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SLANT CURVES IN CONTACT PSEUDO-HERMITIAN 3-MANIFOLDS

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Abstract

By using the pseudo-Hermitian connection (or Tanaka–Webster connection) $\widehat{\nabla}$, we construct the parametric equations of Legendre pseudo-Hermitian circles (whose $\widehat{\nabla}$-geodesic curvature $\widehat{\kappa}$ is constant and $\widehat{\nabla}$-geodesic torsion $\widehat{\tau}$ is zero) in S^3. In fact, it is realized as a Legendre curve satisfying the $\widehat{\nabla}$-Jacobi equation for the $\widehat{\nabla}$-geodesic vector field along it.

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1. Introduction

Given a contact structure η, we have two compatible structures. One is a Riemannian structure (or metric) g, and then we call $(M; \eta, g)$ a *contact Riemannian manifold*. The other is an *almost CR-structure* (η, L), where L is the *Levi form* associated with an endomorphism J on D such that $J^2 = -I$. In particular, if J is integrable, then we call it the (integrable) CR-structure. The associated almost CR-structure is said to be *pseudo-Hermitian, strongly pseudo-convex* if the Levi form is Hermitian and positive definite. We call such a manifold a *contact strongly pseudo-convex pseudo-Hermitian (or almost CR-)manifold*. There is a one-to-one correspondence between the two associated structures by the relation

$$g = L + \eta \otimes \eta,$$

where we denote by the same letter L the natural extension of the Levi form to a (0, 2)-tensor field on M. From this point of view, we have two geometries for a given contact structure, that is, one is formed by the Levi-Civita connection ∇, the other is derived by the *Tanaka–Webster connection* $\widehat{\nabla}$ (or *the pseudo-Hermitian connection*), which is a canonical affine connection on a strongly pseudo-convex CR-manifold.

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In the present paper, we study the contact pseudo-Hermitian geometry in a three-dimensional Sasakian space form whose holomorphic sectional curvature with respect to $\widehat{\nabla}$ is $2c$. Generalizing a Legendre curve in a three-dimensional contact metric manifold, we consider a *slant curve* whose tangent vector field has constant angle with the characteristic direction ξ (see [9]).

Corresponding to biharmonicity for ∇ we investigate the $\widehat{\nabla}$-Jacobi equation for a $\widehat{\nabla}$-geodesic vector field:

$$(C)\quad \begin{cases} \widehat{\nabla}_{\dot{\gamma}}\dot{\gamma} = \widehat{\mathfrak{T}}(\gamma), \\ \widehat{\nabla}^2_{\dot{\gamma}}\widehat{\mathfrak{T}}(\gamma) + \widehat{\nabla}_{\dot{\gamma}}\widehat{T}(\widehat{\mathfrak{T}}(\gamma), \dot{\gamma}) + \widehat{R}(\widehat{\mathfrak{T}}(\gamma), \dot{\gamma})\dot{\gamma} = 0, \end{cases}$$

where $\widehat{T}$, $\widehat{R}$ denotes the pseudo-Hermitian torsion tensor and the pseudo-Hermitian curvature tensor, respectively. Then we prove that no nongeodesic slant curve satisfying (C) exists when $c \leq 0$ (Corollary 3.10). In Section 4 we determine a slant curve satisfying (C) in S^3. In particular, a Legendre curve satisfying (C) in S^3 is realized as a pseudo-Hermitian circle, whose pseudo-Hermitian curvature $\widehat{\kappa} = 2$ and pseudo-Hermitian torsion $\widehat{\tau} = 0$. We obtain their explicit parametric equations in Theorem 4.4. It is notable [10, 11] that there does not exist a Legendre proper biharmonic curve in S^3.

2. Preliminaries

We start by collecting some fundamental material about contact metric geometry. We refer to [3] for further details.

A three-dimensional manifold M^3 is said to be a *contact manifold* if it admits a global 1-form η such that $\eta \wedge (d\eta) \neq 0$ everywhere. Given a contact form η, there exists a unique vector field ξ, the *characteristic vector field*, which satisfies $\eta(\xi) = 1$ and $d\eta(\xi, X) = 0$ for any vector field X. It is well known that there exists an associated Riemannian metric g and a $(1, 1)$-type tensor field φ such that

$$\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \varphi Y), \quad \varphi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

where X and Y are vector fields on M. From (2.1), it follows that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

A Riemannian manifold M equipped with the structure tensors (η, ξ, φ, g) satisfying (2.1) is said to be a *contact Riemannian manifold*. We denote it by $M = (M; \eta, \xi, \varphi, g)$. Given a contact Riemannian manifold M, we define an operator h by $h = \frac{1}{2}L_\xi\varphi$, where L_ξ denotes Lie differentiation in the characteristic direction ξ. Then we may observe that the *structural operator* h is symmetric and satisfies

$$h\xi = 0, \quad h\varphi = -\varphi h,$$

$$\nabla_X\xi = -\varphi X - \varphi hX, \tag{2.2}$$

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where ∇ is the Levi-Civita connection. A contact Riemannian manifold for which ξ is a Killing vector field, is called a K-contact manifold. It is at once shown that a contact Riemannian manifold is K-contact if and only if $h = 0$. We note that three-dimensional K-contact manifold is Sasakian (or normal contact Riemannian manifold) (see [3, p. 76]).

