A CONFORMAL PROOF OF A JORDAN CURVE PROBLEM

G. Spoar and N.D. Lane

The following theorem appears in [1].

THEOREM. Let R be a closed simply connected region of the inversive plane bounded by a Jordan curve J, and let J be divided into three closed arcs A_1, A_2, A_3. Then there exists a circle contained in R and having points in common with all three arcs.

An elegant metric proof was given by Paul Erdős [1, p. 568]. The theorem, however, belongs to the inversive plane and therefore it may be of interest to indicate how a slight modification of Erdős' proof avoids the use of metric concepts.

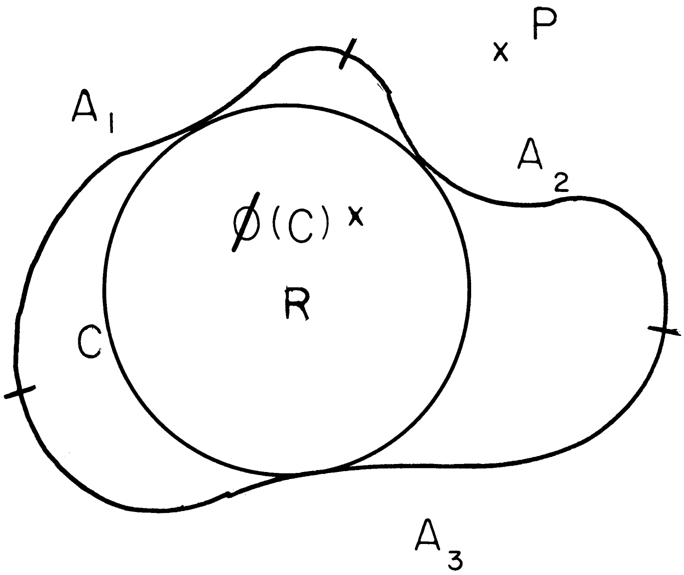
Proof. Let S_i be the set of circles lying in R which have a point in common with A_i, $i = 1, 2, 3$. We include in S_i the point circles of A_i. The sets S_i are closed and connected. Since $S_i \cap S_j \neq \phi$, $S_1 \cup S_2$ is a closed connected set and so is $S = S_1 \cup S_2 \cup S_3$.

Let P be any fixed point $P \notin R$. Let ϕ be the mapping: $S \to R$ which takes a non-degenerate circle C of S into that point of R which is the image of P under inversion in the circle C. If C is a point circle of S, take $\phi(C) = C$. The mapping ϕ is a homeomorphism and both ϕ and ϕ^{-1} take closed connected sets into closed connected sets. Also $\phi[S] = R$.

It is well known that the set of points of R is unicoherent (i.e., if R is written as a sum of two closed connected sets R_1 and R_2, then $R_1 \cap R_2$ is also closed and connected). Hence S is also unicoherent.

Suppose that $S_1 \cap S_2 \cap S_3 = \phi$. Then $S_3 \cap S_1$ and $S_1 \cap S_2$ are disjoint. They are also non-empty. Hence $S_3 \cap (S_1 \cup S_2) = (S_3 \cap S_1) \cup (S_3 \cap S_2)$ consists of two non-empty disjoint closed sets and is therefore not connected. This contradicts the unicoherence of S. Hence there is some circle C in R that has points in common with each of the arcs A_1, A_2, A_3.

https://doi.org/10.4153/CMB-1969-087-8 Published online by Cambridge University Press



REFERENCE

1. S.B. Jackson, Vertices of plane curves. Bull. Amer. Math. Soc. 60 (1944) 564-578.

McMaster University.

https://doi.org/10.4153/CMB-1969-087-8 Published online by Cambridge University Press