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LEFT IDEALS AND 0-PRIMITIVITY IN MATRIX NEAR-RINGS

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Maximal left ideals in matrix rings were studied by Stone [**10**]. Similar results are not necessarily valid in the general near-ring case and one of the objectives of this paper is to study these differences. Furthermore, although much is known about 2-primitivity in general matrix near-rings (Van der Walt [**11**]), quite the opposite is true for 0-primitivity and the other objective of this paper is to present some results on 0-primitivity in matrix near-rings in certain restricted cases.

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0. Introduction

Matrix near-rings were introduced in 1984 by Meldrum and Van der Walt [**5**]. Since then several papers ([**8, 12, 11, 13, 6, 2, 3**]) and theses ([**7, 1**]) were devoted to matrix near-rings and as this field of study is still very immature, many more publications are expected to follow.

The purpose of this paper is to study 0-primitivity in matrix near-rings. A good survey on 2-primitivity in matrix near-rings over any zero-symmetric near-ring has been done by Van der Walt [**11**]. Some results on 0-primitivity are also contained in Abbasi, Meldrum and Meyer [**2**], but only for a very special class of near-rings, namely the weakly distributive d.g. near-rings. Because of some complexities, we could only manage to obtain certain results in restricted cases such as finite near-rings, or near-rings having the DCCR. It seems that a considerable amount of work still needs to be done to obtain similar results in the general zero-symmetric case.

The first section merely introduces some of the basic definitions, results and techniques in matrix near-rings which will be used in this paper. For more details the interested reader should consult [**5**], [**7**] and [**1**]. Section 2 deals with maximal left ideals in matrix near-rings and the connections they have (or do not have) with maximal left ideals in the base near-ring. A counter-example is given to show that the near-ring case does not always necessarily follow the same pattern as in the ring case.

The final section is devoted, for the greater part, to finite zero-symmetric near-rings and 0-primitivity. It becomes clear from this section that in order to have a reasonable understanding of modules over matrix near-rings, it is useful if one knows whether or not such modules can be embedded into a direct sum of finitely many copies of the additive group of the base near-ring.

1. Definitions and preliminaries

Throughout this paper R will denote a zero-symmetric right near-ring. Unless otherwise specified, R will also be assumed to contain an identity element. For any natural number n, R^n denotes the direct sum of n copies of the (not necessarily abelian) group $(R, +)$. From now on, n will always denote an arbitrary but fixed natural number. We write the elements of R^n in the form $\langle r_1, r_2, \ldots, r_n \rangle$ where $r_i \in R$ for all $i = 1, 2, \ldots, n$. In particular, $\bar{0} := \langle 0, 0, \ldots, 0 \rangle$ where the symbol $:=$ means "is defined by". The functions $\pi_i : R^n \to R$ and $\iota_i : R \to R^n$ will denote the ith co-ordinate projection and injection functions respectively.

Definition 1.1. The *near-ring of* $n \times n$*-matrices over* R, denoted by $\mathbb{M}_n(R)$, is defined to be the subnear-ring of $M(R^n)$, generated by the set of functions $\{f^r_{ij} : R^n \to R^n \mid r \in R, 1 \leqq i, j \leqq n\}$ where $f^r_{ij}\langle r_1, r_2, \ldots, r_n \rangle := \langle s_1, s_2, \ldots, s_n \rangle$ with $s_i = rr_j$ and $s_k = 0$ if $k \neq i$. The elements of $\mathbb{M}_n(R)$ will be referred to as $n \times n$*-matrices over* R.

It follows that $\mathbb{M}_n(R)$ is a zero-symmetric right near-ring with identity $I = f^1_{11} + f^1_{22} + \cdots + f^1_{nn}$. If R happens to be a ring, then $\mathbb{M}_n(R)$ is isomorphic to the usual full matrix ring over R. Sometimes, because of typographical problems, we write f^r_{ij} as $[r; i, j]$.

It happens frequently that we need to know a specific way in which a matrix is compiled in terms of the functions f^r_{ij}. We therefore introduce the following concept.

Definition 1.2. Let S denote the free semigroup over the alphabet of symbols $\{f^r_{ij} \mid r \in R, 1 \leqq i, j \leqq n\} \cup \{(,), +\}$. The set $\mathbb{E}_n(R)$ of *matrix expressions* is the subset of S, recursively defined by the following rules:

(a) $f^r_{ij} \in \mathbb{E}_n(R)$ for all $r \in R$ and $1 \leqq i, j \leqq n$;

(b) if $X, Y \in \mathbb{E}_n(R)$, then $X + Y \in \mathbb{E}_n(R)$;

(c) if $X, Y \in \mathbb{E}_n(R)$, then $(X)(Y) \in \mathbb{E}_n(R)$;

(d) nothing else is in $\mathbb{E}_n(R)$.

Clearly, each element of $\mathbb{E}_n(R)$ represents a matrix in $\mathbb{M}_n(R)$. On the other hand, each matrix has infinitely many expressions representing it. For example, the expressions X and $X + f^0_{11}$, for any $X \in \mathbb{E}_n(R)$, represent the same matrix. Also, when we write down an expression, we usually discard any redundant parentheses without disturbing unambiguity. For example, the expression $(f^r_{11})(f^s_{11} + f^t_{12})$ would be written (mostly) as $f^r_{11}(f^s_{11} + f^t_{12})$. If $X \in \mathbb{E}_n(R)$, $m(X)$ will denote the matrix in $\mathbb{M}_n(R)$ represented by X.

Definition 1.3. Let $X \in \mathbb{E}_n(R)$ and $U \in \mathbb{M}_n(R)$. The *length*, $l(X)$, of X is defined to be the number of f^r_{ij} in it. The *weight*, $w(U)$, of U is defined to be the length of an expression Y of minimal length such that $m(Y) = U$.

One way to relate (two-sided) ideals in $\mathbb{M}_n(R)$ to those in R, is by means of Noetherian quotients: If A is an ideal of R then we define A^* to be the ideal

$(A^n:R^n)=\{U\in\mathbb{M}_n(R)\,|\,U\alpha\in A^n$ for all $\alpha\in R^n\}$, where A^n is the set $\{\langle a_1,a_2,\ldots,a_n\rangle\in R^n\,|\, a_i\in A, i=1,2,\ldots,n\}$. As a matter of fact, if L is a left ideal of R, then $(L^n:R^n)$ is also a two-sided ideal of $\mathbb{M}_n(R)$ and is equal to A^*, where A is the largest two-sided ideal contained in L. We prove this in the following lemma.