The sectional curvature function of holomorphic planes invariant by φ is called the *holomorphic sectional curvature*. In particular, Sasakian 3-manifolds of constant holomorphic sectional curvature are called three-dimensional *Sasakian space forms*. Simply connected and complete three-dimensional Sasakian space forms $\mathcal{M}^3(H)$ of constant holomorphic sectional curvature H are classified as one of the following unimodular Lie groups with left invariant Sasakian structures: the special unitary group SU(2) for $H > -3$, the Heisenberg group for $H = -3$ or the universal covering group $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ of the special linear group $\mathrm{SL}(2, \mathbb{R})$ for $H < -3$. The three-dimensional Sasakian space forms are naturally reductive homogeneous spaces. In particular, $\mathcal{M}^3(1)$ is the unit 3-sphere S^3 with the canonical Sasakian structure.

Let c be a real number and set

$$\mathcal{D} = \left\{(x, y, z) \in \mathbb{R}^3(x, y, z) \;\middle|\; 1 + \frac{c}{2}(x^2 + y^2) > 0\right\}.$$

Note that $\mathcal{D}$ is the whole $\mathbb{R}^3(x, y, z)$ for $c \geq 0$. On the region $\mathcal{D}$, we equip the following Riemannian metric:

$$g_c = \frac{dx^2 + dy^2}{\{1 + (c/2)(x^2 + y^2)\}^2} + \left(dz + \frac{y\,dx - x\,dy}{1 + (c/2)(x^2 + y^2)}\right)^2. \tag{2.3}$$

Take the following orthonormal frame field on $(\mathcal{D}, g_c)$:

$$u_1 = \left\{1 + \frac{c}{2}(x^2 + y^2)\right\}\frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \quad u_2 = \left\{1 + \frac{c}{2}(x^2 + y^2)\right\}\frac{\partial}{\partial y} + x\frac{\partial}{\partial z},$$
$$u_3 = \frac{\partial}{\partial z}.$$

Then the Levi-Civita connection ∇ of this Riemannian 3-manifold is described as

$$\begin{aligned} &\nabla_{u_1}u_1 = c\,yu_2, \quad \nabla_{u_1}u_2 = -c\,yu_1 + u_3, \quad \nabla_{u_1}u_3 = -u_2,\\ &\nabla_{u_2}u_1 = -c\,xu_2 - u_3, \quad \nabla_{u_2}u_2 = c\,xu_1, \quad \nabla_{u_2}u_3 = u_1,\\ &\nabla_{u_3}u_1 = -u_2, \quad \nabla_{u_3}u_2 = u_1, \quad \nabla_{u_3}u_3 = 0, \end{aligned} \tag{2.4}$$

$$[u_1, u_2] = -c\,yu_1 + c\,xu_2 + 2u_3, \quad [u_2, u_3] = [u_3, u_1] = 0. \tag{2.5}$$

Define the endomorphism field φ by

$$\varphi u_1 = u_2, \quad \varphi u_2 = -u_1, \quad \varphi u_3 = 0.$$

The dual one-form η of the vector field $\xi = u_3$ is a contact form on $\mathcal{D}$ and satisfies

$$d\eta(X, Y) = g(X, \varphi Y), \quad X, Y \in \mathfrak{X}(\mathcal{D}).$$

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Then we see that the structure $(\varphi, \xi, \eta, g_c)$ is Sasakian and further that $(\mathcal{D}, g_c)$ is of constant holomorphic sectional curvature $H = -3 + 2c$ (see [1, 13]). Hereafter we denote this model $(\mathcal{D}, g_c)$ of a Sasakian space form by $\mathcal{M}^3(H)$. The one-parameter family of Riemannian 3-manifolds $\{\mathcal{M}^3(H)\}_{H\in\mathbb{R}}$ is classically known by Bianchi [2], Cartan [6] and Vranceanu [16] (see also Kobayashi [12]). The model $\mathcal{M}^3(H)$ of Sasakian 3-space form is called the *Bianchi–Cartan–Vranceanu model* of three-dimensional Sasakian space form.

The Reeb flows are the translations in the z-directions. Hence the orbit space $\overline{\mathcal{M}^2}(H+3) = \mathcal{M}^3(H)/\xi$ is given explicitly by

$$\overline{\mathcal{M}^2} = \left(\left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; 1 + \frac{c}{2}(x^2 + y^2) > 0 \right\}, \frac{dx^2 + dy^2}{\{1 + (c/2)(x^2 + y^2)\}^2} \right).$$

The natural projection $\pi : \mathcal{M}^3(H) \to \overline{\mathcal{M}^2}(H+3)$ is

$$\pi(x, y, z) = (x, y).$$

We briefly recall the harmonic or the biharmonic maps. Let (N, h) and (M, g) be Riemannian manifolds. For a smooth map $\phi : N \to M$, the Levi-Civita connection ∇ of (N, h) induces a connection ∇^ϕ on the pull-back bundle $\phi^* TM = \bigcup_{p\in N} T_{\phi(p)}M$. The section $\mathfrak{T}(\phi) := \operatorname{tr} \nabla^\phi \, d\phi$ is called the *tension field* of ϕ. Then ϕ is said to be harmonic if its tension field vanishes identically. The *bitension field* $\mathfrak{T}_2(\phi)$ of ϕ is defined by

$$\mathfrak{T}_2(\phi) = -\triangle_\phi \mathfrak{T}(\phi) + \operatorname{tr} R(\mathfrak{T}(\phi), d\phi)\, d\phi,$$

where R is the Riemannian curvature tensor of M defined by $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. The operator $\triangle_\phi$ is the *rough Laplacian* acting on $\Gamma(\phi^* TM)$ defined by

$$\triangle_\phi := -\sum_{i=1}^{n} (\nabla^\phi_{e_i} \nabla^\phi_{e_i} - \nabla^\phi_{\nabla^N_{e_i} e_i}),$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field of N. It is obvious that every harmonic map is biharmonic. Nonharmonic biharmonic maps are called *proper biharmonic maps*.

