivariant fixed point index* G-$\mathrm{ind}(f, \Omega) = \{i_H\}$ of f with respect to Ω. The numbers i_H will also be denoted by G-$\mathrm{ind}_H(f, \Omega)$.

The equivariant fixed point index has the following properties.

(1) *(Existence of Fixed Points)*

If G-$\mathrm{ind}_H(f, \Omega) \neq 0$ then there exists $x = f(x) \in \Omega$ such that $H \subset G_x$.

(2) *(Excision)*

If $\Omega_1 \subset \Omega$ is an invariant open subset such that $f(x) \neq x$ for all $x \in \bar{\Omega} \setminus \Omega_1$ then

$$G\text{-}\mathrm{ind}(f, \Omega) = G\text{-}\mathrm{ind}(f, \Omega_1).$$

(3) *(Homotopy)*

If $h: \bar{\Omega} \times [0, 1] \longrightarrow V$ is a continuous map such that

(i) $h(\cdot, t)$ is equivariant for each $t \in [0, 1]$,

(ii) $h(x, t) \neq x$ for all $x \in \partial\Omega$ and $t \in [0, 1]$, then

$$G\text{-}\mathrm{ind}\big(h(\cdot, 0), \Omega\big) = G\text{-}\mathrm{ind}\big(h(\cdot, 1), \Omega\big).$$

(4) *(Additivity)*

If Ω_1, Ω_2 are two disjoint bounded open subsets such that $f(x) \neq x$ for $x \in \partial\Omega_1 \cup \partial\Omega_2$ then

$$G\text{-}\mathrm{ind}(f, \Omega_1 \cup \Omega_2) = G\text{-}\mathrm{ind}(f, \Omega_1) + G\text{-}\mathrm{ind}(f, \Omega_2).$$

(5) *(Product Formula)*

Suppose that $V = V_0 \oplus V_1$ and $\Omega_0 \subset V_0$, $\Omega_1 \subset V_1$ are invariant bounded and open subsets. Suppose further that $f_0: \bar{\Omega}_0 \longrightarrow V_0$ is an equivariant map such that $f_0(x) \neq x$ for $x \in \partial\Omega_0$. Define $f: \bar{\Omega}_0 \times \bar{\Omega}_1 \longrightarrow V$ by $f(x, y) = \big(f_0(x), 0\big)$, $x \in \bar{\Omega}_0$, $y \in \bar{\Omega}_1$. Then $f(x, y) \neq (x, y)$ for $(x, y) \in \partial(\Omega_0 \times \Omega_1)$ and

$$G\text{-}\mathrm{ind}(f, \Omega_0 \times \Omega_1) = G\text{-}\mathrm{ind}(f_0, \Omega_0).$$

(6) *(Normalization)*

If $H \in \mathcal{H}(G)$ and $G_x = H$ for all $x \in \Omega$ then

$$G\text{-}\mathrm{ind}_K(f, \Omega) = \begin{cases} \frac{1}{|H|}\,\mathrm{ind}(f, \Omega) & \text{if } K = H \\ 0 & \text{if } K \neq H \end{cases}$$

where $\mathrm{ind}(f, \Omega)$ denotes the classical fixed point index.

For a more detailed description and other properties of the equivariant fixed point index we refer to K. Komiya [6].

2. **Orthogonal representations of the cyclic group $G = \mathbb{Z}_n$.** Let V be a finite dimensional linear space over $\mathbb{R}$. For a linear map $A: V \longrightarrow V$ we denote by $\sigma(A)$ the spectrum of A.

Assume that $X \subset \mathbb{C}$ is a subset satisfying the condition

$$(*) \qquad \text{if } \lambda \in X \text{ then } \bar{\lambda} \in X.$$

Then we denote by $\Lambda(A, X)$ the linear subspace of V determined by $\sigma(A) \cap X$, i.e. $\Lambda(A, X)$ is the linear subspace of V generated by all generalized eigenspaces corresponding to eigenvalues in X. Let $\lambda(A, X)$ denote the dimension of the subspace $\Lambda(A, X)$.

