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## On the Solution of Non-linear Partial Differential Equations of the Second Order.

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§ 1. It is proposed to discuss in this paper partial differential equations involving two independent variables $x$ and $y$, and a dependent variable $z$. The method of reduction which is explained can be applied to certain equations involving more than two independent variables, but such application is subject to too many restrictions to be of much general utility.

The usual notation will be employed, viz. :—

$$\frac{\delta z}{\delta x}=p, \quad \frac{\delta z}{\delta y}=q, \quad \frac{\delta^2 z}{\delta x^2}=r, \quad \frac{\delta^2 z}{\delta x \delta y}=s, \quad \frac{\delta^2 z}{\delta y^2}=t.$$

No general method for the solution of partial differential equations of an order higher than the first has yet been obtained. In the case of equations of the second order little towards a general method has been attempted except in the case of the equation, linear in $r$, $s$, $t$,

$$\mathrm{R}r+\mathrm{S}s+\mathrm{T}t=\mathrm{V} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

or the slightly more general form

$$\mathrm{R}r+\mathrm{S}s+\mathrm{T}t+\mathrm{U}(rt-s^2)=\mathrm{V} \quad \ldots \quad \ldots \quad (2$$

in which R, S, T, U, V are functions of $x$, $y$, $z$, $p$ and $q$.

Methods of more or less generality have been given for the solution of these equations by Monge, Ampere, Laplace, etc. ; and these methods comprehend almost all our knowledge of the subject.

It is the principal object of this paper to show how the solution of equations of the second order which are not linear in $r$, $s$, and $t$, may be made to depend on the solution of an equation of the form (1) : in many important cases, however, the method will lead to the required solution through equations of the first order only.

§ 2. The general method may be said to consist in the substitution for $z$ of a new dependent variable $z'$, which is a function of $x$, $y$, $z$, $p$ and $q$; and in the formation of a new equation in $x$, $y$, $z'$ and its differential co-efficients $p'$, $q'$, $r'$, $s'$ and $t'$, from the original equation and another of the third order derived from it by differentiation, by the elimination of $z$, $p$, $q$, $r$, $s$ and $t$ in virtue of the relations between the two sets of differential co-efficients implied in the choice of $z'$. The equation thus formed will in general be of the second order and linear in $r$, $s$, $t$, but in particular cases it may be of the first order only. From its solution, obtained in any way, the value of $z$ will easily follow by integration.

The nature of the method, however, will be best elucidated by the full consideration of one of the simpler, though remarkably general, cases to which we proceed.

§ 3. Let $$f(x, y, z, p, q, r, s, t) = 0 \quad \ldots \quad \ldots \quad (3)$$

be any non-linear equation of which the solution is required.

Take $z' = q$ for a new dependent variable, write $\delta z'/\delta x = p'$, $\delta^2 z/\delta x^2 = r'$, etc.; then, noting that $s = p'$, $q = t'$, (3) may be written

$$f(x, y, z, p, z', r, p', q') = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Differentiate (4) with respect to $y$, writing $f_x$, $f_p$, $f_r$ etc., for the partial differential co-efficients of $f$ with respect to $x$, $p$, $r$, etc., and we get

$$f_y + z' f_z + p' f_p + q' f_{z'} + r' f_r + s' f_{p'} + t' f_{q'} = 0 \quad \ldots \quad \ldots \quad (5)$$

The next step is to eliminate $z$, $p$, and $r$ between the equations (4) and (5): it is clear that in general this cannot be done, and thus a limitation is imposed on the form of the equation (3), which so far has been unrestricted. We see at once that for the elimination to be possible, $r$, $p$ and $z$ must enter into $f$ only as a group $r + \mathrm{P}p + \mathrm{Z}z$, where P and Z are functions of $x$ only.

Then the result of the elimination of this group, between (4) and (5) will be of the form

$$\mathrm{R}r' + \mathrm{S}s' + \mathrm{T}t' = \mathrm{V} \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

where R, S, T and V are functions of $x$. $y$. $z'$, $p'$ and $q'$ only.

If $z'$, that is $q$, can be obtained by the solution of (6), we have at once $z = \int q\,dy + \mathrm{X}$, where the quantity X, so far an arbitrary

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function of $x$, may be easily determined by substitution in the original equation.