Lemma 1.4. *If L is a left ideal of R and A is the largest two-sided ideal of R contained in L, then $L^*=A^*$.*

Proof. Since $A\subseteq L$, $A^*\subseteq L^*$. Now suppose $U\notin A^*$. Then $\pi_i U\alpha\notin A$ for some i, $1\leqq i\leqq n$, and $\alpha\in R^n$. Therefore, $(\pi_i U\alpha)r\notin L$ for some $r\in R$. But $(\pi_i U\alpha)r=\pi_i U(\alpha r)$, where αr means multiply each co-ordinate of α by r on the right. (See Meyer [7, Lemma 2.1.]) Hence, $U\notin L^*$. □

Note that there are other (non-equivalent) ways of relating ideals in $\mathbb{M}_n(R)$ with those of R, resulting in a vital difference between ring matrices and near-ring matrices, namely that there is in general not a bijection between the set of ideals of R and the set of ideals of $\mathbb{M}_n(R)$—even if R is a finite weakly distributive d.g. near-ring with identity. More details are contained in [12], [7] and [3].

Given an R-module G, one can ask the question: If G^n is the direct sum of n copies of G, how can we define an $\mathbb{M}_n(R)$-module structure on G^n? We need the following definition.

Definition 1.5. Let G be an R-module. Then G is said to be *locally monogenic* if for any finite subset H of G there exists $g\in G$ such that $H\subseteq Rg$.

This idea was introduced by Van der Walt [11] and he used the term *connected*. Clearly, if G is finite, then G is locally monogenic if and only if G is monogenic.

Now, if G is a locally monogenic R-module, then we define the action of $\mathbb{M}_n(R)$ on G^n as follows: Let $U\in\mathbb{M}_n(R)$ and $\langle g_1,g_2,\ldots,g_n\rangle\in G^n$. Then, by Definition 1.5, there are $g\in G$ and $r_1,r_2,\ldots,r_n\in R$ such that $g_i=r_ig, i=1,2,\ldots,n$. Let $U\langle g_1,g_2,\ldots,g_n\rangle:=(U\langle r_1,r_2,\ldots,r_n\rangle)g$, where $\langle s_1,s_2,\ldots,s_n\rangle g:=\langle s_1g,s_2g,\ldots,s_ng\rangle$ for any $\langle s_1,s_2,\ldots,s_n\rangle\in R^n$. It is shown in Van der Walt [11] that this action is well-defined and it makes G^n an $\mathbb{M}_n(R)$-module.

Also note that R^n can be viewed as an $\mathbb{M}_n(R)$-module in a natural way, since $\mathbb{M}_n(R)$ is a subnear-ring of $M(R^n)$. If L is a left ideal of R, then the action of R on R/L, namely $r(s+L):=rs+L$ for all $r,s\in R$, can be used to define $(R/L)^n$ as an $\mathbb{M}_n(R)$-module as follows: Let $U\in\mathbb{M}_n(R)$ and $\langle r_1+L,r_2+L,\ldots,r_n+L\rangle\in(R/L)^n$ and suppose $U\langle r_1,r_2,\ldots,r_n\rangle=\langle t_1,t_2,\ldots,t_n\rangle$. Then $U\langle r_1+L,r_2+L,\ldots,r_n+L\rangle:=\langle t_1+L,t_2+L,\ldots,t_n+L\rangle$. An easy induction argument on the weight of matrices in $\mathbb{M}_n(R)$ shows that this action is well-defined and turns $(R/L)^n$ into an $\mathbb{M}_n(R)$-module. Furthermore, L^n is an $\mathbb{M}_n(R)$-ideal of R^n and we can therefore also consider R^n/L^n as an $\mathbb{M}_n(R)$-module in the usual way. The following lemma states that there is virtually no difference between the $\mathbb{M}_n(R)$-modules $(R/L)^n$ and R^n/L^n.

Lemma 1.6. (Meyer [7]). *If L is a left ideal of R then the $\mathbb{M}_n(R)$-modules R^n/L^n and $(R/L)^n$ are $\mathbb{M}_n(R)$-isomorphic.*

We now state some results which will be useful later on:

Theorem 1.7. (Van der Walt [11]). *If A is a two-sided ideal of R, then $\mathbb{M}_n(R/A)\cong \mathbb{M}_n(R)/A^*$ as near-rings.*

Lemma 1.8. (Van der Walt [11]). *Let G be an R-module and $v\in\{0,2\}$. If R is v-primitive on G, then $\mathbb{M}_n(R)$ is v-primitive on G^n.*

Lemma 1.9. (Van der Walt [11]). *Let $v\in\{0,2\}$. If A is a v-primitive ideal of R, then A^* is a v-primitive ideal of $\mathbb{M}_n(R)$.*

Lemma 1.10. (Van der Walt [11]). *Suppose Γ is a type 2 $\mathbb{M}_n(R)$-module and let $\mathscr{A}:=\mathrm{Ann}_{\mathbb{M}_n(R)}\Gamma$. Then there is an ideal A of R such that $\mathscr{A}=A^*$.*

Lemma 1.11. (Meyer [7]). *An ideal $\mathscr{A}$ of $\mathbb{M}_n(R)$ is 2-primitive if and only if $\mathscr{A}=A^*$ for some 2-primitive ideal A of R.*

Lemma 1.12. (Van der Walt [11]). *If the $\mathbb{M}_n(R)$-module Γ is monogenic, then $\Gamma\cong G^n$ as additive groups for an appropriate R-module G.*

The R-module G of Lemma 1.12 is defined as $f_{11}^1\Gamma=\{f_{11}^1\gamma \mid \gamma\in\Gamma\}$ where $r(f_{11}^1\gamma):=f_{11}^1(f_{11}^r\gamma)$ for all $r\in R$ and $f_{11}^1\gamma\in f_{11}^1\Gamma$.

2. Maximal left ideals

Whilst studying 0-primitivity in matrix near-rings, it would be very handy to have some nice relationships between maximal left ideals of R and those of $\mathbb{M}_n(R)$. Stone [10] characterises all maximal left ideals in matrix rings as follows:

Theorem 2.1. (Stone [10]). *If L is a maximal left ideal of a ring R and $\alpha\in R^n\backslash L^n$, then $(L^n:\alpha):=\{U\in\mathbb{M}_n(R) \mid U\alpha\in L^n\}$ is a maximal left ideal of $\mathbb{M}_n(R)$. Moreover, every maximal left ideal of $\mathbb{M}_n(R)$ is of this form.*

Unfortunately, in the near-ring case the situation is not the same. We will show that under certain conditions, $(L^n:\alpha)$ is indeed a maximal left ideal of $\mathbb{M}_n(R)$, where R is a zero-symmetric near-ring with identity (Theorem 2.4), but not under the general conditions of Theorem 2.1 (Example 2.5). Also, we will prove that for some "well-behaved" near-rings R, the maximal left ideals of $\mathbb{M}_n(R)$ are indeed of the form $(L^n:\alpha)$ as described in Theorem 2.1 (Theorem 2.11). Before we can prove these theorems, we need the following lemmas.