We introduce the following notation

$$d(A) := (-1)^{\lambda\left(A,(-\infty,0)\right)};$$
$$j(A) := (-1)^{\lambda\left(A,(1,\infty)\right)};$$
$$k(A) := (-1)^{\lambda\left(A,(-\infty,-1)\right)}.$$

Note that $d(I - A) = j(A)$, where $I: V \longrightarrow V$ denotes the identity.

LEMMA 2.1. *Let $A_1, A_2, \ldots, A_k$ be a sequence of $n \times n$ matrices such that $1 \notin \sigma(A_i)$, $j = 1, 2, \ldots, k$ and let*

$$M = \begin{bmatrix} 0 & A_1 & 0 & \ldots & 0 \\ 0 & 0 & A_2 & \ldots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \ldots & A_{k-1} \\ A_k & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

Then $j(M) = j(A_1 A_2 \ldots A_k)$ and $k(M) = k(A_1 A_2 \ldots A_k)$.

PROOF. Let us remark that

$$M^n = \begin{bmatrix} A_1 A_2 \cdots A_k & 0 & 0 \\ 0 & A_2 \cdots A_k A_1 & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & A_k A_1 \cdots A_{k-1} \end{bmatrix}$$

therefore $\sigma(A_1 A_2 \cdots A_k) = \{\mu^k : \mu \in \sigma(M)\}$. Let μ be a solution of the equation $\mu^k = \lambda$, where $\mu \in \sigma(M)$. Put $\alpha = \cos\left(\frac{\pi}{k}\right) + i \sin\left(\frac{\pi}{k}\right)$. Then the numbers $\mu_j = \alpha^j \mu$; $j = 0, \ldots, k-1$ are exactly the k roots $\mu_0, \ldots, \mu_{k-1}$ of that equation. Now, let $x := [x_1, \ldots, x_k] \in V^k$, be an eigenvector associated with μ. Then $[x_1, \alpha^j x_2, \ldots, \alpha^{j(k-1)} x_k]$ is the eigenvector associated with $\mu_j = \alpha^j \mu$. Indeed, $Mx = \left[A_1 x_2, A_2 x_3, \ldots, A_k x_1\right] = \mu[x_1, x_2, \ldots, x_k]$, thus $M[x_1, \alpha^j x_2, \ldots, \alpha^{j(k-1)} x_k] = [A_1 \alpha^j x_2, A_2 \alpha^{2j} x_3, \ldots, A_k x_1] = [\alpha^j \mu x_1, \alpha^{2j} \mu x_2, \ldots, \mu x_1] = \alpha^j \mu [x_1, \alpha^j x_2, \ldots, \alpha^{j(k-1)} x_k]$. We can assume without loss of generality that M has no multiple eigenvalues. Therefore the number of real eigenvalues of $\pm M$ greater than 1 is equal to the number of real eigenvalues of $\pm A_1, A_2 \cdots A_k$ greater than 1. That means $j(M) = j(A_1 A_2 \cdots A_k)$ and $k(M) = k(A_1 A_2 \cdots A_k)$. ∎

We put $\mathbb{Z}_n = \{\gamma \in \mathbb{C} : \gamma^n = 1\}$, $n = 2, 3, \ldots$, and let $\gamma_n \in \mathbb{Z}_n$ denote the generator $\gamma_n = e^{\frac{2\pi i}{n}}$.

Let $\varphi \colon \mathbb{Z}_n \longrightarrow O(V) \subset \mathrm{GL}(V)$ be an orthogonal representation of $\mathbb{Z}_n$. We put $T_n := \varphi(\gamma_n) \colon V \longrightarrow V$ and we denote $\gamma v := \varphi(\gamma)(v)$ for $\gamma \in \mathbb{Z}_n$, $v \in V$.