The most general form of equation which can be thus solved by taking $q$ as a new dependent variable is as we have seen

$$F(x, y, q, r+Pp+Zz, s, t)=0 \quad \ldots \quad \ldots \quad (7)$$

where P and Z are functions of $x$ only: but since many equations which are not not of this form may be easily reduced to it by a change of independent variables, the limitation is less than might at first sight appear. Thus, had $p$ been taken as the new dependent variable, we should have obtained for the reducible form

$$F(x, y, p, r, s, t+Qq+Zz)=0 \quad \ldots \quad \ldots \quad (8)$$

where Q and Z are functions of $y$ only: but this result might be regarded as derived from (7) by an interchange of the independent variables.

§ 4. Another interesting case may be briefly considered.

Take $z'=px+qy-z$ for the new dependent variable.

Hence

$$\left.\begin{aligned} \frac{\delta z'}{\delta x}=p'=rx+sy, \quad \frac{\delta z}{\delta y}=q'=sx+ty \\ \frac{\delta^2 z}{\delta x^2}=r'=x\frac{\delta r}{\delta x}+y\frac{\delta r}{\delta y}+r \\ \frac{\delta^2 z}{\delta x \delta y}=s'=x\frac{\delta s}{\delta x}+y\frac{\delta s}{\delta y}+s \\ \frac{\delta^2 z}{\delta y^2}=t'=x\frac{\delta t}{\delta x}+y\frac{\delta t}{\delta y}+t \end{aligned}\right\} \quad \ldots \quad \ldots \quad (9)$$

Suppose now that the solution of an equation of the form

$$s=f(x, y, px+qy-z, rx+sy, sx+ty) \quad \ldots \quad \ldots \quad (10)$$

is required. In terms of the new variable this may be written

$$s=f(x, y, z', p', q') \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

Operate on both sides of this with $x\dfrac{\delta}{\delta x}+y\dfrac{\delta}{\delta y}+1$; then by (9) we get

$$s'=f+xf_x+yf_y+(p'x+q'y)f_{z'}+r'xf_{p'}+t'yf_{q'} \quad \ldots \quad \ldots \quad (12)$$

an equation, linear in $r'$ $s'$ and $t'$, to give $z'$.

When $z'$ has been obtained by the solution of (12), (11) will give $s$, from which $z$ may be obtained by direct integration, except as to an arbitrary function of $x$, and another of $y$, which will, however, be at once determined by substitution in the equation $px+qy-z=z'$.

If the equation proposed had been of the general form (3), we should have operated on both sides with $x\frac{\delta}{\delta x}+y\frac{\delta}{\delta y}$ and, by means of the equation thus obtained and the relations (9), eliminated the quantities $r$, $s$, $t$ and $z$: if the original equation was adapted for this method of solution, $p$ and $q$ would be found to have been eliminated also. It may be noted that all equations which contain $p$, $q$ and $z$, only in the group $px+qy-z$, are reducible by this method.

The more general case where

$$z'=apx+bqy+cz+hp+kq+\phi(x, y) \quad \ldots \quad \ldots \quad (13)$$

is taken as the new dependent variable is almost equally simple and interesting. It may be noted that it gives

$$\left.\begin{aligned} p'&=(a+c)p+(h+ax)r+(k+by)s+\phi_x \\ q'&=(b+c)q+(h+ax)s+(k+by)t+\phi_y \end{aligned}\right\} \quad \ldots \quad (14)$$

§ 5. The most general substitution may now be considered, in which the new dependent variable may be any function $x$, $y$, $z$, $p$ and $q$.

Take then $$z'=\phi(x, y, z, p, q) \quad \ldots \quad \ldots \quad (15)$$

which gives

$$\left.\begin{aligned} p'&=\phi_x+pq_z+r\phi_p+s\phi_q \\ q'&=\phi_y+q\phi_z+s\phi_p+t\phi_q \end{aligned}\right\} \quad \ldots \quad \ldots \quad (16)$$

and therefore

$$\left.\begin{aligned} r'&=\phi_q\frac{\delta r}{\delta x}+\phi_q\frac{\delta r}{\delta y}+\mathrm{R} \\ s'&=\phi_p\frac{\delta s}{\delta x}+\phi_q\frac{\delta s}{\delta y}+\mathrm{S} \\ t'&=\phi_p\frac{\delta t}{\delta x}+\phi_q\frac{\delta t}{\delta y}+\mathrm{T} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (17)$$

where R, S, T are functions of $x$, $y$, $z$, $p$, $q$, $r$, $s$ and $t$, linear in the three last.

