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A NOTE ON WEIGHTED MAXIMAL INEQUALITIES

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In this paper, we obtain some characterizations for the weighted weak type $(1, q)$ inequality to hold for the Hardy-Littlewood maximal operator in the case $0 < q < 1$; prove that there is no nontrivial weight satisfying one-weight weak type (p, q) inequalities when $0 < p \neq q < \infty$, and discuss the equivalence between the weak type (p, q) inequality and the strong type (p, q) inequality when $p \neq q$.

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1. Introduction

Let M be the Hardy-Littlewood maximal operator for locally integrable functions f on R^n defined by

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y)|dy,$$

where Q is a nondegenerate cube with sides parallel to the axes; $|Q|$ is the Lebesgue measure of Q, and the supremum is taken over all cubes Q containing x.

Let $w(x)$, $v(x)$ be weight functions, i.e. nonnegative measurable functions taking values in $[0, \infty]$, and $0 < p, q < \infty$. For a measurable set E, write $w(E) = \int w(x)dx$, and let χ_E denote its characteristic function.

During the past two decades, the two-weight weak type (p, q) inequality

$$(w(\{x : Mf(x) > \lambda\})\lambda^q)^{1/q} \leq C\left(\int_{R^n} |f(x)|^p v(x)dx\right)^{1/p} \tag{1}$$

and the two-weight strong type (p, q) inequality

$$\left(\int_{R^n} Mf(x)^q w(x)dx\right)^{1/q} \leq C\left(\int_{R^n} |f(x)|^p v(x)dx\right)^{1/p}. \tag{2}$$

have been intensely studied (see [3], [4], [8] or [1] and its references).

A pioneering and excellent contribution was made by B. Muckenhoupt [3], who gives a very precise and satisfactory answer to the one-weight inequalities, i.e. $w = v$ in (1) or (2), with $p = q$, that is to find those w for which either

$$\int_{R^n} Mf(x)^p w(x)dx \le C\left(\int_{R^n} |f(x)|^p w(x)dx\right)$$

or the corresponding weak type inequality hold. This answer is provided by what has become known as A_p weight theory.

Later on, E. T. Sawyer made another significant and important progress in the weighted norm inequalities for the Hardy-Littlewood maximal operator. That is, Sawyer characterizes the two-weight strong type (p, q) inequality when $1 < p \le q \le \infty$ (see [4]). Meanwhile, he obtains a characterization for the weak type (p, q) inequality (1) without the restriction $p \le q$ (see [6]).

Recently, further remarkable development has been made by I. E. Verbitsky. Both (1) and (2) are characterized in the case $1 < p \le \infty$ and $0 < q < p$ (see [8]).

This note is a supplementary to the weight theory for the Hardy-Littlewood maximal operator. Here we shall release some new properties of the weighted norm inequalities for M.

This paper is set out as following. The second section contains the characterizations of the weak type $(1, q)$ inequality for M with $0 < q < 1$. This case is not discussed in [8], and our proof is new. In the third section, we shall see that there is no nontrivial weight satisfying the one-weight weak type (p, q) inequality when $p \ne q$. This fact not only shows how delicate Muckenhoupt's Theorem is, but also implies that the one-weight strong type inequality is always equivalent to the corresponding weak type inequality, if $p \ne q$. Therefore we shall discuss whether the weak type (p, q) inequality is strictly weaker than the strong type (p, q) inequality in the cases $w \ne v$ and $p \ne q$ in the final section. Although an example has been given in [3] to show that they are different when $w \ne v$ and $p = q$, there are no further discussions for the cases $p \ne q$. For our discussion, we shall establish some necessary or sufficient conditions for the weak type inequalities, and these are interesting in their own right.

With a little trivial modification, the argument in [1, p. 388] shows that, for any $0 < p$, $q < \infty$, the weak type (p, q) inequality (1) implies that $v(x) > 0$ a.e. unless $w(x) = 0$ a.e., and $w(x)$ is locally integrable unless $v(x) = \infty$ a.e. Hence we shall impose these natural conditions on the pairs (w, v) of weight functions throughout this note. Also we shall keep the usual conventions for multiplication in $[0, \infty]$, namely $\infty \cdot t = t \cdot \infty = \infty$ for $0 < t \le \infty$, $0 \cdot \infty = \infty \cdot 0 = 0$, $1/\infty = 0$ and $1/0 = \infty$.

