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ON ALGEBRAIC DIFFERENTIAL EQUATIONS FOR THE GAMMA FUNCTION AND L-FUNCTIONS IN THE EXTENDED SELBERG CLASS

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Abstract

This paper concerns the problem of algebraic differential independence of the gamma function and $\mathcal{L}$-functions in the extended Selberg class. We prove that the two kinds of functions cannot satisfy a class of algebraic differential equations with functional coefficients that are linked to the zeros of the $\mathcal{L}$-function in a domain $D := \{z : 0 < \mathrm{Re}\, z < \sigma_0\}$ for a positive constant σ_0.

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1. Introduction and main result

This paper is devoted to studying the question of whether or not the gamma function Γ and some other functions, for example, the Riemann zeta function, ζ, or $\mathcal{L}$-functions in the extended Selberg class, are algebraically independent. The functions Γ and ζ have played a very important role in the development of mathematics. The $\mathcal{L}$-functions are Dirichlet series with ζ as the prototype and are important objects in number theory. Selberg introduced a class of Dirichlet series $\mathcal{L}(s) = \sum_{n=1}^{\infty} a(n)/n^s$ of a complex variable $s = \sigma + it$ with $a(1) = 1$ (now called the Selberg class of $\mathcal{L}$-functions), satisfying the following axioms (see, for example, [12]).

(1) *Dirichlet series*: for $\sigma > 1$, the series representation of $\mathcal{L}(s)$ is absolutely convergent.

(2) *Analytic continuation*: for some integer $m > 0$, the function $(s-1)^m\mathcal{L}(s)$ is entire and of finite order.

(3) *Functional equation*: $\mathcal{L}(s)$ satisfies a functional equation of the form

$$\phi(s) = \omega\overline{\phi(1-\overline{s})},$$

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where

$$\phi(s) = Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \mu_j) L(s),$$

with $Q > 0$, $\lambda_j > 0$, $\mathrm{Re}\,\mu_j \geq 0$ and $|\omega| = 1$.

(4) *Ramanujan hypothesis*: for any $\varepsilon > 0$, we have $a(n) \ll n^{\varepsilon}$.

(5) *Euler product*: for σ sufficiently large,

$$\log \mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}, \quad s = \sigma + it,$$

where $b(n) = 0$ unless n is a positive power of a prime, and $b(n) \ll n^{\theta}$ for some $\theta < 1/2$.

The Selberg class includes the Riemann zeta function ζ and, essentially, those Dirichlet series where one might expect the analogue of the Riemann hypothesis. Throughout the paper, all $\mathcal{L}$-functions are assumed to be functions from the extended Selberg class of Dirichlet series satisfying the axioms (1)–(3) (see, for example, [4]).

The degree $d_{\mathcal{L}}$ of such an $\mathcal{L}$-function $\mathcal{L}$ is defined to be $d_{\mathcal{L}} = 2\sum_{j=1}^{K} \lambda_j$. Let $\lambda = \prod_{j=1}^{K} \lambda_j^{2\lambda_j}$. Then the famous Riemann–von Mangoldt formula (see, for example, [12, page 145]) for $\mathcal{L}$-functions can be stated as

$$N^0_{\mathcal{L}}(T) = \frac{d_{\mathcal{L}}}{\pi} T \log \frac{T}{e} + \frac{T}{\pi} \log(\lambda Q^2) + O(\log T),$$

where $N^0_{\mathcal{L}}(T)$ denotes the number of zeros of the $\mathcal{L}$-function in the region $|\mathrm{Im}\, s| \leq T$ and $0 < \mathrm{Re}\, s < \sigma_0$, where σ_0 is a large enough positive constant.

A classical theorem of Hölder [3] states that the gamma function does not satisfy any nontrivial algebraic differential equation whose coefficients are rational functions in $\mathbb{C}$. Bank and Kaufman [1] generalised the theorem to coefficients being meromorphic functions which grow more slowly than Γ. In his famous list of 23 problems, Hilbert [2] stated the analogous problem for ζ and Mordykhai-Boltovskoi [10] and Ostrowski [11] proved that ζ does not satisfy any nontrivial algebraic differential equation whose coefficients are rational functions. It is natural to study whether the functions Γ and ζ are related by any nontrivial algebraic differential equation. In 2007, Markus [9] showed that Γ and the composition function $\zeta(\sin(2\pi z))$ are differentially independent over $\mathbb{C}$. That is to say, $\zeta(\sin(2\pi z))$ cannot satisfy any nontrivial algebraic differential equations whose coefficients are polynomials of Γ and its derivatives. In the same paper, Markus conjectured that ζ does not satisfy any nontrivial algebraic differential equations whose coefficients are polynomials of Γ and its derivatives. Li and Ye [6, 7] partially solved the conjecture by proving that ζ is not a solution of any nontrivial algebraic differential equation, even allowing coefficients that are polynomials in Γ, Γ' and Γ''. Very recently, Li and Ye [8] made use of the well-known fact that ζ has an infinity of zeros on the critical line to show that $P(z, \Gamma, \Gamma', \ldots, \Gamma^{(n)}, \zeta) \not\equiv 0$ in $\mathbb{C}$ for any nontrivial distinguished polynomial P whose