Now let $\gamma(s) : I \to M$ be a curve parametrized by arc length s and denote the tangent vector field by $T = \dot\gamma$. Then the harmonic equation becomes $\mathfrak{T}(\gamma) = \nabla_T T = 0$ and the biharmonic equation reduces to

$$\mathfrak{T}_2(\gamma) = \nabla^3_T T + R(\nabla_T T, T)T = 0. \tag{2.6}$$

Obviously, every geodesic is biharmonic. A nongeodesic biharmonic curve is called a *proper biharmonic curve*. For the facts and related results regarding biharmonic maps, we refer the interested reader to [4, 5, 10, 7]. The biharmonicity (for ∇) in S^3 is studied and the following results are obtained.

THEOREM 2.1 [4]. *Let γ be a proper $(\nabla-)$biharmonic curve in S^3. Then $\kappa \leq 1$ and we have two cases:*

(i) $\kappa = 1$ *and γ is a circle of radius* $1/\sqrt{2}$;

(ii) $0 < \kappa < 1$ *and γ is a helix, which is a geodesic in the Clifford minimal torus* $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$.

THEOREM 2.2 [10, 11]. *There exists no nongeodesic biharmonic Legendre curve (for ∇) in S^3.*

3. Pseudo–Hermitian contact 3-manifolds

For a three-dimensional contact Riemannian manifold $M = (M^3; \eta, \xi, \varphi, g)$, the tangent space T_pM of M at a point $p \in M$ can be decomposed as the direct sum $T_pM = D_p \oplus \{\xi\}_p$, with $D_p = \{v \in T_pM \mid \eta(v) = 0\}$. Then $D : p \to D_p$ defines a two-dimensional distribution orthogonal to ξ, called the *contact distribution*. We see that the restriction $J = \varphi|_D$ of φ to D defines an almost complex structure on D. Then the associated almost CR-structure of the contact Riemannian manifold M is given by the holomorphic subbundle

$$\mathcal{H} = \{X - iJX \mid X \in D\}$$

of the complexification $TM^{\mathbb{C}}$ of the tangent bundle TM. Then we see that each fiber $\mathcal{H}_p$ is of complex dimension one, $\mathcal{H} \cap \overline{\mathcal{H}} = \{0\}$, and $\mathbb{C}D = \mathcal{H} \oplus \overline{\mathcal{H}}$. Furthermore, the associated almost CR-structure is always integrable, that is $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. For $\mathcal{H}$ we define the Levi form by

$$L : D \times D \to \mathcal{F}(M), \quad L(X, Y) = -d\eta(X, JY),$$

where $\mathcal{F}(M)$ denotes the algebra of differential functions on M. Then we see that the Levi form is Hermitian and positive definite. We call the pair (η, L) a *contact strongly pseudo-convex, pseudo-Hermitian structure* on M. Now, we review the *Tanaka–Webster connection* [14, 17] on a contact strongly pseudo-convex CR-manifold $M = (M; \eta, L)$ with the associated contact Riemannian structure (η, ξ, φ, g). The Tanaka–Webster connection $\widehat{\nabla}$ is defined by

$$\widehat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi$$

for all vector fields X, Y on M. Together with (2.2), $\widehat{\nabla}$ may be rewritten as

$$\widehat{\nabla}_X Y = \nabla_X Y + A(X, Y), \tag{3.1}$$

where we have put

$$A(X, Y) = \eta(X)\varphi Y + \eta(Y)(\varphi X + \varphi h X) - g(\varphi X + \varphi h X, Y)\xi. \tag{3.2}$$

We see that the Tanaka–Webster connection $\widehat{\nabla}$ has the torsion

$$\widehat{T}(X, Y) = 2g(X, \varphi Y)\xi + \eta(Y)\varphi h X - \eta(X)\varphi h Y. \tag{3.3}$$

In particular, for a Sasakian manifold (3.2) and the above equation, we reduce as follows:

$$A(X, Y) = \eta(X)\varphi Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi,$$
$$\widehat{T}(X, Y) = 2g(X, \varphi Y)\xi. \tag{3.4}$$

Furthermore, the following result was proved in [15].

PROPOSITION 3.1. *The Tanaka–Webster connection* $\widehat{\nabla}$ *on a three-dimensional contact Riemannian manifold* $M = (M^3; \eta, \varphi, \xi, g)$ *is the unique linear connection satisfying the following conditions:*

(i) $\widehat{\nabla}\eta = 0$, $\widehat{\nabla}\xi = 0$;
(ii) $\widehat{\nabla}g = 0$, $\widehat{\nabla}\varphi = 0$;
(iii-1) $\widehat{T}(X, Y) = -\eta([X, Y])\xi$, $X, Y \in D$;
(iii-2) $\widehat{T}(\xi, \varphi Y) = -\varphi\widehat{T}(\xi, Y)$, $Y \in D$.