Let $J := \{j \in \mathbb{N} : j \text{ divides } n\}$. For every $j \in J$ we put $\Pi_j := \{\lambda \in \mathbb{C} : \lambda^j = 1$ and $\lambda^r \neq 1$ for $0 < r < j\}$ and we define

$$V_j := \Lambda(T_n, \Pi_j).$$

As an immediate consequence of the above definition we obtain

PROPOSITION 2.2. *We have*

(i) $V = \bigoplus_{j \in J} V_j$;

(ii) If $A \colon V \longrightarrow V$ *is an equivariant linear map then* $A(V_j) \subset V_j$ *for all* $j \in J$.

(iii) If $A \colon V \longrightarrow V$ *is an equivariant isomorphism then* $A(V_j) = V_j$ *for all* $j \in J$. ∎

Let $\mathrm{GL}_G(V)$ denote the group of all linear equivariant automorphisms of V.

PROPOSITION 2.3. *Two equivariant automorphisms* $A_0, A_1 \in \mathrm{GL}_G(V_j)$, *for* $j \in J$, *are in the same connected component of* $\mathrm{GL}_G(V_j)$ *if and only if* $d(A_0) = d(A_1)$. *Moreover* $\mathrm{GL}_G(V_j)$ *has two connected components only if* $j = 1$ *or* 2, *otherwise* $\mathrm{GL}_G(V_j)$ *is connected.*

PROOF. For $j = 1$ the action of G on V_j is trivial, i.e. $\gamma x = T_n x = x$ for all $x \in V_1$. Therefore $\mathrm{GL}_G(V_1) = \mathrm{GL}(\dim V_1, \mathbb{R})$ and our claim follows immediately from the well-known properties of $\mathrm{GL}(p, \mathbb{R})$, $p = \dim V_1$. In the case $j = 2$, $T_n x = -x$ for $x \in V_2$, thus any linear automorphism $A \colon V_2 \longrightarrow V_2$ commutes with T_n, and consequently we obtain again $\mathrm{GL}_G(V_2) = \mathrm{GL}(\dim V_2, \mathbb{R})$.

Suppose now $j > 2$. For an equivariant linear map $A \colon V_j \longrightarrow V_j$ let

$$L_- = \Lambda(A, (-\infty, 0)),$$
$$L_+ = \Lambda\big(A, \mathbb{C} \setminus (-\infty, 0)\big).$$

Since A commutes with T_n, both L_- and L_+ are invariant subspaces of V_j and $V_j = L_- \oplus L_+$. Let $t \in [0, 1]$, we define

$$H_t(x) = \begin{cases} (1-t)A(x) - tx & \text{if } x \in L_-; \\ (1-t)A(x) + tx & \text{if } x \in L_+, \end{cases}$$

and extend H_t to a linear map $H_t \colon V_j \longrightarrow V_j$. Now, we put $\eta(t) = H_t$. It follows from the definition that $\eta \colon [0, 1] \longrightarrow \mathrm{GL}_G(V_j)$ is a path in $\mathrm{GL}_G(V_j)$ such that $\eta(0) = A$ and $\eta(1) = H_1$. Next, we define

$$G_t(x) = \begin{cases} -(1-t)x + tT_n x & \text{if } x \in L_-; \\ (1-t)x + tT_n x & \text{if } x \in L_+. \end{cases}$$

Since neither 1 nor -1 is an eigenvalue of T_n, the mapping $\mu(t) := G_t$ defines a path in $\mathrm{GL}_G(V_j)$ such that $\mu(0) = H_1$ and $\mu(1) = T_n$. Finally $\omega(t) = (1-t)T_n + tI$ defines a path in $\mathrm{GL}_G(V_j)$ such that $\omega(0) = T_n$ and $\omega(1) = I$. This completes the proof. ∎

Let us denote by $J_G(V)$ the set of all linear equivariant maps $A \colon V \longrightarrow V$ such that $1 \notin \sigma(A)$. Clearly $A \in J_G(V)$ if and only if $I - A \in \mathrm{GL}_G(V)$. As a direct consequence of Proposition (2.3) we have

