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THE UNIFORM CONTINUITY OF FUNCTIONS IN SOBOLEV SPACES

BY
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ABSTRACT. Functions in $W^{m,p}(\Omega)\cap W_0^{1,q}(\Omega)$, $mp>\dim\Omega$, $q\geq 1$, may have to be uniformly continuous on Ω even if Ω is not a Lipschitz domain.

1. **Introduction.** Let Ω be a domain (an open set) in n-dimensional Euclidean space $\mathbf{R}^n$. We denote the boundary of Ω by $\partial\Omega$. The Sobolev space $W^{m,p}(\Omega)$ consists of (equivalence classes of) functions u in $L^p(\Omega)$ whose distributional derivatives $D^\alpha u$ also belong to $L^p(\Omega)$ whenever $|\alpha|\leq m$. (m is a positive integer; p is real, $p\geq 1$; $\alpha=(\alpha_1,\ldots,\alpha_n)$ is an n-tuple of nonnegative integers; $|\alpha|=\alpha_1+\cdots+\alpha_n$; $D^\alpha=(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n}$.) $W^{m,p}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{m,p,\Omega}=\left\{\sum_{|\alpha|\leq m}\int_\Omega |D^\alpha u(x)|^p\,dx\right\}^{1/p}.$$

$W_0^{m,p}(\Omega)$ is the closure in $W^{m,p}(\Omega)$ of the space $C_0^\infty(\Omega)$ of infinitely differentiable functions having compact support in Ω.

We denote by $C(\bar{\Omega})$ the space of functions u bounded and uniformly continuous on Ω and having, therefore, unique continuous extensions to the closure $\bar{\Omega}$ of Ω, and by $C_B(\Omega)$ the space of functions bounded and continuous on Ω. Both are Banach spaces with respect to the norm $\sup_{x\in\Omega}|u(x)|$.

The domain Ω has the *cone property* if there exists an open, finite, right spherical cone C such that each point $x\in\Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C. Ω is a *Lipschitz domain* if each point $x\in\partial\Omega$ has a neighbourhood U_x such that, for some rectangular coordinate system ξ in U_x, $U_x\cap\Omega$ is specified by an inequality of the form $\xi_n<f\{\xi_1,\ldots,\xi_{n-1})$ where f is a Lipschitz continuous function.

Many imbedding results for $W^{m,p}(\Omega)$ can be obtained under the fairly mild requirement that Ω should have the cone property. For instance, for such Ω, $W^{m,p}(\Omega)$ is imbedded in $C_B(\Omega)$ provided $mp>n$. (This is a part of the "Sobolev

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Imbedding Theorem"—see e.g. [1], theorem 5.4.) Certain imbeddings, however, require more regularity of Ω. One cannot in general expect to imbed $W^{m,p}(\Omega)$ into $C(\bar{\Omega})$ if Ω has only the cone property. Two obvious counterexamples are the split squares:

$$\Omega_1 = \{x = (x_1, x_2) \in \mathbf{R}^2 : -1 < x_1 < 1, 0 < |x_2| < 1\}$$
$$\Omega_2 = \Omega_1 \cup \{x \in \mathbf{R}^2 : -1 < x_1 < 0, x_2 = 0\}.$$

Both Ω_1 and Ω_2 have the cone property and Ω_2 is connected. However the reader may readily construct a function u belonging to $W^{m,p}(\Omega)$ ($\Omega = \Omega_1$ or Ω_2) for every m, p, but which satisfies $\lim_{x_2 \to 0-} u(x) \neq \lim_{x_2 \to 0+} u(x)$ for $x_1 > 0$, and hence cannot be uniformly continuous on Ω.

If Ω is a bounded Lipschitz domain then the Sobolev imbedding theorem assures us that $W^{m,p}(\Omega)$ is imbedded in $C(\bar{\Omega})$ provided $mp > n$. We examine circumstances under which the Lipschitz property can be weakened. It is clear, at least for bounded Ω, that elements of $C_B(\Omega)$ which also happen to tend to zero on $\partial\Omega$ belong to $C(\bar{\Omega})$. Since for any q the elements of $W_0^{1,q}(\Omega)$ may be regarded as vanishing "in a generalized sense" on $\partial\Omega$ (see Lemma 2 below) one is led to the conjecture:

$$\mathrm{W}^{\mathrm{m,p}}(\Omega) \cap \mathrm{W}_0^{1,\mathrm{q}}(\Omega) \subset \mathrm{C}(\bar{\Omega}).$$

There is good reason to suspect that this conjecture is true for arbitrary domains Ω (see section 5 below) but this writer has been unable to discover a general proof. We can prove it for arbitrary domains with the cone property using a well-known theorem of E. Gagliardo [4] on the decomposition of such domains into unions of Lipschitz domains.