Lemma 2.2. *Let $A=\{s_1,s_2,\dots,s_n\}$ be a finite subset of R and let S be the R-subgroup of R generated by A. Furthermore, let T be the subset of R recursively defined by the following rules:*

(*a*) *$s_i\in T$ for all $i=1,2,\dots,n$;*

(*b*) *if $t_1,t_2\in T$, then $t_1-t_2\in T$;*

(*c*) *if $t\in T$ and $r\in R$, then $rt\in T$;*

(*d*) *nothing else is in T.*

Then $S=T$.

Proof. First of all, that T is an R-subgroup of R, follows directly from (b) and (c). Since $A\subseteq T$ (by (a)), we must have $S\subseteq T$.

Before showing that $T\subseteq S$, let us introduce some more terminology. Each $t\in T$ is always constructed (in many ways) by a finite number of applications of the rules (a)–(c), starting always with rule (a). A unique number $c_A(t)$ which is in effect the minimum number of applications of the rules (a)–(c) needed to construct t, will be assigned to t in the following way:

We call a sequence $t_1,t_2,\dots,t_m$ of elements of T a *generating sequence of length m for t with respect to A* if $t_1\in A$, $t_m=t$ and for each $k=2,3,\dots,m$, one of the following applies:

(i) $t_k\in A$;

(ii) $t_k=t_i-t_j$, $1\leqq i,j<k$;

(iii) $t_k=rt_i$, $1\leqq i<k$ and $r\in R$.

The *complexity of t with respect to A*, denoted by $c_A(t)$, is the length of a generating sequence of minimal length for t with respect to A. Note that $c_A(t)=1$ if and only if $t\in A$. We can now finish the proof of Lemma 2.2.

Let $t\in T$. We will show that $t\in S$ by using induction on $c_A(t)$. If $c_A(t)=1$, then $t\in A\subseteq S$. Suppose $c_A(t)=m>1$ and that all $t'\in T$ with $c_A(t')<m$ are contained in S. We have two possibilities:

1. $t=t_1-t_2$ where $t_1,t_2\in T$ and $c_A(t_1)$, $c_A(t_2)<m$. Since $t_1,t_2\in S$, we must have $t=t_1-t_2\in S$.
2. $t=rt_1$, where $t_1\in T$, $r\in R$ and $c_A(t_1)<m$. Since $t_1\in S$, we have $t=rt_1\in S$.

By induction all elements of T are contained in S and the proof of the lemma is accomplished. □

Lemma 2.3. *Suppose S is an R-subgroup of R generated (as an R-subgroup) by the elements $s_1,s_2,\dots,s_n$ in R. Let $\alpha:=\langle s_1,s_2,\dots,s_n\rangle\in R^n$. Then*

$$\mathbb{M}_n(R)\alpha=S^n$$

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where $\mathbb{M}_n(R)\alpha := \{U\alpha \mid U \in \mathbb{M}_n(R)\}$ *and* $S^n := \{\langle x_1, x_2, \ldots, x_n\rangle \in R^n \mid x_i \in S, i = 1, 2, \ldots, n\}$.

Proof. To show that $\mathbb{M}_n(R)\alpha \subseteq S^n$, we use induction on the weight of matrices in $\mathbb{M}_n(R)$. Let $U \in \mathbb{M}_n(R)$ and suppose $w(U) = 1$, i.e. $U = f^r_{ij}$ for some $r \in R$ and $1 \leqq i, j \leqq n$. Then $U\beta = \iota_i(r\pi_j\beta) \in S^n$, for all $\beta \in S^n$. In particular $U\alpha \in S^n$. Now suppose $w(U) = m > 1$ and $V\beta \in S^n$ for all $\beta \in S^n$ and for all $V \in \mathbb{M}_n(R)$ with $w(V) < m$. There are two cases to consider:

1. $U = V_1 + V_2$ with $V_1, V_2 \in \mathbb{M}_n(R)$ and $w(V_1), w(V_2) < m$. It follows that $U\beta = V_1\beta + V_2\beta \in S^n + S^n \subseteq S^n$.
2. $U = V_1V_2$ with $V_1, V_2 \in \mathbb{M}_n(R)$ and $w(V_1), w(V_2) < m$. In this case $U\beta = (V_1V_2)\beta = V_1(V_2\beta) = V_1\gamma$ for some $\gamma \in S^n$ so that $V_1\gamma \in S^n$.

In both cases it follows that $U\alpha \in S^n$, since $\alpha \in S^n$. From induction it follows now that $\mathbb{M}_n(R)\alpha \subseteq S^n$.

In order to prove that $S^n \subseteq \mathbb{M}_n(R)\alpha$, we will show that $\iota_1\pi_1(S^n) = \langle S, \{0\}, \{0\}, \ldots, \{0\}\rangle \subseteq \mathbb{M}_n(R)\alpha$. The same method can then be used to show that $\iota_i\pi_i(S^n) \subseteq \mathbb{M}_n(R)\alpha$ for all $i = 1, 2, \ldots, n$. Since $\mathbb{M}_n(R)\alpha$ is an $\mathbb{M}_n(R)$-subgroup of the $\mathbb{M}_n(R)$-module R^n, it follows that $\sum_{i=1}^n \iota_i\pi_i(S^n) = S^n \subseteq \mathbb{M}_n(R)\alpha$.

Since S is the R-subgroup of R generated by $A = \{s_1, s_2, \ldots, s_n\}$, we can apply Lemma 2.2 and so each element of S has a complexity with respect to A. Now let $s \in S$ such that $c_A(s) = 1$. Then $s \in A$, i.e. $s = s_j$ for some $j, 1 \leqq j \leqq n$. But then $\iota_1(s) = \langle s, 0, 0, \ldots, 0\rangle = f^1_{1j}\alpha \in \mathbb{M}_n(R)\alpha$. Now suppose $s \in S$ with $c_A(s) = m > 1$ and that $\iota_1(t) \in \mathbb{M}_n(R)\alpha$ for all $t \in S$ with $c_A(t) < m$. Consider the following possibilities:

1. $s = t_1 - t_2$ with $t_1, t_2 \in S$ and $c_A(t_1)$, $c_A(t_2) < m$. But then $\iota_1(s) = \iota_1(t_1) - \iota_1(t_2) \in \mathbb{M}_n(R)\alpha - \mathbb{M}_n(R)\alpha \subseteq \mathbb{M}_n(R)\alpha$.
2. $s = rt$ where $r \in R, t \in S$ and $c_A(t) < m$. In this case $\iota_1(s) = f^r_{11}\iota_1(t) \in f^r_{11}\mathbb{M}_n(R)\alpha \subseteq \mathbb{M}_n(R)\alpha$.