2. Weak type $(1, q)$ inequality

Theorem 1. *Let $0 < q < 1$. The following statements are equivalent.*

(i) *The weak type $(1, q)$ inequality (1) holds.*

(ii) *There exists a constant $B > 0$ such that*

$$w\Big(\bigcup_j Q_j\Big) \le B^q \Big(\sum_j \alpha(Q_j)|Q_j|\Big)^q \tag{3}$$

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holds for all sequences $\{Q_j\}$ *with pairwise disjoint cubes, where* $\alpha(Q) = \text{ess}\inf_{x\in Q}\{v(x)\}$.

(iii) *The function*

$$\Phi(x) = \sup_{t>0} \frac{w(Q(x,t))}{\alpha(Q(x,t))|Q(x,t)|}$$

is in $L^{r,\infty}(w)$, *i.e.*

$$\|\Phi\|_{r,\infty,w} = \sup_{\lambda>0} \lambda w(\{\Phi(x) > \lambda\})^{1/r} < \infty, \tag{4}$$

where $Q(x,t)$ *are the cubes centred at* x *with side length* t, $\alpha(Q)$ *is defined as above, and* $r = q/(1-q)$.

(iv) *The function*

$$\tilde{\Phi}(x) = \sup_{Q\ni x} \frac{w(Q)}{\alpha(Q)|Q|}$$

is in $L^{r,\infty}(w)$, *where the notation* $\alpha(Q)$ *and* r *are the same as above.*

Proof. (i)$\Rightarrow$(ii). For $\lambda > 1$ arbitrary, let $E_j = \{x \in Q_j : v(x) \le \lambda\alpha(Q_j)\}$, and $f = \sum |Q_j|\alpha(Q_j)\chi_{E_j}/v(E_j)$. It is obvious that $\cup Q_j \in \{Mf(x) > 1/\lambda\}$. Then it follows from (1) that

$$w(\cup Q_j) \le C^q\lambda^q\left(\int fv\right)^q = (\lambda C)^q\left(\sum \alpha(Q_j)|Q_j|\right)^q.$$

Then we get (2.1) with $B \le C$, since $\lambda > 1$ is arbitrary.

(ii)$\Rightarrow$(iii). Let $\Omega \subset \{\Phi(x) > \lambda\}$ be bounded. For every $x \in \Omega$, there exists a cube $Q(x,t)$ such that

$$\frac{w(Q(x,t))}{|Q(x,t)|\alpha(Q(x,t))} > \lambda. \tag{5}$$

By virtue of Besicovitch's covering lemma (see [2]), from $\{Q(x,t)\}_{x\in\Omega}$ one can select a sequence $\{Q_j\}$ such that

(i) $\Omega \subset \cup Q_j$;

(ii) the sequence $\{Q_j\}$can be split into ξ_n subsequences $\{Q_i^k\}_i$ $(k = 1, 2, \ldots, \xi_n)$ of disjoint cubes, where ξ_n depends only on the dimension n.

Fix k, write $\{Q_i^k\} = \{Q_i\}$. For any finite N, by use of (3) and (5) we have

$$w\left(\bigcup_1^N Q_i\right) \le B^q\left(\sum_1^N \alpha(Q_i)|Q_i|\right)^q \le B^q\lambda^{-q}\left(\sum_1^N w(Q_i)\right)^q.$$

This is $\sum_1^N w(Q_i) \le B^r \lambda^{-r}$, then (iii) with $\|\Phi\|_{r,\infty,w} \le \xi_n^{1/r} B$ follows.

The implication (iii)$\Rightarrow$(iv) follows from the fact that for any $Q(y,t)$ containing x we have

$$x \in Q(y,t) \subset Q(x,3t) \subset Q(y,9t).$$

We shall omit the details.