THEOREM 1. *Let Ω be a domain in $\mathbf{R}^n$ having the cone property. If $mp > n$ and $q \geq 1$ then $W^{m,p}(\Omega) \cap W_0^{1,q}(\Omega) \subset C(\bar{\Omega})$. More generally, for any nonnegative integer j, $W^{m+j,p}(\Omega) \cap W_0^{1+j,q}(\Omega) \subset C^j(\bar{\Omega})$.*

Here, of course, $C^j(\bar{\Omega})$ denotes the space of functions u for which $D^\alpha u \in C(\bar{\Omega})(|\alpha| \leq j)$, normed by $\max_{|\alpha| \leq j} \sup_{x \in \Omega} |D^\alpha u(x)|$. Theorem 1 need only be proved for $j = 0$ as it then follows for general j be application of the special case to derivatives $D^\alpha u$, $|\alpha| \leq j$. We give a proof in sections 3 and 4 below. At this point we can make several remarks.

(i) Theorem 1 is only of interest when $q \leq n$. If $q > n$ it is a trivial consequence of the Sobolev imbedding theorem that $W_0^{1+j,q}(\Omega)$ is imbedded in $C^j(\bar{\Omega})$ for arbitrary domains Ω (since zero extension outside Ω imbeds $W_0^{k,q}(\Omega)$ into $W^{k,q}(\mathbf{R}^n)$). Several useful characterizations of $W_0^{k,q}(\Omega)$ for $q > n$ are known (see Burenkov [2, 3]) but these are of no avail in the context of our problem.

(ii) It is not difficult to find examples of domains Ω not having the cone property for which, at least for some of the appropriate values of m, p and q

the conclusion of Theorem 1 holds. (See section 5.) It is for this reason that we conjecture that Theorem 1 may hold for arbitrary domains, but a different sort of proof will be necessary to show this.

(iii) The (generalized) vanishing of functions is not really required on the whole of the boundary of Ω for Theorem 1 to hold. One might consider replacing $W_0^{1,q}(\Omega)$ by the larger space $W_0^{1,q}(\Omega^*)$, the closure in $W^{1,q}(\Omega)$ of the space of infinitely differentiable functions of compact support in $\mathbf{R}^n$ which vanish near $\partial\Omega \sim \partial\bar{\Omega}$. It is clear, for instance, that such is the case for the two examples Ω_1 and Ω_2 given above, where for each we have $\partial\Omega \sim \partial\bar{\Omega} = \{x \in \partial\Omega : x_2 = 0 \text{ and } -1 < x_1 < 1\}$.

(iv) Weak solutions of null Dirichlet problems for elliptic partial differential equations on Ω are known *a priori* to belong to spaces of the form $W_0^{k,q}(\Omega)$ (usually with $q = 2$). Theorem 1 thus enables us to obtain "up to the boundary" regularity of solutions in $W^{m,p}(\Omega)$ for suitably large mp even if Ω has only the cone property.

2. **A preliminary lemma.** Before proving Theorem 1 we prepare the following lemma. It is well-known, at least for smoothly bounded domains, and asserts that continuous functions in $W_0^{1,1}(\Omega)$ do in fact vanish on sufficiently well-behaved parts of $\partial\Omega$.