Let $\gamma : I \to M^3$ be a curve parameterized by arc length in M^3. We may define the Frenet frame fields (T, N, B) along γ for the pseudo-Hermitian connection $\widehat{\nabla}$. Then they satisfy the following Frenet–Serret equations for $\widehat{\nabla}$:

$$\begin{cases} \widehat{\nabla}_T T = \widehat{\kappa} N \\ \widehat{\nabla}_T N = -\widehat{\kappa} T + \widehat{\tau} B \\ \widehat{\nabla}_T B = -\widehat{\tau} N \end{cases} \tag{3.5}$$

where $\widehat{\kappa} = |\widehat{\nabla}_T T|$ is the *pseudo-Hermitian curvature* of γ and $\widehat{\tau}$ its *pseudo-Hermitian torsion*. A *pseudo-Hermitian helix* is a curve where both its pseudo-Hermitian curvature and pseudo-Hermitian torsion are constants. In particular, curves with constant nonzero pseudo-Hermitian curvature and zero pseudo-Hermitian torsion are called *pseudo-Hermitian circles*. Note that *pseudo-Hermitian geodesics* are regarded as pseudo-Hermitian helices where both their pseudo-Hermitian curvature and pseudo-Hermitian torsion are zero.

Let M be a contact metric 3-manifold and $\gamma(s)$ a Frenet curve parametrized by the arc length s in M. The *contact angle* $\alpha(s)$ is a function defined by $\cos\alpha(s) = g(T(s), \xi)$. A curve γ is said to be a *slant curve* if its contact angle is constant. Slant curves of contact angle $\pi/2$ are traditionally called *Legendre curves*. The Reeb flow is a slant curve of contact angle zero. Let γ be a nongeodesic Frenet curve in a Sasakian 3-manifold. Differentiating the formula $g(T, \xi) = \cos\alpha$ along γ for the pseudo-Hermitian connection $\widehat{\nabla}$, then it follows that

$$-\alpha' \sin\alpha = g(\widehat{\kappa}N, \xi) + g(T, \widehat{\nabla}_T\xi) = \widehat{\kappa}\,\eta(N).$$

This equation implies the following result.

PROPOSITION 3.2. *A nongeodesic curve* γ *for* $\widehat{\nabla}$ *in a three-dimensional Sasakian manifold* M *is a slant curve if and only if it satisfies* $\eta(N) = 0$.

Hence, we put

$$\xi = \cos\alpha_0 T + \sin\alpha_0 B. \tag{3.6}$$

Differentiating (3.6) along γ and using the Frenet–Serret equations, we obtain

$$(\widehat{\kappa}\cos\alpha_0 - \widehat{\tau}\sin\alpha_0)N = 0. \tag{3.7}$$

This implies that the ratio of $\widehat{\tau}$ and $\widehat{\kappa}$ is a constant. Thus, we obtain the following result.

Proposition 3.3. *If a nongeodesic curve for $\widehat{\nabla}$ in a three-dimensional contact Riemannian manifold is a slant curve, then its ratio of $\widehat{\kappa}$ and $\widehat{\tau}$ is constant.*

Let M be a three-dimensional Sasakian manifold. Then for a curve γ in M, from (3.1) and (3.4) we obtain

$$\widehat{\nabla}_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma + 2\eta(\dot\gamma)\varphi\dot\gamma. \tag{3.8}$$

Equation (3.8) says that a Legendre $\widehat{\nabla}$-geodesic is coincident with a ∇-geodesic. We note that the characteristic vector field ξ is a ∇-geodesic and at the same time a $\widehat{\nabla}$-geodesic. However, in general, $\widehat{\nabla}$-geodesic is not coincident with ∇-geodesic.

Now we return to the Bianchi–Cartan–Vranceanu model space $M = \mathcal{M}^3(H)$, where $H = -3 + 2c$. Let $\gamma = \gamma(s)$ be a curve parametrized by arc length s on Sasakian space form M. From (3.1) and (3.4), the Tanaka–Webster connection $\widehat{\nabla}$ of the Bianchi–Cartan–Vranceanu model space is described as

$$\widehat{\nabla}_{u_1}u_1 = c\,yu_2, \quad \widehat{\nabla}_{u_1}u_2 = -c\,yu_1, \quad \widehat{\nabla}_{u_2}u_1 = -c\,xu_2, \quad \widehat{\nabla}_{u_2}u_2 = c\,xu_1, \tag{3.9}$$

all others are zero.