The principle of induction assures us that $\iota_1(S) = \iota_1\pi_1(S^n) \subseteq \mathbb{M}_n(R)\alpha$ and by the arguments above, our proof is complete. □

Theorem 2.4. *Suppose* L *is a maximal left ideal of* R *and* $\alpha = \langle s_1, s_2, \ldots, s_n\rangle \in R^n \setminus L^n$ *is such that the set* $\{s_1, s_2, \ldots, s_n\}$ *generates* R *as an* R*-subgroup of* R *(for example, if at least one* $s_i = 1$*). Then* $(L^n : \alpha)$ *is a maximal left ideal of* $\mathbb{M}_n(R)$*, where* $(L^n : \alpha) := \{U \in \mathbb{M}_n(R) \mid U\alpha \in L^n\}$.

Proof. Consider the $\mathbb{M}_n(R)$-homomorphisms $\phi : \mathbb{M}_n(R) \to R^n$ and $\psi : R^n \to R^n/L^n \cong (R/L)^n$, where $\phi(U) := U\alpha$ for all $U \in \mathbb{M}_n(R)$ and ψ is the canonical $\mathbb{M}_n(R)$-epimorphism. The isomorphism follows from Lemma 1.6. Furthermore, $\mathbb{M}_n(R)\alpha = R^n$ as follows from Lemma 2.3, which means that ϕ is an epimorphism. But then $\psi \circ \phi : \mathbb{M}_n(R) \to R^n/L^n$ is an epimorphism. We deduce that $\mathbb{M}_n(R)/(L^n : \alpha) = \mathbb{M}_n(R)/\mathrm{Ker}(\psi \circ \phi) \cong \mathrm{Im}(\psi \circ \phi) = R^n/L^n$. But since R/L is simple as R-module, $(R/L)^n$ is simple as an $\mathbb{M}_n(R)$-module. (See Meyer [7, Corollary 2.10.]) This means that $\mathbb{M}_n(R)/(L^n:\alpha)\cong R^n/L^n\cong(R/L)^n$ is simple as $\mathbb{M}_n(R)$-module and we deduce that $(L^n:\alpha)$ is maximal in $\mathbb{M}_n(R)$. □

We will now provide an example to show that when $\alpha\in R^n\backslash L^n$, but the co-ordinates of α do not generate R as R-subgroup of R, then Theorem 2.4 is in general not valid.

Example 2.5. Let $G:=\{0,1,2,\ldots,7\}$ denote the cyclic group of order 8. The non-trivial proper subgroups of G are denoted by $H_1:=\{0,2,4,6\}$ and $H_2:=\{0,4\}$. Define R as follows:

$$R:=\{f\in M_0(G)\,|\,f(H_i)\subseteq H_i, i=1,2, \text{ and if } x,y\in H_1 \text{ with } x-y\in H_2,$$

$$\text{then } f(x)-f(y)\in H_2\}.$$

It is routine verification to check that R is a zero-symmetric, abelian near-ring with identity. Moreover, R is finite with $|R|=2^{16}=65536$.

Now consider the following subsets of R:

$$M:=\{f\in R\,|\,f(1)\in H_1\},$$

$$K:=\{f\in R\,|\,f(1)\in H_2\},$$

$$L:=\{f\in R\,|\,f(1)=0\}=\mathrm{Ann}_R(1).$$

Obviously, $\{0\}\subset L\subset K\subset M\subset R$, where "$\subset$" means proper inclusion. We also observe the following facts:

I. *L is a maximal left ideal of R.*

Proof. Being the annihilator of an element in G, L is certainly a left ideal of R. Since $R1=G$, we have that $R/\mathrm{Ann}_R(1)=R/L\cong G$ as R-modules. The only possible non-trivial proper R-ideals of G are H_1 and H_2. But $r(2+1)-r(1)=r(3)-r(1)=1$ if $r(3)=1$ and $r(x)=0$ if $x\neq 3$. Since $2\in H_1$ and $1\notin H_1$, H_1 is not an R-ideal of G. In a similar way it follows that H_2 neither is an R-ideal of G, implying that G is a simple R-module. But then R/L is a simple R-module and so L is a maximal left ideal of R. □

II. *Both K and M are R-subgroups of R (and not R-ideals).*

Proof. Straightforward. □

III. *K is an R-ideal of M.*

Proof. Since $(K,+)$ is a normal subgroup of $(R,+)$, it is a normal subgroup of $(M,+)$ as well. Let $k\in K, m\in M$ and $r\in R$. Then

$$[r(k+m)-rm](1)=r(h_2+h_1)-r(h_1) \text{ where } h_i\in H_i, i=1,2$$

$$\in H_2, \text{ since } h_1, h_1+h_2\in H_1 \text{ and } (h_1+h_2)-h_1\in H_2.$$ □

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IV. *We have the following proper inclusions of R-modules:*

$$L/L \subset K/L \subset M/L.$$

Proof. This is merely a matter of equivalence class arithmetic. □

V. *The R-module M/L is not simple.*

Proof. From III and IV it follows readily that K/L is a non-trivial proper R-ideal of M/L. □

VI. *The R-subgroup M of R is generated (as an R-subgroup) by the two elements m_1 and m_2, where*

$$m_1(x) := \begin{cases} 2 \text{ if } x=1 \\ 0 \text{ if } x=2 \\ 4 \text{ if } x=6 \\ x \text{ otherwise} \end{cases} \qquad m_2(x) := \begin{cases} 2 \text{ if } x=1 \\ 0 \text{ if } x=4 \\ x \text{ otherwise.} \end{cases}$$

Proof. Since $m_1, m_2 \in M$, the R-subgroup generated by m_1 and m_2 is certainly contained in M. Conversely, if $m \in M$, choose $r_1, r_2 \in R$ as follows:

$$r_1(x) := \begin{cases} m(1)-m(2) & \text{if } x=2 \\ m(1)+m(2) & \text{if } x=6 \\ m(x) & \text{otherwise} \end{cases} \qquad r_2(x) := \begin{cases} m(2) & \text{if } x=2 \\ m(6)-m(4) & \text{if } x=6 \\ 0 & \text{otherwise.} \end{cases}$$

Then $r_1m_1 + r_2m_2 = m$, as can be easily verified and so M is contained in the R-subgroup generated by m_1 and m_2. □

VII. *For any $n \geqq 2$ we have that $\mathbb{M}_n(R)\alpha = M^n$, where $\alpha := \langle m_1, m_2, 0, 0, \ldots, 0 \rangle \in R^n$ with m_1 and m_2 as in VI.*

Proof. This result follows directly from VI and Lemma 2.3. □

VIII. *For the α of VII it follows that $\alpha \in R^n \backslash L^n$ and $(L^n : \alpha)$ is not a maximal left ideal of $\mathbb{M}_n(R)$.*