LEMMA 2. *Let Ω be a domain in $\mathbf{R}^n$ and G a bounded Lipschitz domain contained in Ω. Let $u \in W_0^{1,1}(\Omega) \cap C(\bar{G})$ and let $x \in \partial G$. If there exists a neighbourhood N of x such that $N \cap \partial G \subset \partial\Omega$ then $u(x) = 0$.*

Proof. Suppose $u(x) \neq 0$. We may select the neighbourhood N small enough that $|u(x)| \geq \delta > 0$ *for* $x \in N \cap \bar{G}$. By virtue of the Lipschitz property of G we may, again contracting N if necessary, find a nonzero vector y such that for all $z \in N \cap \partial G$ and all s, $0 < s < 1$, we have $z + sy \in G$. Without loss of generality $y = k(0, 0, \ldots, 0, 1)$. Let $V = \{z + sy : z \in N \cap \partial G,\ 0 < s < 1\}$. Writing $z = (z', z_n)$ where $z' = (z_1, \ldots, z_{n-1})$, setting $P = \{z' : (z', z_n) \in N \cap \partial G \text{ for some } z_n\}$, and denoting by z_n^* the unique number which, for given $z' \in P$, satisfies $(z', z_n^*) \in N \cap \partial G$, we have

$$V = \{z = (z', z_n) : z' \in P,\ z_n^* < z_n < z_n^* + k\}.$$

Let $\phi \in C_0^\infty(\Omega)$ and set $v = u - \phi$. Then $|v(z', z_n^*)| = |u(z', z_n^*)| \geq \delta$ for all $z' \in P$. If $z = (z', z_n) \in V$ then

$$v(z', z_n) = v(z', z_n^*) + \int_{z_n^*}^{z_n} \frac{\partial}{\partial s} v(z', s)\, ds$$

whence

$$\delta \leq |v(z', z_n)| + \int_{z_n^*}^{z_n^*+k} \left| \frac{\partial}{\partial s} v(z', s) \right| ds.$$

Integrating z over V we obtain

$$(\text{vol } V)\delta \le \int_V |v(z)|\,dz + k\int_V \left|\frac{\partial}{\partial z_n} v(z)\right| dz$$
$$\le (1+k)\,\|v\|_{1,1,V} \le (1+k)\,\|u-\phi\|_{1,1,\Omega}.$$

Since $u \in W_0^{1,1}(\Omega)$ the right side of the last inequality can be made arbitrarily small for suitable choice of ϕ and we have a contradiction. Thus $u(x)=0$.

We remark that the above lemma extends with no change in proof to more general domains G than bounded Lipschitz ones. For instance, it is sufficient that G have the segment property. (See [1], section 4.2.)

In view of Lemma 2 the proof of Theorem 1 for domains (like Ω_1 above) which are unions of finitely many *pairwise disjoint* bounded Lipschitz domains is trivial. Similar *ad hoc* techniques will yield the result for somewhat more complicated domains (e.g. Ω_2) as well, but for the general case we require the following theorem of E. Gagliardo [4]. (See also, [1], theorem 4.8)

THEOREM 3. (*Gagliardo*) (a) *If Ω is a bounded domain with the cone property then Ω is a finite union of bounded Lipschitz domains.*

(b) *Any domain Ω (bounded or not) having the cone property is a union of finitely many subdomains each of which is a union of parallel translates of some open parallelepiped.*

3. **Proof of Theorem 1 for bounded domains.** For the time being we assume that Ω is bounded. Thus $W_0^{1,q}(\Omega) \subset W_0^{1,1}(\Omega)$ and we may also assume that $q=1$.

As noted above, we may write $\Omega = \bigcup_{V\in\mathcal{F}} V$ where $\mathcal{F}$ is a finite family of bounded Lipschitz subdomains of Ω. Given $u \in W^{m,p}(\Omega)\cap W_0^{1,1}(\Omega)$ we have $u \in C_B(\Omega)$ and $u \in C(\bar{V})$ for every $V \in \mathcal{F}$. We must show that $u \in C(\bar{\Omega})$.

Let ν be the number of elements of $\mathcal{F}$ and let B be an open ball in $\mathbf{R}^n$. Let $V, W \in \mathcal{F}$ be such that $V\cap B \ne \varnothing$ and $W \cap B \ne \varnothing$. By a $(B,\mathcal{F})$-*chain linking V and W* we mean any (finite) sequence $\{U_1,\ldots,U_k\} \subset \mathcal{F}$, $(k \le \nu)$, such that $U_1 = V$, $U_k = W$ and $\bar{U}_j \cap U_{j+1} \cap \Omega \cap B \ne \varnothing$, $1 \le j \le k-1$. Given $V \in \mathcal{F}$ let $\mathcal{A}(V)$ denote the collection of elements $W \in \mathcal{F}$ linked to V by a $(B,\mathcal{F})$-chain. Evidently $W \in \mathcal{A}(V)$ if and only if $V \in \mathcal{A}(W)$.