COROLLARY 2.4. *Suppose* $A_0, A_1 \in J_G(V_j)$, $j \in J$. *Then* A_0, A_1 *are in the same connected component of* $J_G(V_j)$ *if and only if* $j(A_0) = j(A_1)$. *Moreover,* $J_G(V_j)$ *has two connected components only if* $j = 1$ *or* 2, *otherwise* $J_G(V_j)$ *is connected.*

3. **Equivariant fixed point index of generic maps.** Throughout this section we assume that V is an orthogonal representation of $G = \mathbb{Z}_n$, $\Omega \subset V$ is an open bounded invariant set and $f\colon \bar{\Omega} \longrightarrow V$ is a continuous equivariant map. We also assume that

$$\text{(1)} \qquad \mathrm{Fix}(f) = \{x \in \bar{\Omega} : f(x) = x\} \subset \Omega.$$

Further, we assume that f is *generic*, i.e. f is of class C^1 and $1 \notin \sigma(Df(x))$ for every $x \in \mathrm{Fix}(f)$. Finally, we fix $k \in J$, i.e. k divides n, and assume that $\mathrm{Fix}(f) = Gv_0$, where $G_{v_0} = \mathbb{Z}_r$, $k \cdot r = n$. Note that this implies that $\mathrm{Fix}(f)$ has precisely k points.

Let us introduce

$$J_0 := \{j \in J : j \text{ divides } k\},$$
$$J' := \left\{j \in J : \gcd\left(r, \frac{n}{j}\right) = \frac{r}{2}\right\},$$
$$J'' := \left\{j \in J : \gcd\left(r, \frac{n}{j}\right) \neq r, \frac{r}{2}\right\}.$$

Note that $J = J_0 \cup J' \cup J''$. Roughly speaking J_0 denotes the set of $j \in J$ such that $\mathbb{Z}_r$ acts trivially on V_j, J' is the set of $j \in J$ such that $\mathbb{Z}_r$ acts on V_j like $\mathbb{Z}_2$ and J'' is the set of $j \in J$ such that $\mathbb{Z}_r$ acts on V_j like $\mathbb{Z}_m$ for some $m > 2$. Moreover we have the following direct sum decomposition

$$V = W_k \oplus X_k \oplus Y_k$$

where

$$W_k := \bigoplus_{j \in J_0} V_j, \; X_k := \bigoplus_{j \in J'} V_j, \; Y_k := \bigoplus_{j \in J''} V_j.$$

Since the map $f\colon \bar{\Omega} \longrightarrow V$ is equivariant, the derivative $Df(v_0)\colon V \longrightarrow V$ is an equivariant linear map with respect to the action of the isotropy group $G_{v_0} = \mathbb{Z}_r$. Note that $W_k = \Lambda(T_r, \Pi_1)$ and $X_k = \Lambda(T_r, \Pi_2)$, therefore $Df(v_0)$ has the following matrix representation.

$$Df(v_0) = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix} : W_k \oplus X_k \oplus Y_k \longrightarrow W_k \oplus X_k \oplus Y_k.$$

THEOREM 3.1. *Let* $f\colon \bar{\Omega} \longrightarrow V$ *be an equivariant generic map such that* $\mathrm{Fix}(f) = Gv_0 \subset \Omega$, $G_{v_0} = \mathbb{Z}_r$, $v_0 \in \Omega$. *Then, using the same notation as above, we have that* $G\text{-ind}(f, \Omega) = \{i_H\}$ *where*

$$i_H = G\text{-}\mathrm{ind}_H(f, \Omega) = \begin{cases} j(A) & \textit{for } H = \mathbb{Z}_r \\ 0 & \textit{for } H = \mathbb{Z}_{\frac{r}{2}} \textit{ if } j(B) = 1 \\ -j(A) & \textit{for } H = \mathbb{Z}_{\frac{r}{2}} \textit{ if } j(B) = -1 \\ 0 & \textit{for } H \neq \mathbb{Z}_r, \mathbb{Z}_{\frac{r}{2}}. \end{cases}$$