(iv)$\Rightarrow$(i). Given $\lambda > 0$, let $\Omega_\lambda = \{x : Mf(x) > \lambda\}$. Without loss of generality we may assume $\int |f| v dx = 1$. Write $w_1(x) = w(x)\chi_{\{t:\tilde{\Phi}(t) \le \lambda^{q/r}\}}(x)$, and $w_2(x) = w(x) - w_1(x)$. Then

$$w(\Omega_\lambda) = w_1(\Omega_\lambda) + w_2(\Omega_\lambda). \tag{6}$$

For the second term in (6) it follows from (iv) that

$$w_2(\Omega_\lambda) \le w(\{x : \tilde{\Phi}(x) > \lambda^{q/r}\}) \le \lambda^{-q} \|\tilde{\Phi}\|^r_{r,\infty,w}. \tag{7}$$

On the other hand, it is obvious that

$$\sup_Q \frac{w_1(Q)}{\alpha(Q)|Q|} \le \lambda^{q/r},$$

therefore, using the well-known weighted weak type (1, 1) inequality (e.g. see [**1**, p. 151 and p. 390]), we get

$$w_1(\Omega_\lambda) \le 12^n \lambda^{q/r} \lambda^{-1} \int |f(x)| v(x) dx = 12^n \lambda^{-q} \tag{8}$$

Substituting (7) and (8) into (6), we conclude the required assertion. Theorem 1 is proved.

Remark 1. The argument used above is available in the case $p > 1$ and $0 < q < p$ That is, this procedure leads to a new proof of Verbitsky's weak type inequalities for the Hardy-Littlewood maximal operator.

As an application of Theorem 1, we obtain the following interesting fact.

Proposition 1. *For any* $0 < q < 1$ *there is no weight function* $w(x)$ *satisfying the one-weight, weak type* $(1, q)$ *inequality* (1) *except* $w(x) = 0$ *a.e.*

Proof. Suppose $w(x)$ verifies the one-weight weak type $(1, q)$ inequality (1) with some $0 < q < 1$. Let $f = \chi_Q(x)$ in (1); we get $w(Q)^{1/q} \le Cw(Q)$, therefore $w(R^n) < \infty$. To obtain a contradiction, we shall prove that such $w(x)$ violates condition (iv) on Theorem 1.

Write $Q_t = Q(0, t)$, the cube centred at the origin with side length t (c.f. Theorem 1 (iii)), and $\alpha(\Omega) = ess\inf_{x\in\Omega}\{w(x)\}$. Noting that $\alpha(Q_t)|Q_t| \le w(Q_t) \le w(R^n)$, we have $\alpha(Q_t) \to 0$ when $t \to \infty$.

Suppose $\alpha(Q_t) > 0$ for all t. Then for any given Q_{t_0}, there exists $t > t_0$ such that $\alpha(Q_t) = \alpha(Q_t \backslash Q_{t_0})$, since $\alpha(Q_t) = \min\{\alpha(Q_{t_0}), \alpha(Q_t \backslash Q_{t_0})\}$ and $\alpha(Q_t) \to 0$.

Now we can choose a sequence $t_n \to \infty$ increasing and satisfying $\alpha(Q_{t_n}) = \alpha(Q_{t_n} \backslash Q_{t_{n-1}})$. Given $\epsilon > 0$, there exists an m such that $w(Q_{t_n} \backslash Q_{t_m}) < \epsilon/2$ for all $n > m$, since $w(R^n) < \infty$. Observing that

$$\alpha(Q_{t_n})|Q_{t_n}| \leq w(Q_{t_n} \backslash Q_{t_m}) + \alpha(Q_{t_n})|Q_{t_m}|,$$

and combining this with the fact that $\alpha(Q_{t_n}) \to 0$, we conclude $\alpha(Q_{t_n})|Q_{t_n}| \to 0 (n \to \infty)$. Then it follows that the associated $\tilde{\Phi}(x)$, defined in Theorem 1 (iv), is infinite for all x. Thus $w(\{\Phi(x) > \lambda\}) = w(R^n)$ for all $\lambda > 0$, and $\tilde{\Phi}(x)$ is not in $L^{r,\infty}(w)$ except $w(R^n) = 0$. This contradiction completes the proof of Proposition 1.

3. One-weight weak type inequality

Proposition 1 suggests the following statement.