We put $\gamma'(s) = T(s) = T_1u_1 + T_2u_2 + T_3u_3$. Then by using (3.9) we have the geodesic equation for γ:

$$\widehat{\nabla}_T T = \{T_1' - T_2(cyT_1 - cxT_2)\}u_1 + \{T_2' + T_1(cyT_1 - cxT_2)\}u_2 + T_3'u_3 = 0.$$

Hence, γ is a $\widehat{\nabla}$-geodesic if and only if

$$\begin{cases} T_1' - T_2(cyT_1 - cxT_2) = 0, \\ T_2' + T_1(cyT_1 - cxT_2) = 0, \\ T_3' = 0. \end{cases}$$

We may put $T_1(s) = \sin\alpha(s)\cos\beta(s)$, $T_2(s) = \sin\alpha(s)\sin\beta(s)$, $T_3(s) = \cos\alpha(s)$. Here we call the angle function α of T and ξ the *contact angle* of γ. Then γ is a $\widehat{\nabla}$-geodesic if and only if

$$\begin{cases} \alpha'\cos\alpha\cos\beta - \sin\alpha\sin\beta(\beta' + cy\sin\alpha\cos\beta - cx\sin\alpha\sin\beta) = 0, \\ \alpha'\cos\alpha\sin\beta + \sin\alpha\cos\beta(\beta' + cy\sin\alpha\cos\beta - cx\sin\alpha\sin\beta) = 0, \\ \alpha'\sin\alpha = 0. \end{cases} \tag{3.10}$$

From the third equation in the above, it follows that the contact angle $\alpha = \alpha_0$ is constant. So, we have the following result.

PROPOSITION 3.4. *A $\widehat{\nabla}$-geodesic in a three-dimensional Sasakian space form is a slant curve.*

In the next step, we study the $\widehat{\nabla}$-Jacobi equation for a $\widehat{\nabla}$-geodesic vector field $\widehat{\mathfrak{T}}(\gamma)$. Actually, we investigate the following system of the second-order ordinary differential equations (ODEs) for the Tanaka–Webster connection $\widehat{\nabla}$:

$$\begin{cases} \widehat{\nabla}_{\dot\gamma}\dot\gamma = \widehat{\mathfrak{T}}(\gamma), \\ \widehat{\nabla}^2_{\dot\gamma}\widehat{\mathfrak{T}}(\gamma) + \widehat{\nabla}_{\dot\gamma}\widehat{T}(\widehat{\mathfrak{T}}(\gamma), \dot\gamma) + \widehat{R}(\widehat{\mathfrak{T}}(\gamma), \dot\gamma)\dot\gamma = 0. \end{cases} \tag{3.11}$$

Since $\widehat{\nabla}$ parallelizes the characteristic vector field ξ ($\widehat{\nabla}\xi = 0$) and the metric tensor g ($\widehat{\nabla}g = 0$) we obtain

$$g(\widehat{R}(X, Y)Z, \xi) = g(\widehat{R}(X, Y)\xi, Z) = 0,$$

for any vector fields X, Y and Z. Thus, we have the following result.

LEMMA 3.5. *For a slant curve γ, $\widehat{\nabla}_{\dot\gamma}\dot\gamma$, $\widehat{\nabla}^3_{\dot\gamma}\dot\gamma$ and $\widehat{R}(\widehat{\nabla}_{\dot\gamma}\dot\gamma, \dot\gamma)\dot\gamma$ are all orthogonal to ξ along γ.*

Hence, together with (3.4) and (3.11) we have a system of the $\widehat{\nabla}$-Jacobi equations for $\widehat{\mathfrak{T}}(\gamma)(= \widehat{\nabla}_{\dot\gamma}\dot\gamma)$ along a slant curve γ:

$$\begin{cases} g(\varphi\widehat{\nabla}^2_{\dot\gamma}\dot\gamma, \dot\gamma) = 0, \\ \widehat{\nabla}^3_{\dot\gamma}\dot\gamma + \widehat{R}(\widehat{\nabla}_{\dot\gamma}\dot\gamma, \dot\gamma)\dot\gamma = 0. \end{cases} \tag{3.12}$$

By using (3.5), we calculate

$$\begin{aligned} \widehat{\nabla}^3_T T &= \widehat{\nabla}_T(\widehat{\nabla}_T(\widehat{\nabla}_T T)) \\ &= -3\widehat{\kappa}\widehat{\kappa}'T + (\widehat{\kappa}'' - \widehat{\kappa}^3 - \widehat{\kappa}\widehat{\tau}^2)N + (2\widehat{\tau}\widehat{\kappa}' + \widehat{\kappa}\widehat{\tau}')B. \end{aligned} \tag{3.13}$$

Together with (3.9), we calculate the Tanaka–Webster curvature tensor:

$$\widehat{R}(X, Y)Z = \widehat{\nabla}_X(\widehat{\nabla}_Y Z) - \widehat{\nabla}_Y(\widehat{\nabla}_X Z) - \widehat{\nabla}_{[X,Y]}Z.$$

Then we find that

$$\widehat{R}(u_1, u_2)u_2 = 2cu_1, \quad \widehat{R}(u_1, u_2)u_1 = -2cu_2, \tag{3.14}$$

all others are zero.

By using these relations we compute

$$\widehat{R}(\widehat{\kappa}N, T)T = 2c\widehat{\kappa}B_3(B_3N - N_3B)$$

and

$$\begin{aligned} &\widehat{\nabla}^3_T T + \widehat{R}(\widehat{\kappa}N, T)T \\ &= (-3\widehat{\kappa}\widehat{\kappa}')T + [\widehat{\kappa}'' - \widehat{\kappa}^3 - \widehat{\kappa}\widehat{\tau}^2 + 2c\widehat{\kappa}B_3^2]N + [2\widehat{\tau}\widehat{\kappa}' + \widehat{\kappa}\widehat{\tau}' - 2c\widehat{\kappa}B_3N_3]B \end{aligned}$$

with respect to $\{u_1, u_2, u_3\}$.