PROOF. Without loss of generality we may assume

(a) there exists $\varepsilon > 0$ such that

$$\Omega = \bigcup_{j=0}^{k-1} \Omega_j,\ \Omega_{j_1} \cap \Omega_{j_2} = \emptyset \text{ for } j_1 \neq j_2$$

where $\Omega_j = \gamma_n^j \Omega_0$, $\Omega_0 = B(v_0, \varepsilon) \times B_X \times B_Y$, $B(v_0, \varepsilon) = \{w \in W_k : |w - v_0| < \varepsilon\}$, $B_X = \{x \in X : |x| < 1\}$ and $B_Y = \{y \in Y_k : |y| < 1\}$;

(b) $f(w, x, y) = \big(A(w_0 - v_0), B(x), C(y)\big)$, where $w \in B(v_0, \varphi)$, $x \in B_X$ and $y \in B_Y$.

Let us remark that if $\varphi : \bar{\Omega}_0 \longrightarrow V$ is a continuous mapping such that $\varphi(w, x, y) = \big(\varphi_1(w), \varphi_2(x), \varphi_3(y)\big)$, where $\varphi_1 : \overline{B(v_0, \varepsilon)} \longrightarrow W_k$, $\varphi_2 : \overline{B}_X \longrightarrow X_k$ and $\varphi_3 : \overline{B}_Y \longrightarrow Y_k$ are continuous and φ_2, φ_3 commute with the action of $\mathbb{Z}_r$ on X_k and Y_k respectively, then φ determines uniquely a $\mathbb{Z}_n$-equivariant map $\psi : \bar{\Omega} \longrightarrow V$ such that the restriction of ψ to $\bar{\Omega}_0$ equals φ. This implies that it is sufficient to construct an appropriate Z_r-equivariant homotopy of f restricted to Ω_0.

From the definition of Y_k and Corollary 2.4 it follows that there exists a continuous map $\eta : [0, 1] \longrightarrow J_{\mathbb{Z}_r}(Y_k)$ such that $\eta(0) = C$ and $\eta(1) = 0$. Therefore f is homotopic to an equivariant map F_1, such that

$$f_1(w, x, y) = \big(A(w - v_0), B(x), 0\big), \quad w \in W_k,\ x \in X_k,\ y \in Y_k.$$

By applying the Product Formula, we may assume that $Y_k = \{0\}$. Let us also remark that if we denote by $\gamma_r := e^{\frac{2\pi i}{r}}$ the generator of $\mathbb{Z}_r$, then $\gamma_r x = -x$ for all $x \in X_k$. Therefore $J_{\mathbb{Z}_r}(X_k)$ has two connected components. Consequently it follows that it is sufficient to consider the following two cases

(i) $B = 0$;

(ii) $B(x_1, x_2, \ldots, x_p) = (2x_1, 0, \ldots, 0)$, where we assume that $\dim X_k = p$ and $(x_1, x_2, \ldots, x_p) \in X_k$.

In the case (i), it follows directly from Corollary 2.4 that

$$G\text{-}\operatorname{ind}_H(f, \Omega) = j(A) \text{ for } H = \mathbb{Z}_r$$
$$G\text{-}\operatorname{ind}_H(f, \Omega) = j(A) \text{ for } H \neq \mathbb{Z}_r.$$

In the case (ii), we replace the mapping $f_1(w, x) = \big(A(w - v_0), 2x_1, 0, \ldots, 0\big)$ by $f_2(w, x) := \big(A(w - v_0), g(x_1), 0, \ldots, 0\big)$, where $g(x_1) = x_1 - x_1\big(x_1 + \frac{1}{2}\big)\big(x_1 - \frac{1}{2}\big)$. The mapping $h(w, x, t) = (1 - t)f_1(w, x) + tf_2(w, x)$, $t \in [0, 1]$, defines a $\mathbb{Z}_r$-equivariant homotopy between f_1 and f_2 such that $h(w, x, t) \neq (w, x)$ for every $(w, x) \in \partial\Omega_0$. The set $\mathrm{Fix}(f_2)$ is the union of two orbits Gv_0 and Gv_1, where $v_1 = (v_0, \frac{1}{2}, 0, \ldots, 0)$, $G_{v_1} = \mathbb{Z}_{\frac{r}{2}}$. Therefore in this case we obtain