Theorem 2. *There is no nontrivial weight satisfying the one-weight weak type* (p, q) *inequality* (1) *when* $p \neq q$ *and* $0 < p, q < \infty$.

While this fact, particularly in the case $p < q$, seems to be well known in the folklore, we have been unable to find any explicit proof, and so for the convenience of the reader we include a complete proof.

Proof. We divide the proof into the following two propositions. The statement for the case of $0 < p < q < \infty$ follows from Proposition 3 immediately. Meanwhile, suppose $0 < q < p < \infty$ and $p \neq 1$, it is easy to verify that the one-weight weak type (p, q) inequality implies $w \in L^1(dx)$ and it contradicts Proposition 2. This completes Theorem 2.

Proposition 2. *Let* $0 < q < p < \infty$ *and* $p \neq 1$. *If* $v \in L^1(dx)$, *then the weak type* (p, q) *inequality* (1) *holds only if* $w(x) = 0$ *a.e.*

Proof. Suppose v is integrable, and (w, v) verifies the weak type (p, q) inequality (1), then it is obvious that the inequality

$$\frac{w(Q)^{1/q}\sigma(Q)^{1/p'}}{|Q|} \leq C < \infty \tag{9}$$

holds for all cubes Q, where $\sigma = v^{1/(1-p)}$ and $p' = p/(p-1)$.

Let Q_t be the same as that in the proof of Proposition 1, and $B_t = Q_{rt} \backslash Q_t$. Adopting an idea in [7], for each B_t we define a median value associated with weight v by the expression

$$v_{B_t} = (s_1 s_2)^{1/2},$$

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where

$$s_1 = \sup\{s > 0 : |\{x \in B_t : v(x) < s\}| \le |B_t|/2\}$$

and

$$s_2 = \inf\{s > 0 : |\{x \in B_t : v(x) > s\}| \le |B_t|/2\}.$$

It is easy to see that

$$\int_{B_t} v(x)^\alpha dx/|B_t| \ge (v_{B_t})^\alpha/2 \tag{10}$$

for all real numbers α (c.f. [7, p. 6]).

Since $v \in L^1(dx)$, it follows from (10) that

$$v_{B_t}|B_t| \le 2\int_{B_t} v(x)dx \to 0 \quad (t \to \infty). \tag{11}$$

Given t_0, for every $t > t_0$ it follows from (9) and (10) that

$$\begin{aligned} C &\ge \frac{w(Q_{2t})^{1/q}\sigma(Q_{2t})^{1/p'}}{|Q_{2t}|} \\ &\ge \frac{w(Q_{t_0})^{1/q}(\int_{B_t} v(x)^{-(p'/p)}dx)^{1/p'}}{|Q_{2t}|} \\ &\ge \frac{w(Q_{t_0})^{1/q}}{4^{1+1/p'}(v_{B_t}|B_t|)^{1/p}}. \end{aligned}$$

Hence $w(Q_{t_0})$ must be 0 according to (11). The proof of Proposition 2 is completed.

Proposition 3. *Given $0 < p < q < \infty$. Suppose the weak type (p, q) inequality* (1) *holds for all measurable functions f and $\lambda > 0$, then for a.e. $x \in R^n$ either $w(x) = 0$ or $v(x) = \infty$.*

Proof. Let $0 < p < q < \infty$. Suppose, in order to derive a contradiction, $|\{x \in R^n : w(x) > 0, v(x) < \infty\}| > 0$. Then there exists an $M > 0$ such that

$$|E| = \left|\left\{x \in R_n : w(x) > \frac{1}{M}, v(x) < M\right|\right\} > 0.$$