Thus, we have the following result.

PROPOSITION 3.6. *Let M be a three-dimensional Sasakian space form and let $\gamma : I \to M$ be a nongeodesic slant curve for $\widehat{\nabla}$ parametrized by arc length, then γ satisfies $\widehat{\nabla}^3_{\dot\gamma}\dot\gamma + \widehat{R}(\widehat{\nabla}_{\dot\gamma}\dot\gamma, \dot\gamma)\dot\gamma = 0$ if and only if γ satisfies*

$$\begin{cases} \widehat{\kappa} = \text{constant} \neq 0, \\ \widehat{\kappa}^2 + \widehat{\tau}^2 = 2c\eta(B)^2, \\ \widehat{\tau}' = 2c\eta(N)\eta(B). \end{cases}$$

From Propositions 3.3 and 3.6, we have the following result.

PROPOSITION 3.7. *Let M be a three-dimensional Sasakian space form and let $\gamma : I \to M$ be a nongeodesic slant curve for $\widehat{\nabla}$ parametrized by arc length, then γ satisfies $\widehat{\nabla}^3_{\dot\gamma}\dot\gamma + \widehat{R}(\widehat{\nabla}_{\dot\gamma}\dot\gamma, \dot\gamma)\dot\gamma = 0$ if and only if γ is a pseudo-Hermitian helix such that $\widehat{\kappa}^2 + \widehat{\tau}^2 = 2c\eta(B)^2$.*

Using (3.5) a direct computation gives

$$\widehat{\nabla}^2_T T = -\widehat{\kappa}^2 T + \widehat{\kappa}' N + \widehat{\kappa}\tau B,$$

and hence, it follows that

$$\begin{aligned} g(\varphi\widehat{\nabla}^2_T T, T) &= -g(\widehat{\kappa}^2, \varphi T) \\ &= g(\widehat{\kappa}^2 T - \widehat{\kappa}' N - \widehat{\kappa}\tau B, \varphi T). \end{aligned} \tag{3.15}$$

However, since γ is a slant curve we may put

$$\begin{aligned} T &= \sin\alpha_0\{\cos\beta(s)e_1 + \sin\beta(s)e_2\} + \cos\alpha_0\xi, \\ N &= -\sin\beta(s)e_1 + \cos\beta(s)e_2 \end{aligned}$$

for any unit vector field $e_1 \perp \xi$. From these and $e_2 = \varphi e_1$, we have the following relation:

$$\varphi T = \sin\alpha_0 N.$$

Then together with (3.15) we obtain

$$g(\varphi\widehat{\nabla}^2_T T, T) = -\widehat{\kappa}' \sin\alpha_0.$$

Thus, we have the following result.

PROPOSITION 3.8. *A slant curve γ in a Sasakian 3-manifold M satisfies $g(\varphi\widehat{\nabla}^2_{\dot\gamma}\dot\gamma, \dot\gamma) = 0$ if and only if $\widehat{\kappa} = \text{constant}$ or γ is a integral curve of ξ.*

In view of (3.12), from Propositions 3.7 and 3.8 we have the following result.

THEOREM 3.9. *Let M be a three-dimensional Sasakian space form and let $\gamma : I \to M$ be a nongeodesic slant curve for $\widehat{\nabla}$ parametrized by arc length, then γ satisfies the $\widehat{\nabla}$-Jacobi equations for a $\widehat{\nabla}$-geodesic vector field $\widehat{\mathfrak{T}}(\gamma)$ if and only if γ is a pseudo-Hermitian helix such that $\widehat{\kappa}^2 + \widehat{\tau}^2 = 2c\eta(B)^2$.*

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COROLLARY 3.10. *Let M be a three-dimensional Sasakian space form with $c \leq 0$. Then there no nongeodesic slant curve for $\widehat{\nabla}$ exists satisfying the $\widehat{\nabla}$-Jacobi equations for a $\widehat{\nabla}$-geodesic vector field.*

The above corollary implies that there no nongeodesic slant curve exists for $\widehat{\nabla}$ satisfying the $\widehat{\nabla}$-Jacobi equations for a $\widehat{\nabla}$-geodesic vector field in the Heisenberg group $\mathbb{H}_3$ or the special linear group $SL_2\mathbb{R}$.

4. Pseudo-Hermitian circles in S^3

In this section, we study a slant curve satisfying $\widehat{\nabla}$-Jacobi equation for a $\widehat{\nabla}$-geodesic vector field $\widehat{\mathfrak{T}}(\gamma)$ in S^3.

First of all, it follows from (3.9) that

$$\widehat{\nabla}_{u_1}u_1 = 2yu_2, \quad \widehat{\nabla}_{u_1}u_2 = -2yu_1, \quad \widehat{\nabla}_{u_2}u_1 = -2xu_2, \quad \widehat{\nabla}_{u_2}u_2 = 2xu_1, \tag{4.1}$$

all others are zero. By using the above data, we obtain

$$\widehat{R}(u_1, u_2)u_1 = -4u_2, \quad \widehat{R}(u_1, u_2)u_2 = 4u_1, \tag{4.2}$$

all others are zero for $i, j, k = 1, 2, 3$. This yields that the unit sphere S^3 has a constant holomorphic sectional curvature 4 for $\widehat{\nabla}$, namely, $L(\widehat{R}(X, \varphi X)\varphi X, X) = 4$ for any unit vector $X \perp \xi$ (see [8]).