Let the equation to be solved be of the general form (3).

Operate on it with $\phi_p \frac{\delta}{\delta x} + \phi_q \frac{\delta}{\delta y}$ : then, by means of the resulting equation and equations (15) to (17), $z$, $r$, $s$, and $t$ can be eliminated from the original equation (3). If on performing this elimination the resulting equation is found to be free of $p$ and $q$ also, we shall have obtained for the determination of $z'$ an equation of the type (1). When, by the solution of this equation, $z'$ has been determined, $z$ may be got by the solntion of (15), and the substitution of the resulting value of $z$ in the original equation (3), to determine the precise value of the arbitrary function which would appear in the general solution of (15).

§ 6. One important case of the general method deserves special notice. It may be found that the given equation is of such a form that the quantities $z$, $p$, $q$, $r$, $s$, $t$ may be eliminated from it by the relations (15) and (16) without proceeding to a differentiation and the use of the relations (17). In this case the equation proposed reduces to the form

$$f(x, y, z', p', q') = 0, \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

an equation of the first order only to determine $z'$, and thence $z$ by the solution of (15), which is of the first order also.

To illustrate the power of this method one example may be taken. Let the equation proposed for solution be

$$x^2r + 2xys + y^2t = px + qy - z + \mathrm{F}(rx + sy, sx + ty) \quad \ldots \quad (19)$$

Take $z' = px + qy - z$, then by (9), (19) may be written

$$p'x + q'y = z' + \mathrm{F}(p', q') \quad \ldots \quad \ldots \quad (20)$$

an equation belonging to a well known type. The general solution of this equation is got at once by eliminating $\theta$ between the equations

$$\left.\begin{aligned} z' &= x\theta + y\Theta - \mathrm{F}(\theta, \Theta) \\ 0 &= x + y\frac{d\Theta}{d\theta} - \frac{d}{d\theta}\mathrm{F}(\theta, \Theta) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (21)$$

where $\Theta$ is an arbitrary function of $\theta$. Then, if $z' = \psi(x, y)$ is the result of this elimination, $z$ is given by

$$px + qy - z = \psi(x. y)$$

and therefore

$$z = \sqrt{(x^2+y^2)}f\left(\frac{y}{x}\right) + \sqrt{(x^2+y^2)}\int \frac{\psi(x, y)}{x^2+y^2} \cdot d. \sqrt{(x^2+y^2)} \ldots \quad (22)$$

where in the integration the ratio $y/x$ is to be taken as a constant.

§ 7. It is hardly necessary to remark that the practical application of these methods will often be much facilitated by suitably changing the independent variables. It would be superfluous to give examples of the simplification of equations by the application of the ordinary case of transformation, where the variables are changed from $x$ and $y$ to certain other definite functions of $x$ and $y$; but there is a peculiar case, sometimes called reciprocation, which deserves special notice owing to its intimate connection with the methods of this paper. The method of reciprocation is well known, and calls therefore for but brief description: it is due, I believe, to Legendre, though De Morgan's name is most commonly associated with it. It consists essentially in the change of the independent variables from $x$ and $y$ to $p$ and $q$, say $x_1$ and $y_1$ for clearness, and the simultaneous change of the dependent variable from $z$ to $px+qy-z$, say $z_1$. It follows at once that

$$\left.\begin{aligned} p_1 &= \frac{\delta z_1}{\delta x_1} = x \\ q_1 &= \frac{\delta z_1}{\delta y_1} = y \\ r_1 &= \frac{\delta^2 z_1}{\delta x_1^2} = t/J \\ s_1 &= \frac{\delta^2 z_1}{\delta x_1 \delta y_1} = -s/J \\ t_1 &= \frac{\delta^2 z_1}{\delta y_1^2} = r/J \\ J_1 &= r_1 t_1 - s_1^2 = 1/J \end{aligned}\right\} \quad \ldots \quad \ldots \quad (23)$$

where $\quad J = rt - s^2.$

If the given equation, after transformation by (23) can be solved so as to give $z_1$ in terms of $x_1$ and $y_1$, then $z$ can be obtained in terms of $x$ and $y$, by the elimination of $x_1$ and $y_1$ between the equations

$$\left.\begin{aligned} z_1 = xx_1 + yy_1 - z \\ \frac{\delta z_1}{\delta x_1} = x, \quad \frac{\delta z_1}{\delta y_1} = y \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (24).$$

Two instances will suffice to show the use of reciprocation in connection with the method developed in this paper. For the first, consider the general equation, not explicitly containing the independent variables,

$$f(z, p, q, r, s, t) = 0 \qquad \dots \qquad \dots \qquad (25).$$

If this is transformed by means of the relations (23), it becomes

$$f\{p_1x_1 + q_1y_1 - z_1,\ x_1,\ y_1,\ t_1/\mathrm{J}_1,\ -s_1/\mathrm{J}_1,\ r_1/\mathrm{J}_1\} = 0 \qquad \dots \qquad (26),$$

which is an equation of the general type (10), and is therefore reducible by the substitution considered in § 4.