Put $v_E(x) = v(x)\chi_E(x)$; then v_E is locally integrable. One can choose $x_0 \in E$ being a Lebesgue point of both w and v_E and also a point of density of E. Meanwhile we may assume $v(x_0) > 0$. Then for every positive integer n there exists a cube Q_n centred at

x_0 and satisfying $|Q_n| \to 0 (n \to \infty)$ and

$$1 - \frac{1}{n} < \frac{|E \cap Q_n|}{|Q_n|} \le 1. \tag{12}$$

Let $f(x) = \chi_{E \cap Q_n}(x)$. It follows from (1) and (12) that

$$\begin{aligned}\left(1 - \frac{1}{n}\right)(w(Q_n))^{1/q} &\le \left[\left(\frac{|E \cap Q_n|}{|Q_n|}\right)^q w\left(\left\{x : Mf(x) > \frac{|E \cap Q_n|}{|Q_n|}\right\}\right)\right]^{1/q} \\ &\le C\left(\int_{R^n} f(x)^p v(x)dx\right)^{1/p} = C(v_E(Q_n))^{1/p}.\end{aligned}$$

Therefore

$$\left(1 - \frac{1}{n}\right)\left(\frac{w(Q_n)}{|Q_n|}\right)^{1/q} \le C\left(\frac{v_E(Q_n)}{|Q_n|}\right)^{1/p} |Q_n|^{1/p - 1/q}. \tag{13}$$

Keep $x_0 \in E$ in mind. It follows from (13) and the Lebesgue Differentiation Theorem that

$$\left(\frac{1}{M}\right)^{1/q} < w(x_0)^{1/q} \le Cv(x_0)^{1/p} \cdot 0 = 0.$$

This contradiction completes the proof of Proposition 3.

Remark 2. *The idea used in Proposition 3 deduces the following statement, we shall omit the proof.*

Proposition 4. *Given* $0 < p < \infty$. *Suppose the weak type* (p, p) *inequality* (1) *holds for all measurable functions* f *and* $\lambda > 0$. *Then* $w(x) \le Bv(x)$ *a.e. Furthermore, for the best constants* B *and* C *in* (1), *we have* $B \le C^p$.

4. Discussion of equivalence between (1) and (2)

Theorem 2 indicates that the one-weight weak type (p, p) inequalities are always equivalent to the corresponding strong type inequalities, when $0 < p \neq q < \infty$. What would happen when $w \neq v$? That is, whether the weak type (p, q) inequalities with $p \neq q$ are strictly weaker than the strong type (p, q) inequalities in general? The answer is positive, and we shall give some examples. The example for the case $p < q$ might have been known but for which we can find no statement in literature. Therefore we include it for completeness.

In the following examples and further discussion we shall use the well known $A(p, q)$ condition $(1 \le p \le q < \infty)$:

$$A = \sup_Q \frac{w(Q)^{1/q}\sigma(Q)^{1/p'}}{|Q|} < \infty, \quad (p > 1)$$

$$A = \sup_Q \frac{w(Q)^{1/q}}{|Q| ess \inf_{x\in Q}\{v(x)\}} < \infty \quad (p = 1), \tag{14}$$

and the $S(p, q)$ condition $(1 < p \le q < \infty)$:

$$\left(\int_Q (M(\chi_Q\sigma)(x))^q w(x)dx\right)^{1/q} \le C(\sigma(Q))^{1/p} < \infty \tag{15}$$

for all cubes Q, where $p' = p/(p-1)$ and $\sigma = v^{-(1/p-1)}$. These two conditions characterize the weak type inequality (1) and the strong type inequality (2) respectively (see [**1**]).

Example 1. Suppose $1 \le p < q < \infty$. Consider the real line R^1. Let $w(x) = x^{q-1}\chi_{[0,\infty)}(x)$, and $v(x) = 1$ on $[-1, 0]$ and ∞ elsewhere. We shall verify that $(w, v) \in A(p, q)$, but the function $f(x) = \chi_{[-1,0]}(x)$ shows that they do not satisfy the strong type (p, q) inequality.

To prove $(w, v) \in A(p, q)$, we only need to verify the $A(p, q)$ condition for all intervals (a, b) with $-1 \le a < 0$ and $0 < b < \infty$. Then it is obvious that

$$\frac{w(a,b)^{1/q}\sigma(a,b)^{1/p'}}{b-a} = \frac{\left(\frac{1}{q}\right)^{1/q} b|a|^{1/p'}}{b-a} \le \left(\frac{1}{q}\right)^{1/q}.$$

This completes Example 1.

In order to create examples in the case $q < p$, we introduce a sufficient condition for the weak type inequalities with $q < p$, because Verbitsky's weak type condition concerns some sort of maximal function, and it is difficult to be calculated in practice.