$$G\text{-}\operatorname{ind}_H(f, \Omega) = j(A) \text{ for } H = \mathbb{Z}_r;$$
$$G\text{-}\operatorname{ind}_H(f, \Omega) = -j(A) \text{ for } H = \mathbb{Z}_{\frac{r}{2}};$$
$$G\text{-}\operatorname{ind}_H(f, \Omega) = 0 \text{ for } H \neq \mathbb{Z}_r, \mathbb{Z}_{\frac{r}{2}}.$$

This completes the proof. ∎

https://doi.org/10.4153/CJM-1991-042-4 Published online by Cambridge University Press

4. **Period-doubling cascades of periodic points.** Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset and let $f : \bar{\Omega} \longrightarrow \mathbb{R}^n$ be a continuous map such that $f^k(x) \neq x$ for all $x \in \partial\Omega$ and all $k = 1, 2, \ldots$.

For $k \in \mathbb{N}$ we set

$$\Omega^k = \underbrace{\Omega \times \cdots \times \Omega}_{k\text{-fold}} \subset \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k\text{-fold}} = \mathbb{R}^{nk}$$

and

$$\hat{f}_k : \bar{\Omega}^k \longrightarrow \mathbb{R}^{nk} \text{ is given by}$$
$$\hat{f}_k(x_1, x_2, \ldots, x_k) = \big(f(x_k), f(x_1), \ldots, f(x_{k-1})\big).$$

We say that $\delta = \{a_1, a_2, \ldots, a_k\}$, $a_i \in \Omega$, is a *periodic orbit* for f if $f(a_i) = a_{i+1}$, for $i = 1, 2, \ldots, k-1$ and $f(a_k) = a_1$. Note that $\delta = \{a_1, a_2, \ldots, a_k\}$ is a periodic orbit for f if and only if $(a_1, a_2, \ldots, a_k) \in \Omega^k$ is a fixed point for $\hat{f}_k$. We say that the *least period* of δ equals m if $f^m(a_1) = a_1$ and $f^j(a_1) \neq a_1$ for $0 < j < m$. It is clear that in this case m divides k.

Let us remark that $\hat{f}_k$ is $\mathbb{Z}_k$-equivariant with respect to the action of $\mathbb{Z}_k$ defined on $V = \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ by

$$T_k(x_1, x_2, \ldots, x_k) = (x_k, x_1, \ldots, x_{k-1}), \quad x_i \in \mathbb{R}^n,$$

where T_k corresponds to the generator $\xi_k = \exp\left(\frac{2\pi i}{k}\right) \in \mathbb{Z}_k$.

DEFINITION 4.1. Suppose that $f : \bar{\Omega} \longrightarrow \mathbb{R}^n$ is a continuous map such that $f^k(x) \neq x$ for all $x \in \partial\Omega$ and all $k \in \mathbb{N}$. Then for every $k \in \mathbb{N}$ we define

$$c\text{-}\operatorname{ind}_k(f, \Omega) = c_k := \mathbb{Z}_k\text{-}\operatorname{ind}_{\mathbb{Z}_1}(\hat{f}_k; \Omega^k).$$

We call $c\text{-ind}(f, \Omega) := \{c_k\}_{k \in \mathbb{N}}$ the *c-index* of f in Ω.

It follows directly from the definition and the homotopy invariance of the equivariant fixed point index that the c-index satisfies the following homotopy invariance property.

PROPOSITION 4.2. *Suppose that $h : \bar{\Omega} \times [0, 1] \longrightarrow \mathbb{R}^n$ is a continuous map such that $h_t^k(x) \neq x$ for all $x \in \partial\Omega$, $k \in \mathbb{N}$ and $t \in [0, 1]$. Then*

$$c\text{-}\operatorname{ind}(h_0, \Omega) = c\text{-}\operatorname{ind}(h_1, \Omega).$$

We say that a periodic orbit $\delta = \{a_1, a_2, \ldots, a_k\}$ with the least period k is a *transverse periodic orbit* if $Df^k(a_1)$ does not have 1 as an eigenvalue.