Let $\varepsilon > 0$ be given. For each $V \in \mathcal{F}$ there exists $\delta_V > 0$ such that if $x, y \in \bar{V}$ and $|x-y| < \delta_V$ then $|u(x)-u(y)| < \varepsilon/\nu$. (In this context we regard u as its unique continuous extension to $\bar{V}$.) Let $\delta = \min_{V\in\mathcal{F}} \delta_V$ Let $x, y \in \Omega$ satisfy $|x-y| < \delta$. We show that $|u(x)-u(y)| < \varepsilon$ and hence complete the proof.

Let B be an open ball in $\mathbf{R}^n$ having diameter δ and containing x and y. There exist elements $V, W \in \mathcal{F}$ such that $x \in V$ and $y \in W$.

Case I. $W \in \mathcal{A}(V)$. In this case there exists a $(B,\mathcal{F})$-chain $\{U_1,\ldots,U_k\}$ linking $V = U_1$ and $W = U_k$. Select points $z_1,\ldots,z_{k-1}$ with $z_j \in$

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$\bar{U}_j \cap \bar{U}_{j+1} \cap \Omega \cap B$. Evidently

$$|u(x)-u(y)| \le |u(x)-u(z_1)+\sum_{j=1}^{k-2} |u(z_j)-u(z_{j+1})|+|u(z_{k-1})-u(y)|<\varepsilon.$$

Case II. $W \notin \mathcal{A}(V)$. Then $\mathcal{A}(W) \cap \mathcal{A}(V) = \varnothing$. Let λ, μ be the numbers of elements in $\mathcal{A}$(V) and $\mathcal{A}(W)$ respectively, so that $\lambda+\mu \le \nu$. Let $S = \bigcup_{U \in \mathcal{A}(V)} U$, $T = \bigcup_{U \in \mathcal{A}(W)} U$. We show that there exist points $z \in B \cap \bar{S}$, $\rho \in B \cap \bar{T}$ such that $u(z)=u(\rho)=0$. Granted this, for the moment, we have $z \in B \cap \bar{U}$ for some $U \in \mathcal{A}(V)$. Hence there exists a $(B, \mathcal{F})$-chain $\{U_1, \ldots, U_k\}$ $(k \le \lambda)$ linking $U_1 = V$ and $U_k = U$. Selecting $z_1, \ldots, z_{k-1}$ as in case I we conclude that

$$|u(x)| = |u(x)-u(z)| < \lambda\varepsilon/\nu.$$

A similar argument yields $|u(y)| < \mu\varepsilon/\nu$ whence $|u(x)-u(y)|<\varepsilon$ as required It is sufficient, therefore, to show the existence of $z \in B \cap \bar{S}$ with $u(z)=0$.

Let $G \in \mathcal{A}(V)$ and let $\tilde{G} = \bigcup_{U \in \mathcal{A}(V), U \ne G} U$. Thus $S = G \cup \tilde{G}$. Suppose that $t \in B \cap (\partial G \sim \tilde{G})$. Then $t \in \bar{G} \subset \bar{\Omega}$ so that either $t \in \partial\Omega$ or $t \in \Omega$. Since G is open $t \notin G$; thus $t \notin S$. If $t \in \Omega$ then $t \in U$ for some $U \in \mathcal{F}$. Thus $t \in \bar{U} \cap \bar{G} \cap \Omega \cap B$ whence $U \in \mathcal{A}(V)$ and $U \subset S$, a contradiction. Thus $t \in \partial\Omega$ and we have proved

$$B \cap (\partial G \sim \tilde{G}) \subset \partial\Omega \quad \text{for every } G \in \mathcal{A}(V).$$

Now $\partial S = \bigcup_{G \in \mathcal{A}(V)} (\partial G \sim \tilde{G})$ so that

$$B \cap \partial S \subset \partial\Omega.$$

Let k be the largest integer such that every point of $B \cap \partial S$ belongs to the boundaries of at least k distinct elements of $\mathcal{A}(V)$. Clearly $1 \le k \le \lambda$. Then there exists $z \in \partial S \cap B$ and elements $G_1, \ldots, G_k \in \mathcal{A}(V)$ such that $z \in \partial G_1 \cap \cdots \cap \partial G_k$ but $z \notin \bar{G}$ for any $G \in \mathcal{A}(V)$, $G \ne G_1, \ldots, G_k$. Since $B \sim \bigcup_{G \in \mathcal{A}(V), G \ne G_1, \ldots, Gk} \bar{G}$ is open, there exists a neighbourhood N of z with $N \subset B$ such that $N \cap \partial S = N \cap \partial G_1 \cap \cdots \cap \partial G_k$ and $N \cap S = N \cap (G_1 \cup \cdots \cup G_k)$. We show that $N \cap \partial S = N \cap \partial G_1$.