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coefficients can be allowed to be any polynomials of ζ over $\mathbb{C}$, over the ring of polynomials or, more generally, over the class L_δ (see Definition 1.1). Before giving their theorem, we need the following definitions.

Definition 1.1. Let L_δ be the set of the zero function and all nonzero functions f from $\mathbb{C}$ to $\mathbb{C} \cup \infty$ with the following property: there exist infinitely many zeros $z_n = \frac{1}{2} + iy_n$ of ζ on the critical line L such that $\{|f(z_n)|\}$ has a positive lower bound and $|f(z_n)|e^{-\delta|y_n|} = o(1)$, as $n \to \infty$, where $\delta < \pi/2$ is a positive number.

Example 1.2. In [8], the authors pointed out some important classes of functions belonging to L_δ.

(1a) For any $\delta > 0$, L_δ contains the ring of all polynomials in $\mathbb{C}$.
(1b) L_δ may contain entire functions or meromorphic functions of finite or infinite order, such as the functions e^z and e^{e^z}.
(1c) The functions in L_δ are not even required to be continuous or meromorphic. For example, given any complex function f with $f(0) \neq 0, \infty$ (even not continuous), the composite $f(\zeta(z))$ belongs to L_δ.

Definition 1.3. Let $I = (i_0, i_1, \ldots, i_n)$ be a multi-index with $|I| = i_0 + i_1 + \cdots + i_n$. A polynomial in the variables $u_0, u_1, \ldots, u_n$ with functional coefficients in a set $\mathcal{S}$ can always be written as

$$P(u_0, u_1, \ldots, u_n) = \sum_{I \in \Lambda} a_I(z) u_0^{i_0} u_1^{i_1} \cdots u_n^{i_n},$$

where the coefficients a_I are functions in $\mathcal{S}$ and Λ is an index set. We call P a distinguished polynomial in $u_0, u_1, \ldots, u_n$ with coefficients in $\mathcal{S}$, or simply an $\mathcal{S}$-distinguished polynomial, if the index set Λ has the property that $|I_i| \neq |I_j|$ for distinct indices I_i, I_j in Λ.

Li and Ye [8] obtained the following result.

Theorem 1.4. *Let $P(z, u_0, u_1, \ldots, u_n, v) = \sum_{k=0}^{m} P_k(z, u_0, u_1, \ldots, u_n) v^k$, where the P_k, not all identically zero, are L_δ-distinguished polynomials. Then, for $z \in \mathbb{C}$,*

$$P(z, \Gamma, \Gamma', \ldots, \Gamma^{(n)}, \zeta) \not\equiv 0.$$

In the same paper, Li and Ye [8] mentioned that it would be natural to extend the above study to $\mathcal{L}$-functions. However, it remains an open problem of whether a general $\mathcal{L}$-function in the Selberg class has infinitely many zeros on the critical line $L = \{z \in \mathbb{C} : \operatorname{Re} z = 1/2\}$. In this paper, we study the problem of whether Γ and $\mathcal{L}$-functions in the extended Selberg class are related by any nontrivial distinguished polynomial. In fact, we consider a more general class of functions.

Definition 1.5. Let $\mathcal{F}$ be a set of functions with infinitely many zeros in the domain $D = \{z : |\operatorname{Re} z| < \sigma_0\}$, where σ_0 is a positive constant. For $F \in \mathcal{F}$, let $\mathfrak{L}_{\delta,F}$ be the set of the zero function and all nonzero functions f from $\mathbb{C}$ to $\mathbb{C} \cup \infty$ with the following

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property: there exist infinitely many zeros $z_n = x_n + iy_n$ of F in the domain D such that $\{|f(z_n)|\}$ has a positive lower bound and

$$|f(z_n)|e^{-\delta|y_n|} = o(1),$$

as $n \to \infty$, where $\delta < \pi/2$ is a positive number.

Example 1.6. The set $\mathcal{F}$ contains some important classes of functions.