Definition 1. Suppose $1 \le p$, $0 < q < p < \infty$ and (w, v) is a pair of weight functions. Let $1/r = 1/q - 1/p$. We say that $(w, v) \in B(p, q)$ if

$$\left\{\sum_j \left(\frac{w(Q_j)^{1/q}\sigma(Q_j)^{1/p'}}{|Q_j|}\right)^r\right\}^{1/r} \le B < \infty \quad (p > 1)$$

$$\left\{\sum_j \left(\frac{w(Q_j)^{1/q}}{|Q_j| ess \inf_{x\in Q_j} v(x)}\right)^r\right\}^{1/r} \le B < \infty \quad (p = 1) \tag{16}$$

hold for all sequences $\{Q_j\}$ of disjoint cubes, where $\sigma = v^{-1/(p-1)}$.

We give now a class of $B(p, q)$ functions.

Example 2. Given $p \geq 1$, suppose $(u, v) \in A(p, p)$ and $\Omega \subset R^n$ such that $u(\Omega) < \infty$. Write $w(x) = u(x)\chi_\Omega(x)$, then $(w, v) \in B(p, q)$ for all $0 < q < p$ with $B \leq A(u(\Omega)^{1/r})$, where A and B are the constants in (14) and (16) respectively.

Indeed, for any cube Q, it follows from the $A(p, p)$ condition (14) that

$$\left(\frac{w(Q)^{1/q}\sigma(Q)^{1/p'}}{|Q|}\right)^r \leq \left[(w(Q\cap\Omega)^{1/q-1/p}\left(\frac{u(Q)^{1/p}\sigma(Q)^{1/p'}}{|Q|}\right)\right]^r$$

$$\leq A^r w(Q\cap\Omega) \quad (p > 1)$$

$$\left(\frac{w(Q)^{1/q}}{|Q| ess\inf_{x\in Q} v(x)}\right)^r \leq \left[(w(Q\cap\Omega)^{1/q-1/p}\left(\frac{u(Q)^{1/q}}{|Q| ess\inf_{x\in Q} v(x)}\right)\right]^r$$

$$\leq A^r w(Q\cap\Omega) \quad (p = 1).$$

Therefore $(w, v) \in B(p, q)$ and $B \leq A(u(\Omega))^{1/r}$.

Theorem 3. *If* $(w, v) \in B(p, q)$, *then the weak type inequality* (1) *holds for all* $\lambda > 0$ *and measurable* f. *Moreover, for the best constants B in* (16) *and C in* (1), *we have*

$$C \leq \xi_n B,$$

where the absolute constant ξ_n *depends only on the dimension n and is associated with Besicovitch's Covering Lemma (see the proof of Theorem 1).*

Proof. Let $Q(x, t)$ be the cube Q centred at x with side length t. Let

$$M^c f(x) = \sup_{t>0} \frac{1}{|Q(x, t)|}\int_{Q(x,t)} |f(y)|dy$$

be the Hardy-Littlewood centred maximal operator. It is well known that

$$M^c f(x) \leq Mf(x) \leq 3^n M^c f(x).$$

From this observation, we shall prove (1) for $M^c f$ instead of Mf. Let $\Omega \subset \{x : M^c f(x) > \lambda\}$ be bounded. For every $x \in \Omega$, there exists a cube $Q(x, t)$ such that

$$\frac{1}{|Q(x, t)|}\int_{Q(x,t)} |f(y)|dy > \lambda. \tag{17}$$

Let $\{Q_i^k\}_i$ $(k = 1, 2, \ldots, \xi_n)$ be the sequences selected from $\{Q(x, r)\}_{x\in\Omega}$ according to

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Besicovitch's covering lemma (see the argument following (5)). Fix k. If $p > 1$, then the inequality (17) and Hölder's inequality show

$$\sum_i \lambda^q w(Q_i^k) \le \sum_i \left(\frac{1}{|Q_i^k|}\int_{Q_i^k} |f(y)|dy\right)^q w(Q_i^k) \le \sum_i \frac{w(Q_i^k)}{|Q_i^k|^q}\left(\int_{Q_i^k} |f(y)|^p v(y)dy\right)^{q/p} (\sigma(Q_i^k))^{q/p'}. \tag{18}$$

