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A classification of groups with a centralizer condition II

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Let G be a finite group. A nontrivial proper subgroup M of G is called a *CC-subgroup* if M contains the centralizer in G of each of its nonidentity elements. In this paper groups containing a CC-subgroup of order divisible by 3 are completely determined.

1. Introduction

The purpose of this paper is to prove the following:

THEOREM 1. *Let G be a finite group and let M be a CC-subgroup of G. Assume that $3 \mid |M|$. Then one of the following statements is true:*

(i) $G \cong \mathrm{PSL}(2, q)$;

(ii) G *is a Frobenius group with* M *as the Frobenius kernel or a Frobenius complement;*

(iii) M *is a noncyclic elementary abelian Sylow* 3*-subgroup of* G *;*

(iv) M *is a cyclic subgroup of* G *of odd order.*

Groups satisfying *(iii)* or *(iv)* were completely classified in [2] and [6], respectively. Simple groups satisfying the assumptions of Theorem 1 were listed in Theorem B of [1]. In order to prove Theorem 1 it suffices, in view of [1, Theorem A], to establish:

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THEOREM 2. *Let* G *be a finite group and let* M *be a* CC*-subgroup of* G. *Assume that* $N_G(M) = M$ *and* $3 \mid |M|$. *Then either*

$G \cong \mathrm{PSL}(2, q)$ *or* G *is a Frobenius group with* M *as a Frobenius complement.*

Sections 3 and 5 contain related results of independent interest.

In this paper all groups are finite. If G is a group, then $\pi(G)$, $G^{\#}$, and S_p denote, respectively, the set of primes p dividing $|G|$, the nonidentity elements of G, and a Sylow p-subgroup of G. If π is a set of primes, $O_{\pi'}(G)$ denotes the maximal normal π'-subgroup of G. The signs $\subseteq$ and $\subset$ will denote containment and proper containment of subgroups, respectively. By a simple group we mean a nonabelian simple group. We shall use freely the bar-convention for images in a quotient group.

2. Two lemmas

The following lemmas are necessary for induction arguments in the next section. The letter G denotes a group.

LEMMA 1. *Let* $H \triangleleft G$ *and let* $x \in G$ *satisfy* $(|x|, |H|) = 1$. *Denote* $G/H = \overline{G}$. *Then*

$$C_{\overline{G}}(\overline{x}) = C_G(x)H/H .$$

Proof. Clearly $\supseteq$ holds. Now let $c \in C_G(x \bmod H)$; then $x^cH = xH$ and consequently $\langle x^c\rangle H = \langle x\rangle H$. By the Schur-Zassenhaus Theorem there exists $h \in H$ such that $\langle x^c\rangle = \langle x\rangle^h$. Let i be an integer satisfying $x^c = (x^i)^h$. Then

$$x^iH = x^ihH = hx^cH = hxH .$$

As $x^{-1}hx \in H$, $x^{i-1} \in H$; hence $x^{i-1} = 1$, $x^i = x$, and $x^c = x^h$. Thus $ch^{-1} \in C_G(x)$, as required.

LEMMA 2. *Let* M *be a Hall* π*-subgroup of* G. *Suppose that* $H \triangleleft G$ *and either* H *is a* π'*-group or* MH *is solvable. Denote* $G/H = \overline{G}$. *Then*

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$$N_{\bar{G}}(\bar{M}) = N_G(M)H/H .$$

Proof. Clearly $\supseteq$ holds. Now let $n \in N_G\ (M \bmod H)$; then $M^nH = MH$, and by the Schur-Zassenhaus Theorem or Hall's Theorem there exists $h \in H$ satisfying $M^n = M^h$. Thus $nh^{-1} \in N_G(M)$, as required.

3. A general theorem

In order to prove Theorem 2 we need the following:

THEOREM 3. *Let G be a finite group containing a CC-subgroup M . Suppose that $N_G(M) = M$. Then one of the following statements is true:*

(i) G is a Frobenius group with a complement M ;

(ii) G has a simple section $K/H = \bar{K}$ satisfying

(a) $M \subseteq N_G(K) \cap N_G(H)$,

(b) MH/H is a CC-subgroup of MK/H ,

(c) $\overline{K \cap M}$ is a (nontrivial) CC-subgroup of $\bar{K}$,

(d) $N_{\bar{K}}(\overline{K \cap M}) = \overline{K \cap M}$.

As an immediate corollary we get the following characterization of soluble Frobenius groups.

THEOREM 4. *Let G be a soluble group containing a CC-subgroup M . Then $N_G(M) = M$ if and only if G is a Frobenius group with a complement M .*

Proof of Theorem 3. Let G be a counter-example of minimal order. It is well known that M is a Hall π-subgroup of G , where $\pi = \pi(M)$. Clearly G is not simple. Thus, by [8, Theorem 1], $2 \nmid |M|$ and by the Feit-Thompson Theorem, M is solvable.