(2a) The Riemann–von Mangoldt formula implies that $\mathcal{L}$-functions have infinitely many zeros in the domain D for a positive σ_0. Therefore, the Riemann zeta function ζ and the $\mathcal{L}$-functions in the extended Selberg class belong to $\mathcal{F}$.

(2b) $\mathcal{F}$ contains the difference shifts $\zeta(z+\eta)$ and $\mathcal{L}(z+\eta)$ of the ζ function and $\mathcal{L}$-functions, respectively, where η is a fixed constant.

(2c) The functions in $\mathcal{F}$ are not even required to be continuous or meromorphic. For example, suppose that the function g satisfies $g(0) = 0$. Then the compound functions $g(\zeta)$ and $g(\mathcal{L})$ belong to $\mathcal{F}$.

(2d) A periodic function with period it may belong to $\mathcal{F}$, where t is a real constant; for example, functions such as $e^z - 1, \sin(iz)$ and so on.

We will prove the following result.

Theorem 1.7. *Let $F \in \mathcal{F}$. Let $P(z, u_0, u_1, \ldots, u_n, v) = \sum_{k=0}^{m} P_k(z, u_0, u_1, \ldots, u_n,)v^k$, where P_k, not all identically zero, are $\mathfrak{L}_{\delta,F}$-distinguished polynomials. Then, for $z \in \mathbb{C}$,*

$$P(z, \Gamma, \Gamma', \ldots, \Gamma^{(n)}, F) \not\equiv 0.$$

A nontrivial polynomial $P(z, u, v)$ can be written as $P(z, u, v) = \sum_{k=0}^{m} P_k(z, u)v^k$, where the $P_k(z, u)$ are distinguished polynomials in one argument u. Thus, the following corollary is an immediate consequence of Theorem 1.7.

Corollary 1.8. *The derivatives, $\Gamma^{(n)}$ $(n \geq 0)$, of the Γ function and a function F in $\mathcal{F}$ are algebraically independent over $\mathfrak{L}_{\delta,F}$. In particular, $P(z, \Gamma^{(n)}, F) \not\equiv 0$ in $\mathbb{C}$ for any nontrivial polynomial $P(z, u, v)$ whose coefficients are polynomial functions.*

Proof of Theorem 1.7. The proof is based on the ideas of Li and Ye in [6–8]. The polynomial $P(z, u_0, \ldots, u_n, v)$ may be written in the form

$$\begin{aligned} P(z, u_0, \ldots, u_n, v) = v^m \sum_{I\in\Lambda_m} a_{m,I} u_0^{i_0} u_1^{i_1} \cdots u_n^{i_n} + v^{m-1} \sum_{I\in\Lambda_{m-1}} a_{m-1,I} u_0^{i_0} u_1^{i_1} \cdots u_n^{i_n} + \cdots \\ + v \sum_{I\in\Lambda_1} a_{1,I} u_0^{i_0} u_1^{i_1} \cdots u_n^{i_n} + \sum_{I\in\Lambda_0} a_{0,I} u_0^{i_0} u_1^{i_1} \cdots u_n^{i_n}, \end{aligned}$$

where m is the highest power of v in the polynomial P and the Λ_j are index sets. The coefficients $a_{i,I}$ are either identically zero in $\mathbb{C}$ or nonzero functions in $\mathfrak{L}_{\delta,F}$.

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Suppose, contrary to the statement of the theorem, that $\Gamma, \Gamma', \ldots, \Gamma^{(n)}, F$ satisfy $P(z, u_0, \ldots, u_n, v) = 0$ in $\mathbb{C}$. That is,

$$\begin{aligned} P(z, \Gamma, \Gamma', \ldots, \Gamma^{(n)}, F) = F^m & \sum_{I\in\Lambda_m} a_{m,I}\Gamma^{i_0}(\Gamma')^{i_1}\cdots(\Gamma^{(n)})^{i_n} \\ & + F^{m-1}\sum_{I\in\Lambda_{m-1}} a_{m-1,I}\Gamma^{i_0}(\Gamma')^{i_1}\cdots(\Gamma^{(n)})^{i_n} + \cdots \\ & + F\sum_{I\in\Lambda_1} a_{1,I}\Gamma^{i_0}(\Gamma')^{i_1}\cdots(\Gamma^{(n)})^{i_n} + \sum_{I\in\Lambda_0} a_{0,I}\Gamma^{i_0}(\Gamma')^{i_1}\cdots(\Gamma^{(n)})^{i_n} \\ = 0. & \end{aligned} \tag{1.1}$$

We will prove that all the coefficients $a_{i,I}$ in (1.1) are identically zero in $\mathbb{C}$ for all possible i, I. This, of course, contradicts the assumption of the theorem.