Proof. Consider the mappings $\phi : \mathbb{M}_n(R) \to R^n$ and $\psi : R^n \to R^n/L^n \cong (R/L)^n$ of $\mathbb{M}_n(R)$-modules as in the proof of Theorem 2.4. It follows that

$$\begin{aligned} \operatorname{Im}(\psi \circ \phi) &= \{U\alpha + L^n \mid U \in \mathbb{M}_n(R)\} \\ &= M^n/L^n \text{ by VII.} \end{aligned}$$

Furthermore , $\mathbb{M}_n(R)/(L^n:\alpha) = \mathbb{M}_n(R)/\operatorname{Ker}(\psi \circ \phi) \cong M^n/L^n$. But M/L is not simple as an R-module (from V) and so $(M/L)^n \cong M^n/L^n$ is not simple as an $\mathbb{M}_n(R)$-module which implies that $(L^n:\alpha)$ is not maximal in $\mathbb{M}_n(R)$. □

It must be emphasised that although K is not a left ideal of R, $(K^n:\alpha)$ is indeed a maximal left ideal of $\mathbb{M}_n(R)$, properly containing $(L^n:\alpha)$. It can be shown that $(K^n:\alpha)$ is of the form $(T^n:\beta)$ where T is a maximal left ideal of R and $\beta \in R^n \backslash T^n$: Take T as $\{f \in R \mid f(2), f(6) \in H_2\}$, and $\beta = \langle 1, 1, 0, 0, \ldots, 0 \rangle \in R^n$.

If Γ is a faithful type 0 $\mathbb{M}_n(R)$-module, then Γ is $\mathbb{M}_n(R)$-isomorphic to $\mathbb{M}_n(R)/\mathscr{L}$ for some maximal left ideal $\mathscr{L}$ of $\mathbb{M}_n(R)$. It follows from faithfulness that the largest two-sided ideal in $\mathscr{L}$ is $\{0\}$ and hence, if $\mathscr{L}=(L^n:\alpha)$ for some maximal left ideal L of R and $\alpha\in R^n\backslash L^n$, then $L^*=\{0\}$, because $L^*=(L^n:R^n)\subseteq(L^n:\alpha)=\mathscr{L}$ and L^* is two-sided. Consequently, if we can find an R with $\mathbb{M}_n(R)$ 0-primitive and such that no maximal left ideal L of R has the property $L^*=\{0\}$, then at least one maximal left ideal of $\mathbb{M}_n(R)$ cannot be written in the form $(L^n:\alpha)$ where $\alpha\in R^n\backslash L^n$. It is not known whether such an R exists. In Theorem 2.11, however, it will be shown that when R is a weakly distributive d.g. near-ring, then every maximal left ideal of $\mathbb{M}_n(R)$ can be expressed in this form.

Recall that a d.g. near-ring R is weakly distributive if its distributor series $\{D^i(R)\}$ terminates in $\{0\}$, where

$$D^0(R):=R, \text{ and}$$

$$D^{i+1}(R):=Gp\langle\{x(a+b)-xb-xa \mid x\in R, a,b\in D^i(R)\}\rangle^R \text{ if } i\geqq 0.$$

Here $Gp\langle X\rangle^R$ denotes the normal subgroup of $(R,+)$ generated by $X\subseteq R$. The interested reader should consult Meldrum [4] for a comprehensive study on this subject. We also quote the following lemmas from [4]:

Lemma 2.6. (Meldrum [4, Theorem 9.45]). *Let R be a d.g. near-ring with $R^2=R$. Then $D^n(R)=\delta_n(R)$ for all $n\geqq 0$ where $\delta_n(R)$ denotes the nth term of the derived series of the group $(R,+)$.*

Lemma 2.7. (Meldrum [4, Corollary 9.46]). *If R is a d.g. near-ring with $R^2=R$, then R is weakly distributive if and only if $(R,+)$ is soluble.*

Lemma 2.8. (Meldrum [4, Corollary 9.34]). *If R is a d.g. near-ring then $\delta_i(R)$ is an ideal of R for all $i\geqq 0$.*

Lemma 2.9. (Meldrum [4, Corollary 9.49]). *If R is a d.g. near-ring with $(R,+)$ soluble, then $\delta_1(R)$ is multiplicatively nilpotent.*

It was shown in Abbasi, Meldrum and Meyer [2] that if R is a weakly distributive d.g. near-ring, then so is $\mathbb{M}_n(R)$. By Lemmas 2.7, 2.8 and 2.9 it follows that $\delta_1(\mathbb{M}_n(R))$ is a multiplicatively nilpotent ideal of $\mathbb{M}_n(R)$. Consequently, $\delta_1(\mathbb{M}_n(R))$ is contained in $\mathscr{T}_{1/2}(\mathbb{M}_n(R))$ from which it follows that $\delta_1(\mathbb{M}_n(R))\subseteq\mathscr{L}$ for any maximal left ideal $\mathscr{L}$ of $\mathbb{M}_n(R)$, since $\mathscr{T}_{1/2}(\mathbb{M}_n(R))=\cap\{\mathscr{L}\mid\mathscr{L}$ is a maximal left ideal of $\mathbb{M}_n(R)\}$. This leads us to the following lemma:

Lemma 2.10. *Suppose R is a weakly distributive d.g. near-ring and let $\mathscr{L}$ be a maximal left ideal of $\mathbb{M}_n(R)$. Then there exists an $\alpha\in R^n$ such that the set of co-ordinates of α generates R as an R-subgroup and such that $(\mathscr{L}\alpha:\alpha):=\{U\in\mathbb{M}_n(R)\mid U\alpha\in\mathscr{L}\alpha\}\subset\mathbb{M}_n(R)$, where $\mathscr{L}\alpha:=\{L\alpha\mid L\in\mathscr{L}\}$.*

Proof. Since $\mathbb{M}_n(R)$ is d.g., each matrix can be represented by an expression involving only f^r_{ij} and plus-signs (Abbasi [1, Theorem 4.1]). In fact, since $\mathbb{M}_n(R)$ is also weakly distributive, any $U\in\mathbb{M}_n(R)$ can be expressed as

$$
\begin{aligned}
U = \quad & f_{11}^{r_{11}} + f_{12}^{r_{12}} + \cdots + f_{1n}^{r_{1n}} \\
& + f_{21}^{r_{21}} + f_{22}^{r_{22}} + \cdots + f_{2n}^{r_{2n}} \\
& \quad \vdots \qquad \vdots \qquad \ddots \qquad \vdots \\
& + f_{n1}^{r_{n1}} + f_{n2}^{r_{n2}} + \cdots + f_{nn}^{r_{nn}} + U', \text{ where } U' \in \delta_1(\mathbb{M}_n(R)) \subseteq \mathscr{L}.
\end{aligned}
$$

Now suppose the lemma is not true. Then $(\mathscr{L}\alpha:\alpha) = \mathbb{M}_n(R)$ for all $\alpha \in R^n$ of which the co-ordinates form a generating set for R as R-subgroup; in particular, for all α with $\pi_i\alpha = 1$ for some i, $1 \leqq i \leqq n$. Consequently, $\mathscr{L}\alpha = R^n$ for all such α. To simplify matters, we shall stick to the case $n=2$. A similar (but much more clumsy) procedure applies for the case $n>2$.