Suppose that $a \in N \cap \partial G_1$ but $a \notin \partial S$. Evidently $a \in S$ (since otherwise $a \in \operatorname{ext} S$ so a would have a neighbourhood contained in N, containing a point of G_1 but disjoint from S). It follows that $k \ge 2$ and $a \in G_j$ for some j, $2 \le j \le k$. We may assume that N has been chosen so small that $N \cap G_1$ lies on one side of a Lipschitz graph in N. Let $s \in N \sim S$. We may find a continuous path in N going from s to a which meets $\bar{G}_1$ for the first time at a. The path meets $\bar{S}$ for the first time at a point of $\partial S \cap N \subset \partial G_1 \cap \cdots \cap \partial G_k$ so this point must be a. Since $a \notin \partial S$ we have a contradiction. Hence

$$N \cap \partial G_1 = N \cap S \subset \partial\Omega.$$

It follows from Lemma 2 that $u(z)=0$ and the proof of Theorem 3 for bounded domains is complete.

4. **Extension to unbounded domains.** By Theorem 3(b) even an unbounded domain Ω can be written as a union of *finitely many* subdomains $\Omega_j (1 \le j \le k)$ each of which is a union of *parallel translates* of a fixed open parallelepiped P_j having one vertex at the origin; say

$$\Omega_j = \bigcup_{x \in A_j} (x + P_j), \qquad 1 \le j \le k.$$

The dimensions of P_j depend only on the cone C determining the cone property for Ω.

Let $\mathbf{R}^n = \bigcup_\beta Q_\beta$ be a tesselation of $\mathbf{R}^n$ into closed cubes of edge length ρ and set

$$A_{j\beta} = A_j \cap Q_\beta; \qquad \Omega_{j\beta} = \bigcup_{x \in A_{j\beta}} (x + P_j).$$

Evidently $\Omega = \bigcup_{j,\beta} \Omega_{j\beta}$ (no longer necessarily a finite union) and for any $\delta > 0$ there exists an integer $R = R(n, \rho, \delta, C)$ such that any ball of diameter δ intersects at most R of the sets $\Omega_{j\beta}$. It is also shown in the proof of Gagliardo's theorem that for ρ sufficiently small (depending only on the dimensions of the parallelepipeds P_j and thus on C) each $\Omega_{j\beta}$ is a bounded Lipschitz domain; in fact $(x + P_j) \cap (y + P_j) \ne 0$ for every $x, y \in A_{j\beta}$.

For given $u \in W^{m,p}(\Omega)$, $mp > n$, and given $\varepsilon > 0$ it is shown in the proof of the imbedding theorem (see, for example, [1] lemma 5.17) that there exists $\delta > 0$ depending only on ε, $\|u\|_{m,p,\Omega}$, and the cone C, such that if $x, y \in \Omega_{j\beta}$ for some j, β and $|x - y| < \delta$ then $|u(x) - u(y)| < \varepsilon$.

With these observations the proof of Theorem 1 for bounded domains extends to arbitrary domains—one uses in place of ν the number $R = R(n, \rho, 1, C)$; in place of $\mathcal{F}$ the collection $\{\Omega_{j\beta} : \Omega_{j\beta} \cap B_1 \ne \emptyset\}$ where B_1 is a ball of unit diameter containing B. (We assume $\delta \le 1$.) The remaining details are left to the reader.

5. **An example.** We conclude by showing that Theorem 1 may hold, at least in part, for domains not having the cone property. Specifically, we consider 2-dimensional domains of the following type:

$$\Omega = \{x = (x_1, x_2) \in \mathbf{R}^2 : 0 < x_1 < a, 0 < x_2 < f(x_1)\}$$

where the positive, increasing function f satisfies

$$\lim_{x_1 \to 0+} \frac{f(x_1)}{x_1} = 0,$$

so that Ω has a cusp at the origin.