Firstly, we will show that $a_{0,I} \equiv 0$ in the last sum of (1.1). Suppose that the index set Λ_0 contains t indices $I_1, I_2, \ldots, I_t$, which we arrange so that $|I_1| < |I_2| < \cdots < |I_t|$. The last sum of (1.1) can be written as

$$\sum_{j=1}^{t} a_{0,I_j}\Gamma^{i_0}(\Gamma')^{i_1}\cdots(\Gamma^{(n)})^{i_n}. \tag{1.2}$$

Suppose that $a_{0,I_1} \not\equiv 0$. We will derive a contradiction below. Notice that there exist infinitely many zeros $z_l = x_l + iy_l$ $(l = 1, 2, \ldots)$ of F on D satisfying

$$|a_{0,I_j}(z_l)| > \kappa, \quad |a_{0,I_j}(z_l)|e^{-\delta|y_l|} = o(1),$$

where κ is a fixed positive constant. By taking a subsequence if necessary, we may assume that $|y_l| \to \infty$ and $x_l \to x_0$ as $l \to \infty$. Without loss of generality, we assume that $|x_l| < |x_0| + 1/2$ for all l.

Note that $F(z_l) = 0$. From (1.1) and (1.2),

$$\sum_{j=1}^{t} a_{0,I_j}\Gamma^{i_0}(\Gamma')^{i_1}\cdots(\Gamma^{(n)})^{i_n}(z_l) = 0. \tag{1.3}$$

Dividing both sides of the above equality by Γ^{I_1} gives

$$\sum_{j=1}^{t} a_{0,I_j}\Big(\frac{\Gamma'}{\Gamma}\Big)^{i_1}\cdots\Big(\frac{\Gamma^{(n)}}{\Gamma}\Big)^{i_n}\Gamma^{|I_j|-|I_1|}(z_l) = 0.$$

Instead of the formula $\Gamma(1/2 + iy) = (1 + o(1))e^{-\pi|y|/2}\sqrt{2\pi}$ in [8], we will make use of Stirling's formula (see, for example, [13, page 151]),

$$\Gamma(z) = \sqrt{2\pi}e^{-z}e^{(z-1/2)\log z}\Big[1 + O\Big(\frac{1}{z}\Big)\Big] \quad \text{as } |z| \to \infty,\ z \in \mathbb{C}\backslash\{z : |\arg z - \pi| \le \varepsilon\}.$$

We adapt the method of Kilbas and Saigo [5] to prove that

$$|\Gamma(x + iy)| \le M(1 + o(1))e^{\pi|y|/2}|y|^{|x_0|+1}$$

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for $|x| < |x_0| + 1/2$ and $|y| \to \infty$, where $M = \sqrt{2\pi} e^{|x_0|+1/2}$ is a constant. From Stirling's formula, for $|x| < |x_0| + 1/2$, as $|y| \to \infty$,

$$\begin{aligned} |\Gamma(x+iy)| &= \sqrt{2\pi}\, |e^{-x-iy} e^{(x-1/2+iy)[\log|x+iy| + i\arg(x+iy)]}| \Big[1 + O\Big(\frac{1}{x+iy}\Big)\Big] \\ &= \sqrt{2\pi} e^{-x} e^{(x-1/2)\log|x+iy|} e^{-y\arg(x+iy)} \Big[1 + O\Big(\frac{1}{x+iy}\Big)\Big]. \end{aligned} \tag{1.4}$$

Estimating more precisely the terms in (1.4), when $|y| \to \infty$,

$$\begin{aligned} e^{(x-1/2)\log|x+iy|} &= e^{(x-1/2)\log\sqrt{x^2+y^2}} = e^{(x-1/2)\log|y|} e^{(1/2)(x-1/2)\log(1+x^2/y^2)} \\ &= |y|^{x-1/2}\Big(1 + \frac{x^2}{y^2}\Big)^{(1/2)(x-1/2)} \le |y|^{x-1/2}\Big(1 + \frac{(|x_0|+1/2)^2}{y^2}\Big)^{(|x_0|+1)/2} \\ &= |y|^{x-1/2}\Big(1 + O\Big(\frac{1}{|y|^2}\Big)\Big) \le |y|^{|x_0|+1}\Big(1 + O\Big(\frac{1}{|y|^2}\Big)\Big). \end{aligned}$$