Suppose that $O_{\pi'}(G) \neq 1$. As $\bar{G} = G/O_{\pi'}(G)$ is not isomorphic to M , it follows by Lemmas 1 and 2 that $\bar{M}$ is a CC-subgroup of $\bar{G}$ satisfying $N_{\bar{G}}(\bar{M}) = \bar{M}$. Hence, by induction, *(ii)* holds; a contradiction.

Assume, from now on, that $O_{\pi'}(G) = 1$. Let N be a minimal normal

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subgroup of G . Clearly $M \cap N \neq 1$.

Case 1. N is an elementary abelian p-group. Clearly $N \subseteq M$ and, defining V by

$$V \equiv \cap\{M^x \mid x \in G\} ,$$

we have $1 \subset V \subset M$. It follows that V is a normal CC-subgroup of G . Thus both G and M are Frobenius groups with the kernel V . Let C be a complement of V in M . Then C is a CC-subgroup of G and, as $N_G(C) \subseteq N_G(M) = M$, $N_G(C) = C$. By Lemmas 1 and 2 we may apply induction to $\bar{G} = G/V$ and $\bar{C}$. As G is a counterexample, $\bar{G}$ is a Frobenius group with a complement $\bar{C}$. However, since G is a Frobenius group with V as its kernel, by [5, Theorem V, 8.18], $\bar{G} = G/V$ has a nontrivial center, a contradiction.

Case 2. N is a direct product of n isomorphic simple groups. As $R \equiv M \cap N$ is a CC-subgroup of N , it is a Hall subgroup of N and consequently $n = 1$. Suppose that $N_N(R) \neq R$; then $T \equiv N_G(R) \supset M$. If $T = G$, then a contradiction is reached as in Case 1. Thus $T \subset G$; hence, by the minimality of G , T is a Frobenius group with a complement M , contradicting $R \subset T$. Thus we have shown that $N_N(R) = R$. But then G satisfies *(ii)* with $K = N$ and $H = 1$, a final contradiction.

4. Proof of Theorem 2

Let G be a counterexample of minimal order. Thus *(ii)* of Theorem 3 holds. If $2 \mid |M|$, then, by [8, Theorem 1], $G \cong \mathrm{PSL}(2, 2^{2n})$. So suppose, from now on, that $2 \nmid |M|$.

Case 1. Suppose that $3 \mid |\overline{K \cap M}|$. By Theorem B of [1], $\bar{K}$ is one of a known list of simple groups, none of which except PSL(2, q) satisfies $N_{\bar{K}}(\overline{K \cap M}) = \overline{K \cap M}$ and $2 \nmid |\overline{K \cap M}|$.

Case 2. Suppose that $3 \nmid |\overline{K \cap M}|$. Thus $3 \nmid |\bar{K}|$ and, by Thompson's 3'-theorem, $\bar{K}$ is isomorphic to $Sz(2^{2n+1})$. Let m be an element of M of order 3 . Then, by *(ii)* *(a)* of Theorem 3 and by [3, Theorem 6.2.2 (i)], m normalizes the center of an S_2 of $\bar{K}$, which has

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order 2^{2n+1} . As $3 \nmid 2^{2n+1}-1$, m centralizes an involution in $\bar{K}$. Since $2 \nmid |M|$, we have reached a final contradiction to *(ii) (b)* of Theorem 3.

5. A generalization

The result of this section generalizes [*1*, Theorem B], [*4*, Theorem 11], [*6*, Theorem B], and [*7*, Theorems 1 and 2].

THEOREM 5. *Let* G *be a simple group containing a subgroup* $X \times Y$ *which satisfies the following conditions:*

(i) whenever $x \in X^{\#}$ *then* $C_G(x) = X \times Y$ *;*

(ii) $3 \mid |X|$ *and* $2 \nmid |Y|$ *;*

(iii) if $2 \nmid |X|$ *then* $3 \nmid |Y|$.

Then G *is isomorphic to one of the following groups:*

(a) PSL(3, 4) *;*

(b) PSL(2, q) *for some* q *;*

(c) PSU$(3, 2^n)$ *for some* n .

Proof. Suppose, first, that $2 \mid |X|$. As $2 \nmid |Y|$, X contains an S_2 of G . Hence G has an abelian S_2 and by [9] either *(b)* holds or G is isomorphic to one of the following groups:

(A) $J(11)$, Janko's smallest group, or

(B) a group of Ree-type.

However, groups of type (A) or (B) have a self-centralizing S_2 , in contradiction to $3 \mid |X|$.

Suppose, finally, that $2 \nmid |X|$. Then X contains an S_3 of G and consequently G has no elements of order 6 . Recent and as yet unpublished results of Stewart and Fletcher, Glauberman, and Stellmacher, classifying groups without elements of order 6 , imply then that *(a)*, *(b)*, or *(c)* holds.

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