Given X, $0 < X < a$, we set $\Omega_X = \{x \in \Omega : x > X\}$. Then Ω_X is a bounded Lipschitz domain, and if we are given $u \in W^{m,p}(\Omega) \cap W_0^{1,1}(\Omega)$ where $mp > 2$ we may conclude at once that for any X we have $u \in C(\overline{\Omega_X})$ and $u(x)=0$ for $x \in \partial\Omega_X \cap \partial\Omega$. In order to conclude that $u \in C(\bar{\Omega})$ it is evidently sufficient to show that $\lim_{x\in\Omega, x\to 0} u(x)=0$.

First suppose that $p>2$. Let $x=(x_1, x_2) \in \Omega$ be given. For x_1 sufficiently small the open triangle T with vertices at (x_1, x_2), $(x_1, 0)$ and $(x_1+x_2, 0)$ lies in Ω. Let (r, θ) denote polar coordinates of an arbitrary point of Ω with respect to x as pole. The bottom edge of T has equation $r=g(\theta)$, $-\pi/2 \le \theta \le -\pi/4$, where $0<g(\theta)<\sqrt{2}x_2<\sqrt{2}f(x_1)$. Denoting by v the function u expressed in terms of these polar coordinates, and applying Hölder's inequality to the identity

$$u(x)=v(0, \theta)=-\int_0^{g(\theta)} \frac{d}{dt} v(t, \theta)\, dt$$

we obtain

$$\begin{aligned} |u(x)|^p &\le \int_0^{g(\theta)} \left|\frac{d}{dt} v(t, \theta)\right|^p t\, dt \cdot \left\{\int_0^{g(\theta)} t^{-1/(p-1)}\, dt\right\}^{p-1} \\ &\le K_p[f(x_1)]^{p-2} \int_0^{g(\theta)} \left|\frac{d}{dt} v(t, \theta)\right|^p t\, dt, \end{aligned}$$

where K_p depends only on p. Integration of θ from $-\pi/2$ to $-\pi/4$ leads to the estimate

$$\begin{aligned} |u(x)|^p &\le \frac{4K_p}{\pi}[f(x_1)]^{p-2} \int_T |\text{grad}\, u(y)|^p\, dy \\ &\le K_p'[f(x_1)]^{p-2} \|u\|_{m,p,\Omega}^p. \end{aligned}$$

Hence $\lim_{x\in\Omega, x\to 0} u(x)=0$ in this case.

The case $mp>2$, $p\le 2$ remains to be considered; we may assume $m=2$. The technique used above cannot be generalized to involve a repeated integral of the second derivative of v since grad u is not known to vanish on the lower edge of T. The following *ad hoc* argument will yield the desired result providing $p>4/3$. Let R be a rectangle of breadth b and height $h\le 1$. A change of variable mapping R onto a rectangle of breadth b and unit height yields the following form of the norm inequality for the imbedding of $W^{1,p}(R)$ into $L^q(R)$, $q=2p/(2-p)$ (q finite if $p=2$).

$$\|w\|_{0,qR} \le Kh^{-1/2}\|w\|_{1,p,R}$$

where K may depend on b but is independent of h. Note that $q>2$ if $p>1$. For x_1 sufficiently small the open rectangle R having vertices at $(x_1, 0)$, $(x_1, f(x_1))$, $(x_1+(a/2), f(x_1))$ and $(x_1+(a/2), 0)$ is contained in Ω and contains T.

Since $b=a/2$ and $h=f(x_1)$ for this rectangle we obtain

$$\begin{aligned}\|u(x)\|^q &\le K'_q[f(x_1)]^{q-2}\,\|u\|^q_{1,q,R}\\ &\le K'_qK[f(x_1)]^{q-2}[f(x_1)]^{-q/2}\,\|u\|^q_{2,p,R}\\ &\le K''_p[f(x_1)]^{(q-4)/2}\,\|u\|^q_{2,p,\Omega}.\end{aligned}$$

We may conclude that $u(x)\to 0$ as $x\to 0$, $x\in\Omega$ provided $q>4$, that is, provided $p>4/3$.

The method of this example can, of course, be extended to more general cusp domains but it remains uncertain whether the conclusion of Theorem 1 is valid in its entirety for arbitrary domains.

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