Further, since $\arg(x+iy) \to \pi/2$ (respectively $-\pi/2$) as $y \to +\infty$ (respectively $-\infty$) uniformly for all $|x| \le |x_0| + 1/2$, the term $e^{-y\arg(x+iy)}$ can be represented as

$$e^{-y\arg(x+iy)} = e^{-\pi y/2} e^{-y[\arg(x+iy)\mathrm{sgn}(y) - \pi/2]}.$$

From L'Hôpital's rule, for all $|x| \le |x_0| + 1/2$,

$$e^{-y[\arg(x+iy)\mathrm{sgn}(y)-\pi/2]} = 1 + O\Big(\frac{1}{|y|}\Big) \quad \text{uniformly as } |y| \to \infty$$

and, hence,

$$e^{-y\arg(x+iy)} = e^{-\pi y/2}\Big[1 + O\Big(\frac{1}{|y|}\Big)\Big] \quad \text{uniformly as } |y| \to \infty.$$

Furthermore, $|x+iy| \sim |y|$ as $|y| \to \infty$ for $|x| < |x_0| + 1/2$, and so $O(1/|x+iy|) = O(1/|y|)$ as $|y| \to \infty$. The above discussion gives the desired result.

Let us turn back to the proof of Theorem 1.7. As in [8], for any positive integer q,

$$\Gamma^{(q)}(z) = (1 + o(1))(\log z)^q \Gamma(z)$$

uniformly for all $z \in \mathbb{C}\backslash\{z : |\arg z - \pi| \le \varepsilon\}$. For each $j \ge 2$ and as $l \to \infty$,

$$\begin{aligned} &\Big| a_{0,I_j}\Big(\frac{\Gamma'}{\Gamma}\Big)^{i_1} \cdots \Big(\frac{\Gamma^{(n)}}{\Gamma}\Big)^{i_n} \Gamma^{|I_j|-|I_1|}(z_l)\Big| \\ &\quad \le (1+o(1))|a_{0,I_j}(z_l)|(\log|z_l|)^{i_1+2i_2+\cdots+ni_n} M^{|I_j|-|I_i|}|y_l|^{(|x_0|+1)(|I_j|-|I_i|)} e^{-(|I_j|-|I_i|)\pi|y_l|/2} \\ &\quad \le (1+o(1))|a_{0,I_j}(z_l)|e^{-\delta|y_l|}(\log|z_l|)^{i_1+2i_2+\cdots+ni_n} M^{|I_j|-|I_i|}|y_l|^{(|x_0|+1)(|I_j|-|I_i|)} e^{(\delta-\pi/2)|y_l|} \\ &\quad \le o(1)(\log|z_l|)^{i_1+2i_2+\cdots+ni_n} M^{|I_j|-|I_i|}|y_l|^{(|x_0|+1)(|I_j|-|I_i|)} e^{(\delta-\pi/2)|y_l|} \to 0 \end{aligned}$$

because $\delta < \pi/2$. Thus, taking $l \to \infty$,

$$\Big| a_{0,I_1}\Big(\frac{\Gamma'}{\Gamma}\Big)^{i_1} \cdots \Big(\frac{\Gamma^{(n)}}{\Gamma}\Big)^{i_n}(z_l)\Big| \to 0.$$

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On the other hand, as $l \to \infty$,

$$\left| a_{0,I_1}\left(\frac{\Gamma'}{\Gamma}\right)^{i_1}\cdots\left(\frac{\Gamma^{(n)}}{\Gamma}\right)^{i_n}(z_l)\right| = |a_{0,I_1}(z_l)|(\log|z_l|)^{i_1+2i_2+\cdots+ni_n} \to \infty.$$

This is a contradiction. So $a_{0,I_1} \equiv 0$.

Now, since a_{0,I_1} is identically zero, the expression (1.3) reduces to

$$\sum_{j=2}^{t} a_{0,I_j}\Gamma^{i_0}(\Gamma')^{i_1}\cdots(\Gamma^{(n)})^{i_n}(z_l) = 0.$$

This is in the same form as (1.3), except that now j starts from 2. The same argument as above shows that a_{0,I_2} is identically zero. Repeating this argument shows that all the coefficients a_{0,I_j} are identically zero. Therefore, (1.1) becomes

$$P(z,\Gamma,\Gamma',\ldots,\Gamma^{(n)},F) = F^{m-1}\sum_{I\in\Lambda_m} a_{m,I}\Gamma^{i_0}(\Gamma')^{i_1}\cdots(\Gamma^{(n)})^{i_n} + \cdots + \sum_{I\in\Lambda_1} a_{1,I}\Gamma^{i_0}(\Gamma')^{i_1}\cdots(\Gamma^{(n)})^{i_n},$$

which is of the same form as (1.1), except that the highest power of F is now $m-1$. Repeating the argument shows all the coefficients of the polynomial are identically zero, which contradicts the assumption.

This completes the proof of Theorem 1.7. □

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