For every $y \in R$ there is a matrix $U_y \in \mathscr{L}$ such that $U_y\langle 1, y\rangle = \langle 1, 0\rangle$. Since $f_{11}^1 U_y \in \mathscr{L}$ and $f_{11}^1 U_y\langle 1, y\rangle = \langle 1, 0\rangle$, we shall only consider first row matrices in $\mathscr{L}$, i.e. matrices of the form $f_{11}^1 L$, $L \in \mathscr{L}$. Similarly, for every $x \in R$, there is a (first row) matrix $V_x \in \mathscr{L}$ such that $V_x\langle x, 1\rangle = \langle 1, 0\rangle$. Now suppose

$$U_y = [r_1; 1,1] + [s_1; 1,2] + [r_2; 1,1] + [s_2; 1,2] + \cdots + [r_m; 1,1] + [s_m; 1,2].$$

Then

$$U_y = [r_1 + r_2 + \cdots + r_m; 1,1] + [s_1 + s_2 + \cdots + s_m; 1,2] + U'_y \text{ for some } U'_y \in \mathscr{L}.$$

Let $a(y) := r_1 + r_2 + \cdots + r_m$ and $b(y) := s_1 + s_2 + \cdots + s_m$. Then, since $U_y\langle 1, y\rangle = \langle 1, 0\rangle$, it follows that $a(y) + b(y)y + d(y) = 1$ for some $d(y) \in \delta_1(R)$. Consequently, for any $y \in R$, there are $b(y) \in R$ and $d(y) \in \delta_1(R)$ such that

$$[1 - d(y) - b(y)y; 1,1] + [b(y); 1,2] \in \mathscr{L}.$$

But $[-d(y); 1,1] \in \mathscr{L}$ (Abbasi [**1**, Corollary 4.18]) and thus we have that

$$[1 - b(y)y; 1,1] + [b(y); 1,2] \in \mathscr{L}.$$

By a similar argument, for any $x \in R$, there is an $a(x) \in R$ such that

$$[a(x); 1,1] + [1 - a(x)x; 1,2] \in \mathscr{L}.$$

Since $\mathscr{L}$ is a left ideal we deduce that for any $x, y, z, w \in R$, $[z(1 - b(y)y); 1,1] + [zb(y); 1,2] \in \mathscr{L}$ and $[wa(x); 1,1] + [w(1 - a(x)x); 1,2] \in \mathscr{L}$, and so

$$[z(1 - b(y)y) + wa(x); 1,1] + [zb(y) + w(1 - a(x)x); 1,2] \in \mathscr{L}.$$

Let $y=0$, $x = -b(0)$, $w = -b(0)$ and $z = 1 + b(0)a(-b(0))$. Then we have (with $b(0)$ written as b and using the fact that $x(-y) - xy \in \delta_1(R)$ for all $x, y \in R$)

$$[1; 1,1] + [b + ba(-b)b + (-b)(1 + a(-b)b); 1,2] \in \mathscr{L}$$

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and since the expression in a and b is an element of $D^1(R)=\delta_1(R)$, we conclude that

$$f_{11}^1 \in \mathscr{L}.$$

It follows *mutatis mutandis* that $f_{22}^1 \in \mathscr{L}$ and therefore $f_{11}^1+f_{22}^1$, the identity matrix, is an element of $\mathscr{L}$, which is a contradiction. □

Theorem 2.11. *If R is a weakly distributive d.g. near-ring and $\mathscr{L}$ is a maximal left ideal of $\mathbb{M}_n(R)$, then there exists a maximal left ideal L of R such that $\mathscr{L}=(L^n:\alpha)$ for some $\alpha \in R^n \backslash L^n$.*

Proof. From the previous lemma it follows that there is an $\alpha \in R^n$ (of which the co-ordinates generate R as an R-subgroup and can therefore not be in L^n for any proper left ideal L of R) such that $(\mathscr{L}\alpha:\alpha) \subset \mathbb{M}_n(R)$. But since $\mathscr{L} \subseteq (\mathscr{L}\alpha:\alpha)$ and $\mathscr{L}$ is maximal, we must have $\mathscr{L}=(\mathscr{L}\alpha:\alpha)$. Also, $\mathscr{L}\alpha$ is an $\mathbb{M}_n(R)$-ideal of the $\mathbb{M}_n(R)$-module R^n and is thus of the form K^n for some left ideal K of R (Van der Walt [**11**, Lemma 3.7]). But K is contained in a maximal left ideal L which means that $\mathscr{L}=(K^n:\alpha)\subseteq(L^n:\alpha)\subset \mathbb{M}_n(R)$ so that $\mathscr{L}=(L^n:\alpha)$. □

Corollary 2.12. *If the d.g. near-ring R is weakly distributive, then*

$$(\mathscr{T}_{1/2}(R))^* = \mathscr{T}_0(\mathbb{M}_n(R)) = (\mathscr{T}_0(R))^*.$$

Proof.

$$\begin{aligned}
\mathscr{T}_{1/2}(\mathbb{M}_n(R)) &= \cap\, \{\mathscr{L} \mid \mathscr{L} \text{ is a maximal left ideal of } \mathbb{M}_n(R)\} \\
&= \cap\, \{(L^n:\alpha_L) \mid L \text{ is an element of a subset of the set of all maximal} \\
&\qquad \text{left ideals of } R \text{ and } \alpha_L \in R^n\backslash L^n\}, \text{ by Theorem 2.11} \\
&\supseteq \cap\, \{(L^n:\alpha) \mid L \text{ is a maximal left ideal of } R \text{ and } \alpha \in R^n \backslash L^n\} \\
&\supseteq \cap\, \{(L^n:R^n) \mid L \text{ is a maximal left ideal of } R\} \\
&= ((\cap\, \{L \mid L \text{ is a maximal left ideal of } R\})^n : R^n) \text{ by Pilz [9, 1.44]} \\
&= ((\mathscr{T}_{1/2}(R))^n : R^n) \\
&= (\mathscr{T}_{1/2}(R))^*.
\end{aligned}$$

Since $(\mathscr{T}_{1/2}(R))^*$ is two-sided, $(\mathscr{T}_{1/2}(R))^* \subseteq \mathscr{T}_0(\mathbb{M}_n(R))$. Furthermore, $\mathscr{T}_0(\mathbb{M}_n(R)) \subseteq (\mathscr{T}_0(R))^*$, from Meyer [7, Theorem 2.34(a)], and since $(\mathscr{T}_0(R))^* = (\mathscr{T}_{1/2}(R))^*$ (by Lemma 1.4), the result follows. □

3. 0-Primitivity

In this section we will concentrate on those R-modules embeddable into ${}_RR$. We shall see that when R has $DCCR$, i.e. R has the descending chain condition on R-subgroups, then much can be said about simple faithful R-subgroups of R. If R is finite we can even go further and prove a strong relationship between R and $\mathbb{M}_n(R)$, as far as 0-primitivity is concerned. Of course, the next step would be to study this relationship in arbitrary zero-symmetric near-rings.