Now, let $\gamma : I \to S^3$ be a slant curve parametrized by arc length s. Then we may put

$$T = \sin\alpha_0 \cos\beta(s)u_1 + \sin\alpha_0 \sin\beta(s)u_2 + \cos\alpha_0 u_3.$$

By using (4.1), we calculate

$$\widehat{\nabla}_T T = \sin\alpha_0(\beta' + 2y\sin\alpha_0\cos\beta - 2x\sin\alpha_0\sin\beta)(-\sin\beta u_1 + \cos\beta u_2),$$

and we obtain $\widehat{\kappa} = |\sin\alpha_0(\beta' + 2y\sin\alpha_0\cos\beta - 2x\sin\alpha_0\sin\beta)|$. Since γ is a nongeodesic, we may assume that $\sin\alpha_0(\beta' + 2y\sin\alpha_0\cos\beta - 2x\sin\alpha_0\sin\beta) > 0$ without loss of generality. Then by using the first Frenet equation (for $\widehat{\nabla}$),

$$N = -\sin\beta u_1 + \cos\beta u_2,$$

and

$$B = T \times N = -\cos\alpha_0\cos\beta u_1 - \cos\alpha_0\sin\beta u_2 + \sin\alpha_0 u_3.$$

Furthermore, we calculate

$$\widehat{\nabla}_T N = (\beta' + 2y\sin\alpha_0\cos\beta - 2x\sin\alpha_0\sin\beta)(-\cos\beta u_1 - \sin\beta u_2). \tag{4.3}$$

Applying the second Frenet equation, then

$$\widehat{\tau} = \cos\alpha_0(\beta' + 2y\sin\alpha_0\cos\beta - 2x\sin\alpha_0\sin\beta).$$

It is notable that every Legendre curve in S^3 has a vanishing pseudo-Hermitian torsion.

By Theorem 3.9, we have the following result.

PROPOSITION 4.1. *Let $\gamma : I \to S^3$ be a slant curve parametrized by arc length. Then γ is nongeodesic for $\widehat{\nabla}$ and satisfies the $\widehat{\nabla}$-Jacobi equations for a $\widehat{\nabla}$-geodesic vector field $\widehat{\mathfrak{T}}(\gamma)$ if and only if $\sin\alpha_0 \neq 0$ and*

$$\beta' + 2y\sin\alpha_0\cos\beta - 2x\sin\alpha_0\sin\beta - 2\sin\alpha_0 = 0. \tag{4.4}$$

COROLLARY 4.2. *Let $\gamma : I \to S^3$ be a Legendre curve parametrized by arc length. Then γ is nongeodesic (for $\widehat{\nabla}$) and satisfies (3.11) if and only if γ is a pseudo-Hermitian circle with constant $\widehat{\kappa} = 2$, namely,*

$$\beta' + 2y\cos\beta - 2x\sin\beta = 2.$$

For the rest of the paper, our aim is to obtain explicitly the parametric equations of the above nongeodesic slant curve satisfying (3.11). Let $\gamma(s) = (x(s), y(s), z(s))$ be a curve in S^3. Then the tangent vector field T of γ is represented by

$$T = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) = \frac{dx}{ds}\frac{\partial}{\partial x} + \frac{dy}{ds}\frac{\partial}{\partial y} + \frac{dz}{ds}\frac{\partial}{\partial z}.$$

From this it follows that

$$\frac{dx}{ds} = (1 + x^2 + y^2)T_1, \quad \frac{dy}{ds} = (1 + x^2 + y^2)T_2,$$
$$\frac{dz}{ds} = T_3 - \frac{1}{1 + x^2 + y^2}\left(\frac{dx}{ds}y - x\frac{dy}{ds}\right).$$

Hence, we obtain the following result.

LEMMA 4.3. *Let $\gamma : I \to S^3$ be a slant curve parametrized by arc length s in S^3. Then the system of differential equations for γ is as follows:*

$$\frac{dx}{ds}(s) = \sin\alpha_0\cos\beta(s)(1 + x(s)^2 + y(s)^2), \tag{4.5}$$
$$\frac{dy}{ds}(s) = \sin\alpha_0\sin\beta(s)(1 + x(s)^2 + y(s)^2), \tag{4.6}$$
$$\frac{dz}{ds}(s) = \cos\alpha_0 + \sin\alpha_0\{x(s)\sin\beta(s) - y(s)\cos\beta(s)\}. \tag{4.7}$$

We try to solve the above equations. By virtue of (4.4), the equation (4.7) becomes

$$\frac{dz}{ds} = \frac{1}{2}\beta' + \cos\alpha_0 - \sin\alpha_0.$$

Thus,

$$z(s) = \frac{1}{2}\beta(s) + (\cos\alpha_0 - \sin\alpha_0)s + z_0, \tag{4.8}$$

where z_0 is constant.