By another use of Hölder's inequality with exponent p/q and r/q, the right side of (18) is bounded by

$$\left(\sum_i \int_{Q_i^k} |f(y)|^p v(y)dy\right)^{q/p} \left(\sum_i \left(\frac{w(Q_i^k)\sigma(Q_i^k)^{q/p'}}{|Q_i^k|^q}\right)^{r/q}\right)^{q/r} \le A\left(\int_{R^n} |f(y)|^p v(y)dy\right)^{q/p}, \tag{19}$$

since $\{Q_i^k\}_i$ are disjoint and $(w, v) \in B(p, q)$.

On summing over k, (18) and (19) yield

$$\lambda^q w(\Omega) \le \xi_n A\left(\int_{R^n} |f(y)^p v(y)dy\right)^{q/p}.$$

Then we get the required assertion since Ω is arbitrary.

When $p = 1$, the previous argument is still available, if we replace $\sigma(Q_i^k)^{1/p'}$ in (18) and (19) by $\|v^{-1}\chi Q_i^k\|_\infty = 1/(ess\inf_{x\in Q_i^k} v(x))$. Theorem 3 is proved.

Now we give some pairs of weight functions which verify the weak type (p, q) inequality (1) with $q < p$, but not the corresponding strong type inequality (2).

Example 3. Choose $\Omega \subset R^n$ bounded. Set $w = 1$, and $v(x) = 1$ on Ω and ∞ elsewhere.

Note that

$$\left(\frac{w(Q)^{1/q}\sigma(Q)^{1/p'}}{|Q|}\right)^r = (\sigma(Q\cap\Omega))\left(\frac{w(Q)^{1/q}\sigma(Q\cap\Omega)^{1/p'}}{|Q|}\right)^r \quad (p \ne 1)$$

$$\left(\frac{w(Q)^{1/q}}{|Q| ess\inf_{x\in Q} v(x)}\right)^r \le |Q\cap\Omega|\left(\frac{1}{ess\inf_{x\in Q} v(x)}\right)^r \quad (p = 1). \tag{20}$$

Then we have $(w, v) \in B(p, q)$ for all $1 \le q < p$ and $0 < q < p = 1$. But it is obvious that the function $f(x) = \chi_\Omega(x)$ makes the left side of (2) infinite, when $q \le 1$. That is, this (w, v) demonstrates that the weak type inequality (1) is different from the strong type inequality (2), when $q = 1 < p$ or $0 < q < p = 1$.

Example 4. Given $1 < q < p < \infty$. Let $w(x) = x^{q-1}(-\log x)^{q-1}$ on $(0, 1/2]$ and 0 elsewhere, and $v(x) = x^{p-1}(-\log x)^{2p-2}$ on $(0, 1/2]$ and ∞ elsewhere.

Observe that $\sigma(x) = 1/(x(-\log x)^2)$ on $(0, 1/2]$ and 0 elsewhere. It has been verified (see [**3**, p. 218]) that the pair of weight functions $U(x) = 1/(x(-\log x)^2)$ and $V(x) = 1/(x(-\log x))$ satisfies the $A(1, 1)$ condition on $(0, 1/2]$, therefore the $A(q', q')$ condition on $(0, 1/2]$. By use of our notation, this is

$$\sup\left\{(a, b) \subseteq (0, 1/2] : \frac{w(a, b)^{1/q}\sigma(a, b)^{1/q'}}{b-a}\right\} = C < \infty. \tag{21}$$

Combining (21) and (20), we have $(w, v) \in B(p, q)$. But the function $f(x) = \sigma(x)$ violates the strong type (p, q) inequality.

Example 5. Let $0 < q < 1 < p < \infty$. Set $w(x) = |x|^{q-1}(-\log|x|)^{q-1}$ on $[-1/2, 0)$ and 0 elsewhere, and $v(x) = x^{p-1}(-\log x)^{2p-2}$ on $(0, 1/2]$ and ∞ elsewhere.