Lemma 3.1. *Suppose K is an R-subgroup of R. Then*

(*a*) *The R-module K is faithful if and only if the $\mathbb{M}_n(R)$-module K^n is faithful.*

(*b*) *The R-module K is simple if and only if the $\mathbb{M}_n(R)$-module K^n is simple.*

Proof. (a) Suppose ${}_{\mathbb{M}_n(R)}K^n$ is faithful. Let $0 \neq r \in R$. Then f^r_{11} is non-zero in $\mathbb{M}_n(R)$ which means that there is an $\alpha \in K^n$ such that $f^r_{11}\alpha \neq \bar{0}$. This implies that $\pi_1\alpha \in K$ and $r(\pi_1\alpha) \neq 0$. Consequently, ${}_RK$ is faithful.

On the other hand, let ${}_RK$ be faithful. Suppose $U \in \mathbb{M}_n(R)$ is non-zero. Then $U\langle r_1, r_2, \ldots, r_n\rangle = \langle t_1, t_2, \ldots, t_n\rangle$ with $r_i, t_i \in R$ and at least one t_i, say t_1, is non-zero. Since ${}_RK$ is faithful, there is a $k \in K$ such that $t_1k \neq 0$. But then $U\langle r_1k, r_2k, \ldots, r_nk\rangle = \langle t_1k, t_2k, \ldots, t_nk\rangle \neq \bar{0}$, while $\langle r_1k, r_2k, \ldots, r_nk\rangle \in K^n$. In other words, ${}_{\mathbb{M}_n(R)}K^n$ is faithful.

(b) Suppose ${}_RK$ is not simple. Then there exists an R-ideal H of K such that $\{0\} \subset H \subset K$ and so $(H^n, +)$ is a proper non-trivial normal subgroup of $(K^n, +)$. Moreover, H^n is an $\mathbb{M}_n(R)$-ideal of K^n, as follows: Let $\alpha \in H^n$, $\beta \in K^n$ and $f^r_{ij} \in \mathbb{M}_n(R)$. Then $f^r_{ij}(\alpha+\beta) - f^r_{ij}\beta = \gamma$, where $\pi_i\gamma \in H$ and $\pi_k\gamma = 0$ if $k \neq i$. So $\gamma \in H^n$. Now let $w(U) = m > 1$, and suppose $V(\alpha+\beta) - V\beta \in H^n$ for all $\alpha \in H^n$, $\beta \in K^n$ and matrices V with $w(V) < m$. There are two cases to consider:

1. $U = V_1 + V_2$, with $w(V_1)$, $w(V_2) < m$. But then $U(\alpha+\beta) - U\beta = (V_1+V_2)(\alpha+\beta) - (V_1+V_2)\beta = V_1(\alpha+\beta) + V_2(\alpha+\beta) - V_2\beta - V_1\beta = V_1(\alpha+\beta) + \gamma - V_1\beta = V_1(\alpha+\beta) - V_1\beta + \gamma' \in H^n$, for some $\gamma, \gamma' \in H^n$.
2. $U = V_1V_2$, with $w(V_1)$, $w(V_2) < m$. In this case, $U(\alpha+\beta) - U\beta = V_1V_2(\alpha+\beta) - V_1V_2\beta = V_1[V_2(\alpha+\beta) - V_2\beta + V_2\beta] - V_1V_2\beta \in H^n$, since $V_2(\alpha+\beta) - V_2\beta \in H^n$.

From induction it follows that ${}_{\mathbb{M}_n(R)}K^n$ is not simple.

Conversely, suppose ${}_{\mathbb{M}_n(R)}K^n$ is not simple. Then there is a non-trivial $\mathbb{M}_n(R)$-ideal $\mathscr{H} \subset K^n$. But $\mathscr{H}$ is of the form H^n for some R-ideal H of K, where $\{0\} \subset H \subset K$ (take $H = \{\pi_1\alpha \mid \alpha \in \mathscr{H}\}$.) As a consequence, ${}_RK$ is not simple. □

Theorem 3.2. *Suppose R has DCCR and does not necessarily contain an identity. Let K be a non-zero R-subgroup of R. If the R-module K is simple and faithful, then it is monogenic.*

Proof. Since K is faithful, $K \not\subseteq \mathrm{Ann}_R(k_1)$ for some $k_1 \in K$. Moreover, because $K \cap \mathrm{Ann}_R(k_1)$ is an R-ideal of K, we must have $K \cap \mathrm{Ann}_R(k_1) = \{0\}$. Now consider the map $\phi: K \to K$ where $\phi(k) := kk_1$ for all $k \in K$. This map is injective, for if $kk_1 = k'k_1$ where $k \neq k'$, then $0 \neq k - k' \in K \cap \mathrm{Ann}_R(k_1) = \{0\}$, a contradiction. That $\phi(k+k') = \phi(k) +$

$\phi(k')$ and $\phi(rk)=r\phi(k)$, for all $k,k'\in K$ and $r\in R$, follows trivially. We deduce that K and $Kk_1=\mathrm{Im}(\phi)$ are R-isomorphic.

If $Kk_1\subset K$, we can repeat the process with K replaced by Kk_1 and obtain an R-module $Kk_2k_1\subseteq Kk_1$ which is R-isomorphic to Kk_1 (and hence to K). And so we can continue to repeat this process until the containment is not proper any more (because of the $DCCR$) and we end up with a chain of R-subgroups:

$$K\supset Kk_1\supset Kk_2k_1\supset\cdots\supset Kk_ik_{i-1}\ldots k_1=Kk_{i+1}k_i\ldots k_1.$$

This implies that $k_{i+1}k_i\ldots k_1=k'k_{i+1}k_i\ldots k_1$ for some $k'\in K$, whence $Rk_{i+1}k_i\ldots k_1=Rk'k_{i+1}k_i\ldots k_1\subseteq Kk_{i+1}k_i\ldots k_1\subseteq Rk_{i+1}k_i\ldots k_1$ and it follows that $Kk_{i+1}k_i\ldots k_1$ is monogenic over R by $k_{i+1}k_i\ldots k_1$. Since all the subgroups in the chain are R-isomorphic, ${}_RK$ is also monogenic. □

Corollary 3.3. *If R has DCCR and contains a simple faithful R-subgroup, then R is 0-primitive.*

Note that Theorem 3.2 is no longer valid if ${}_RK$ is not faithful: Let, for example, $G:=\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$ and let $H_1:=\{(0,0,0),(0,1,0)\}$, $H_2:=\{(0,0,0),(1,0,0)\}$, $H_3:=\{(0,0,0)(1,1,0)\}$ and $H:=\sum_{i=1}^{3}H_i$. Then define R as follows:

$$R:=\{f\in M_0(G)\,|\,f(H_i)\subseteq H_i \text{ for all } i=1,2,3\}.$$

R is a finite near-ring with identity. If we now take

$$K=\{f\in R\,|\,f(0,0,1)\in H \text{ and } f(\alpha)=(0,0,0) \text{ for all } \alpha\neq(0,0,1)\},$$

then it is easy to verify that ${}_RK$ is simple, not faithful and also not monogenic.