In this instance the efficacy of reciprocation in adapting equations for easy reduction by our method is well shown; the next is designed to show its use after reduction. An equation considered by Legendre gives a simple example, namely

$$r = f(s, t) \qquad \dots \qquad \dots \qquad \dots \qquad (27).$$

This may be regarded as a special case of the general type (7): hence proceeding as in § 3, differentiating with respect to $y$, we get

$$r' = s'f_s + t'f_t \qquad \dots \qquad \dots \qquad \dots \qquad (28)$$

but $$s = p',\ t = q'$$

therefore (28) may be written

$$r' = s'f_{p'} + t'f_{q'} \qquad \dots \qquad \dots \qquad \dots \qquad (29)$$

which is the reduced equation linear in $r'$, $s'$, and $t'$.

This equation we might proceed to solve by any of the methods, such as Mouge's, adapted for the treatment of linear equations, but it is clearly specially suited for reciprocation. The new independent variables will be $p'$ and $q'$, that is $s$ and $t$, and the new dependent variable $p'x + q'y - z'$, that is $sx + ty - q$, say $v$; therefore by (23), (29) becomes

$$\frac{\delta^2 v}{\delta s^2} f_t - \frac{\delta^2 v}{\delta s \delta t} f_s - \frac{\delta^2 v}{\delta t^2} = 0, \qquad \dots \qquad \dots \qquad (30)$$

a result obtained by Legendre by a special method (*Lacroix*, tom. II., p. 631).

It is easy to combine the methods of change of dependent variable and of reciprocation into one transformation of dependent and independent variables, and, in fact, it was this combination that I first considered, in seeking to extend Legendre's mode of solving (27). I am convinced, however, that in practice it is simpler to use the methods successively, as exemplified in the two preceding examples, just as in the ordinary method of changing variables we usually arrive at the ultimate transformation through many intermediate steps.

It need hardly be added that instead of this complete reciprocation, Routh's method of partial reciprocation or modification may be used in many cases with advantage.

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## The Plane Triangle ABC: Intimoscribed Circles, etc.

By R. E. Anderson, M.A.

§ I. *On an infinite series of Triad Circles derived from the inscribed circle. Determination of a direct relation between* r *and the three radii of the* n^th^ *triad.*

Each of the first triad touches two sides of ABC and the inscribed circle: generally, each circle of the $m^{th}$ triad touches two sides of ABC and also touches one of the circles of the $(m-1)^{th}$ triad.

Fig. 27.

In the diagram only one member of each successive triad is indicated—viz., the circles forming a diminishing series between the in-circle and B.

Let ON $= OY = r$

$$\left.\begin{array}{l} O'N' = r_2' \\ O''N'' = r_2'' \\ O'''N''' = r_3''' \\ \text{etc., etc.} \end{array}\right\} \quad \therefore \quad \left\{\begin{array}{lllll} r_2' & r_3' & r_1' & \text{are radii of} & \text{first triad} \\ r_2'' & r_3'' & r_1'' & \ldots\ldots & \text{second} \\ r_2''' & r_3''' & r_1''' & \ldots\ldots & \text{third} \\ & \text{etc.} & & \ldots\ldots & \text{etc.} \end{array}\right.$$

$$\text{and} \quad \rho_2 \quad \rho_3 \quad \rho_1 \quad \ldots\ldots \quad n^{th} \text{ triad,}$$

where in every case the suffix 2 has reference to B, 3 to C, and 1 to A—a rule which also holds for the subsequent sections.

Now $$ON = O'N' + OO'\sin\tfrac{1}{2}B,$$

$\therefore$
$$r = r_2' + (r + r_2')\sin\tfrac{1}{2}B$$
$$r_2' = r_2'' + (r_2' + r_2'')\sin\tfrac{1}{2}B$$
$$\text{etc., for ever}$$