For every sequence $\{(a_j, b_j)\}_j$ of pairwise disjoint intervals on R^1, only at most one among them, which contains 0, makes a contribution to the left sum in (16). Write this interval, if it exists, by (a, b). Furthermore we may assume $-1/2 \le a < 0$ and $0 < b \le 1/2$. Observe that $(-\log x)^{q-1} \le (-\log|a|)^{q-1}$ on $(0, |a|]$. Then we have

$$\begin{aligned}\frac{w(a, b)^{1/q}\sigma(a, b)^{1/p'}}{b-a} &= \frac{(\int_0^{|a|} x^{q-1}(-\log x)^{q-1}dx)^{1/q}(\int_0^b x^{-1}(-\log x)^{-2}dx)^{1/p'}}{b+|a|} \\ &\le C(-\log|a|)^{1/q'}(-\log b)^{-1/p'} \le C(\log 2)^{-1/r}.\end{aligned} \tag{22}$$

It follows from (22) and (20) that $(w, v) \in B(p, q)$. But the function $f(x) = \sigma(x)$ shows that the strong type (p, q) inequality does not hold.

Remark 3. The following theorem shows that our $B(p, q)$ conditions are just between the strong type inequality and the weak type inequality.

Theorem 4. *Suppose* $1 \le p < \infty$, $0 < q < p$ *and* (w, v) *as a pair of weight functions. If there exists a constant C such that the strong type inequality* (2) *holds for all measurable f, then* $(w, v) \in B(p, q)$. *Furthermore, for the best constants C and B in* (2) *and* (16), *we have* $B \le C$.

Proof. Let $\{Q_j\}$ be a sequence of disjoint cubes. When $p > 1$, the proof concerns testing (2) by a function $f(x) = \sum_j f_j(x)$ with

$$f_j(x) = \left(\frac{w(Q_j)^{1/q}\sigma(Q_j)^{1/q'}}{|Q_j|}\right)^{r/p}\sigma(x)\chi_{Q_j}(x),$$

where $1/q' = 0$ if $q = 1$. We shall omit the details.

Suppose $p=1$. Let $a_j = ess\ \inf_{x \in Q_j} v(x)$. For arbitrary $\eta > 1$, set $E_j = \{x \in Q : v(x) < \eta\alpha_j\}$, then $|E_j| > 0$ and $v(E_j) < \infty$. Choose a nondegenerate cube $R_j \subset Q_j$ satisfying $|E_j \cap R_j| > 0$, then $v(E_j \cap R_j) > 0$. Set

$$f_j(x) = \left(\frac{w(Q_j)^{1/q}}{|Q_j|\alpha_j}\right)^r \frac{\chi_{E_j \cap R_j}(x)}{v(E_j \cap R_j)}$$

and $f(x) = \sum_{j=1}^m f_j(x)$. Observe that

$$\left(\frac{\int_{Q_j} f(x)dx}{|Q_j|}\right)^q w(Q_j) = \left[\left(\frac{w(Q_j)^{1/q}}{|Q_j|\alpha_j}\right)^r \frac{|E_j \cap R_j|}{v(E_j \cap R_j)|Q_j|}\right]^q w(Q_j)$$

$$\geq \left(\frac{1}{\eta}\right)^q \left(\frac{w(Q_j)^{1/q}}{|Q_j|\alpha_j}\right)^r.$$

It follows from the strong type inequality (2) that

$$\sum_{j=1}^m \left(\frac{w(Q_j)^{1/q}}{|Q_j|\alpha_j}\right)^r = \int_{R^n} f(x)v(x)dx$$

$$\geq \left(\int_{R^n} (Mf(x))^q w(x)dx\right)^{1/q} \geq \left(\sum_{j=1}^m \left(\frac{\int_{Q_j} f(x)dx}{|Q_j|}\right)^q w(Q_j)\right)^{1/q}$$

$$\geq \frac{1}{\eta}\left(\left(\frac{w(Q_j)^{1/q}}{|Q_j|\alpha_j}\right)^r\right)^{1/q}.$$

Thus we obtain $(w, v) \in B(p, q)$ with $B \leq C$, since $\eta > 1$ is arbitrary. The proof of Theorem 4 is completed.

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