Suppose now that $\delta = \{a_1, a_2, \ldots, a_k\}$ is a transverse periodic orbit with the least period k. Let E^u denote the linear subspace of $\mathbb{R}^n$ spanned by the generalized eigenspaces of $Df^k(a_1)$ corresponding to eigenvalues of absolute value greater than 1. Following the notation of J. Franks (cf. [4]) we put $\mu(\delta) = \dim E^u$. $\mu(\delta)$ is called the *Morse index* of

δ. We say that the orbit δ is *twisted* if $D^k f(a_1)\colon E^u \longrightarrow E^u$ reverses the orientation, and *untwisted* otherwise. We put

$$\tau(\delta) := \begin{cases} 1 & \text{if } \delta \text{ is untwisted} \\ -1 & \text{if } \delta \text{ is twisted.} \end{cases}$$

The above definition yields

$$\tau(\delta) = k\big(Df^k(a_1)\big), \tag{4.3}$$

and

$$(-1)^{\mu(\delta)} = j\big(Df^k(a_1)\big)k\big(Df^k(a_1)\big). \tag{4.4}$$

LEMMA 4.5. *Suppose that $\delta = \{a_1, a_2, \dots, a_k\}$ is a transverse periodic orbit with the least period k and $\Omega_0 \subset \Omega$ is an open subset such that $\delta \subset \Omega_0$ and there is no other periodic orbit in $\overline{\Omega_0}$ with the least period smaller or equal to 2^k. Then*

(i) $c\text{-ind}_k(f, \Omega_0) = (-1)^{\mu(\delta)}\tau(\delta)$;

(ii) $c\text{-ind}_{2k}(f, \Omega_0) = \frac{1}{2}(-1)^{\mu(\delta)}\big(1 - \tau(\delta)\big)$;

(iii) $c\text{-ind}_r(f, \Omega) = 0$ *for all* $0 < r < 2k$ *such that* $r \neq k$.

PROOF. By the definition of the c-index we have that $c\text{-ind}_k(f, \Omega_0) = \mathbb{Z}_k\text{-ind}_{\mathbb{Z}_1}(\hat{f}_k, \Omega_0^k)$. Let $b = (a_1, a_2, \dots, a_k) \in \mathbb{R}^{nk}_\lambda$. Then $G_b = \mathbb{Z}_1$ and, by Theorem 3.1, $Z_k\text{-ind}_{\mathbb{Z}_1}(\hat{f}_k, \Omega_0^k) = j\big(D\hat{f}_k(b)\big)$. Put $A_i = Df(a_i)$, $i = 1, \dots, k$. By Lemma 2.1, we obtain

$$\begin{aligned} jD\hat{f}_k(b) &= j(A_1, A_2 \cdots A_k) = j\big(Df^k(a_1)\big) \\ &= j\big(Df^k(a_1)\big) \cdot k\big(Df^k(a_1)\big) \cdot k\big(Df^k(a_1)\big) \\ &= \tau(\delta) \cdot (-1)^{\mu(\delta)}. \end{aligned}$$

This completes the proof of (i).

Now we proceed to the proof of the statements (ii) and (iii). Let $V = \mathbb{R}^{2kn}$, we define

$$W = \big\{ \big(x_1, \dots, x_k, x_{k+1}, \dots, x_{2k}\big) \in V : x_i = x_{i+k},\ i = 1, 2, \dots, k \big\}.$$

We put $c = (a_1, \dots, a_k, a_1, \dots, a_k) \in W$ and we consider the orthogonal decomposition $V = W \oplus W^\perp$. Then we have

$$D\hat{f}_{2k}(c) = \begin{bmatrix} \hat{A} & 0 \\ 0 & \hat{B} \end{bmatrix} : W \oplus W^\perp \longrightarrow W \oplus W^\perp.$$