Theorem 3.4. *Suppose R is finite. Then $\mathbb{M}_n(R)$ is 0-primitive if and only if R is 0-primitive.*

Proof. If R is 0-primitive then $\mathbb{M}_n(R)$ is 0-primitive by Lemma 1.8. Now suppose ${}_{\mathbb{M}_n(R)}\Gamma$ is a faithful type 0 module with generator γ. Then $\Gamma\cong\mathbb{M}_n(R)/\mathscr{L}$ as $\mathbb{M}_n(R)$-modules where $\mathscr{L}:=\mathrm{Ann}_{\mathbb{M}_n(R)}(\gamma)$ is a maximal left ideal of $\mathbb{M}_n(R)$. Since ${}_{\mathbb{M}_n(R)}\mathbb{M}_n(R)/\mathscr{L}$ is faithful, $\mathscr{L}$ cannot contain any two-sided ideals other than $\{0\}$. Also, since $\mathbb{M}_n(R)$ is finite, it contains minimal left ideals as well as minimal two-sided ideals. Suppose all minimal left ideals of $\mathbb{M}_n(R)$ are contained in $\mathscr{L}$. According to Pilz [**9**, 3.54], every minimal two-sided ideal is a direct sum of minimal left ideals. This would mean that $\mathscr{L}$ contains all the minimal two-sided ideals, which is impossible.

Consequently, there is at least one minimal left ideal, say $\mathscr{B}$, of $\mathbb{M}_n(R)$ such that $\mathscr{B}\nsubseteq\mathscr{L}$. Hence, $\mathscr{B}\gamma\neq\{0\}$. From Pilz [**9**, 3.10], it follows that $\mathscr{B}\cong\Gamma$ as $\mathbb{M}_n(R)$-modules.

Furthermore, since $\mathscr{B}\neq\{0\}$, there is a non-zero $\alpha\in R^n$ such that $\mathscr{B}\alpha$ is a non-zero $\mathbb{M}_n(R)$-subgroup of R^n. This implies that $\mathscr{B}\alpha$ is of the form K^n for some non-zero R-subgroup K of R. (Take $K=\{\pi_1B\alpha\,|\,B\in\mathscr{B}\}$.) The map $\mathscr{B}\to K^n$ which sends $B\in\mathscr{B}$ to $B\alpha$ for all $B\in\mathscr{B}$ assures us of an isomorphism

$$K^n \cong \mathscr{B}/(\mathscr{B} \cap \mathrm{Ann}_{\mathbb{M}_n(R)}(\alpha)) = \mathscr{B}/\{0\} \cong \mathscr{B}$$

of $\mathbb{M}_n(R)$-modules. Consequently, $\Gamma \cong K^n$ as $\mathbb{M}_n(R)$-modules whence ${}_{\mathbb{M}_n(R)}K^n$ is simple and faithful. We therefore must have ${}_RK$ simple and faithful, by Lemma 3.1. Corollary 3.3 now implies that R is 0-primitive. □

Corollary 3.5. *If R is a finite 0-primitive near-ring, then there exist a maximal left ideal $\mathscr{L}$ and a minimal left ideal $\mathscr{B}$ of $\mathbb{M}_n(R)$ such that*

$$\mathbb{M}_n(R) = \mathscr{L} \oplus \mathscr{B}.$$

Proof. Following the same terminology as in the proof of Theorem 3.4, $\mathscr{B} \cap \mathscr{L} = \{0\}$ by the minimality of $\mathscr{B}$ and we therefore must have that $\mathbb{M}_n(R) = \mathscr{L} \oplus \mathscr{B}$, by the maximality of $\mathscr{L}$. □

The following corollary clears up—at least to a certain extent—open problem 5 posed in Meyer [7, p. 105]. For any k, $1 \leqq k \leqq n$, $\mathscr{L}_k$ is defined to be the left ideal of $\mathbb{M}_n(R)$ generated by the matrix f^1_{1k}. We also define

$$\mathscr{M}_k := \mathscr{L}_1 + \mathscr{L}_2 + \cdots + \mathscr{L}_{k-1} + \mathscr{L}_{k+1} + \cdots + \mathscr{L}_n.$$

In Meyer [6] it is shown that if F is a near-field, then, with R replaced by F in the foregoing, $\mathscr{M}_k$ is a maximal left ideal of $\mathbb{M}_n(F)$. Moreover, it is shown that

$$\mathscr{M}_k = \mathrm{Ann}_{\mathbb{M}_n(F)}(\iota_k(1)). \qquad (\dagger)$$

Corollary 3.6. *If F is a finite near-field and with the notation as explained above, there is a minimal left ideal $\mathscr{B}$ of $\mathbb{M}_n(F)$ such that $\mathscr{B} \cap \mathscr{M}_k = \{0\}$ and hence that*

$$\mathbb{M}_n(F) = \mathscr{M}_k \oplus \mathscr{B}.$$

Proof. The module ${}_{\mathbb{M}_n(F)}F^n$ is faithful and of type 0 and we may choose $\gamma := \iota_k(1)$ as generator. But, according to (†), $\mathscr{M}_k$ is the annihilator of γ in the near-ring $\mathbb{M}_n(F)$. Following the proofs of Theorem 3.4 and Corollary 3.5 above, our result is immediate. □

It remains, however, to be seen whether $\mathscr{B} \subseteq \mathscr{L}_k$ in the corollary above, as was suggested by the open problem discussed in the foregoing.

Another question which remains open is whether Lemma 1.10 remains valid if Γ is a type 0 $\mathbb{M}_n(R)$-module. Examples suggest very strongly (at least in the finite case) that this is indeed the case. This would in turn, force Lemma 1.11 to be true in the 0-primitive case and by using Theorem 1.7 one should be able to prove a strong link between $\mathscr{T}_0(R)$ and $\mathscr{T}_0(\mathbb{M}_n(R))$ which we formalise as follows:

Conjecture 3.7. *If R is finite, then*

$$\mathcal{T}_0(\mathbb{M}_n(R)) = (\mathcal{T}_0(R))^*.$$

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