By the definition of c-index we have:

$$c\text{-}\operatorname{ind}_{2k}(f, \Omega_0) = \mathbb{Z}_{2k}\text{-}\operatorname{ind}_{\mathbb{Z}_1}(\hat{f}_{2k}, \Omega_0^{2k}).$$

Since $G_c = \mathbb{Z}_2$, by Theorem 3.1, we have

$$\begin{aligned}\mathbb{Z}_{2k}\text{-}\mathrm{ind}_{\mathbb{Z}_1}\big(\hat{f}_{2k}, \Omega_0^{2k}\big) &= \frac{1}{2} j(\hat{A})\big(j(\hat{B}) - 1\big)\\ &= \frac{1}{2} j(\hat{A})[j(\hat{A} \oplus \hat{B}) j(\hat{A}) - 1]\\ &= \frac{1}{2}\big[j(\hat{A} \oplus \hat{B}) - j(\hat{A})\big]\\ &= \frac{1}{2}\Big[j\big(Df^{2k}(a_1)\big) - j\big(Df^{k}(a_1)\big)\Big]\\ &= \frac{1}{2}[(-1)^{\mu(\delta)} - (-1)^{\mu(\delta)} \cdot \tau(\delta)]\\ &= \frac{1}{2}(-1)^{\mu(\delta)}\big(1 - \tau(\delta)\big).\end{aligned}$$

The proof is complete. ∎

Following J. Franks (cf. [4]) we let $\mathrm{PO}(f, d)$ denote the set of all periodic orbits of f whose least period is $2^k d$ for some $k \geq 0$. Let D^n denote the unit disc in $\mathbb{R}^n$.

THEOREM 4.6 (J. FRANKS, CF. THEOREM A [4]). *Let $f\colon D^n \longrightarrow \mathrm{Int}\ D^n$ be a smooth map with only transverse periodic points. Suppose d is odd and no orbits in $\mathrm{PO}(f, d)$ have even Morse index. If $\delta \in \mathrm{PO}(f, d)$ has least period p, then for each $k \geq 0$ there is a twisted periodic orbit of f with the least period $2^k p$. The same conclusion is valid if $d > 1$ and no orbits in $\mathrm{PO}(f, d)$ have odd Morse index.*

PROOF. Suppose that no orbit in $\mathrm{PO}(f, d)$ has even Morse index and let ρ be an orbit with the least period r. Then $(-1)^{\mu(\rho)} = -1$ and the contribution of ρ to c-index is

$$-\tau(\rho) \text{ for } c\text{-}\mathrm{ind}_r(f, D^n)$$

and

$$-\frac{1}{2}\big(1 - \tau(\rho)\big) \text{ for } c\text{-}\mathrm{ind}_{2r}(f, D^n).$$

Let $u(f, r)$ (resp. $t(f, r)$) denote the number of untwisted (resp. twisted) periodic orbits of f with the least period r. Using the fact that every mapping of D^n into Int D^n is homotopic to a constant map and the homotopy invariance of c-index (Proposition 4.2), we obtain that

$$c\text{-}\mathrm{ind}_s(f, D^n) = \begin{cases} 1 & \text{for } s = 1\\ 0 & \text{for } s > 1\end{cases}$$

and thus

$$0 = c\text{-}\mathrm{ind}_{2r}(f, D^n) = t(f, 2r) - u(f, 2r) - t(f, r)$$

therefore

$$t(f, 2r) = t(f, r) + u(f, 2r) \tag{1}$$

and hence

$$t(f, 2r) \geq t(f, r). \tag{2}$$

Let q be the smallest positive integer such that there is a periodic orbit $\delta_0 \in \mathrm{PO}(f,d)$ with the least period q and $p = 2^{m_0}q$ for some $m_0 \geq 0$. By the assumption such a number q exists and $d \leqq q \leqq p$. It follows from (1) that $t(f,q) = u(f,q) > 0$, thus, by induction, (2) implies that $t(f,2^m q) > 0$ for every $m \geq 0$ and the first part of the theorem is proved.

The proof of the second part is analogous. ■

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