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Next, we compute the $x(s)$ and $y(s)$. We put $h(s) = 1 + x(s)^2 + y(s)^2$. Then (4.5) and (4.6) becomes

$$\frac{dx}{ds} = \sin\alpha_0 \cos\beta(s) h(s), \qquad \frac{dy}{ds} = \sin\alpha_0 \sin\beta(s) h(s).$$

Moreover, we easily see that $h(s)$ satisfies the following ODE:

$$\frac{d}{ds} \ln h(s) = 2 \sin\alpha_0 (\cos\beta(s) x(s) + \sin\beta(s) y(s)). \tag{4.9}$$

If $d\beta/ds = 0$, then $(x(s), y(s))$ is a line in the orbit space. Indeed, we have the following parametrization:

$$x(s) = \sin\alpha_0 \cos\beta_0 \int h(s)\, ds, \tag{4.10}$$

$$y(s) = \sin\alpha_0 \sin\beta_0 \int h(s)\, ds, \tag{4.11}$$

where $\beta = \beta_0$ (constant). Since M is homogeneous, we may choose $x_0 = y_0 = 0$. Then the primitive function $\mathcal{H}(s) = \int_0^s h(t)\, dt$ is a solution of

$$\frac{d}{ds}\mathcal{H}(s) = 1 + \sin^2\alpha_0 \mathcal{H}(s)^2.$$

This is a special case of the well-known Riccati equation. However, we can see that no solution $\mathcal{H}(s)$ exists for the above equation. In general, for a model space $M = \mathcal{M}^3(H)$,

$$\mathcal{H}(s) = \sqrt{-\frac{2}{c}} \left| \frac{1}{\sin\alpha_0} \right| + \frac{1}{a \exp(-\sqrt{-2c}|\sin\alpha_0| s) - \sqrt{-(c/8)}|\sin\alpha_0|}, \qquad a \in \mathbb{R},$$

where $H = -3 + 2c$. So, we conclude that β is not constant along γ.

We differentiate (4.4) again and use (4.9), then

$$\frac{d^2}{ds^2}\beta(s) = \frac{d}{ds}\beta(s) \frac{d}{ds} \ln h(s). \tag{4.12}$$

We assume that $\beta' > 0$, and we readily solve (4.12):

$$h(s) = r \frac{d\beta}{ds}(s), \qquad r \in \mathbb{R}^+. \tag{4.13}$$

Then (4.5) and (4.6) are easily solved:

$$\begin{cases} x(s) = r \sin\alpha_0 \sin\beta(s) + x_0, \\ y(s) = -r \sin\alpha_0 \cos\beta(s) + y_0. \end{cases}$$

In this case, the orbit space is then the whole plane $\mathbb{R}^2(x, y)$. The projected curve $\overline{\gamma}(s)$ is a circle $(x - x_0)^2 + (y - y_0)^2 = r^2 \sin^2 \alpha_0$. We may assume $\overline{\gamma}(s)$ is a circle centered at $(0, 0)$. Then since $h(s) = 1 + r^2 \sin^2 \alpha_0$, from (4.13), we have the angle function $\beta(s)$ for $\overline{\gamma}(s)$:

$$\beta(s) = 2 \sin \alpha_0 (r \sin \alpha_0 + 1)s + \beta_0,$$

where β_0 is constant.

Thus, together with (4.8), we have the following result.

THEOREM 4.4. *Let $\gamma : I \to S^3$ be a nongeodesic slant curve for $\widehat{\nabla}$ parametrized by arc length s. Then the parametric equations of γ satisfying* (3.11) *are given by*

$$\begin{cases} x(s) = r \sin \alpha_0 \sin(2 \sin \alpha_0 (r \sin \alpha_0 + 1)s + \beta_0) + x_0, \\ y(s) = -r \sin \alpha_0 \cos(2 \sin \alpha_0 (r \sin \alpha_0 + 1)s + \beta_0) + y_0, \\ z(s) = (r \sin^2 \alpha_0 + \cos \alpha_0)s + z_0, \end{cases}$$

where $r \in \mathbb{R}^+$, and β_0, x_0, y_0 and z_0 are constants.

COROLLARY 4.5. *Let $\gamma : I \to S^3$ be a nongeodesic Legendre curve parametrized by arc length s. Then the parametric equations of pseudo-Hermitian circles satisfying* (3.11) *are given by*

$$\begin{cases} x(s) = r \sin(2(r + 1)s + \beta_0) + x_0, \\ y(s) = -r \cos(2(r + 1)s + \beta_0) + y_0, \\ z(s) = rs + z_0, \end{cases}$$

where $r \in \mathbb{R}^+$, and β_0, x_0, y_0 and z_0 are constants.

We finally remark that no proper (∇-)biharmonic Legendre curve exists in S^3 (see [10, 11]).

References

[1] M. Belkhelfa, F. Dillen and J. Inoguchi, ‘Surfaces with parallel second fundamental form in Binachi–Cartan–Vranceanu spaces’, in: *PDE’s, Submanifolds and Affine Differential Geometry (Warsaw, 2000)*, Banach Center Publications, 57 (Polish Academy of Science, Warsaw, 2002), pp. 67–87.

[2] L. Bianchi, *Lezioni di Geometrie Differenziale* (E. Spoerri Librao-Editore, 1894).

[3] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, 203 (Birkhäuser, Boston, MA, 2002).

[4] R. Caddeo, S. Montaldo and C. Oniciuc, ‘Biharmonic submanifolds of S^3’, *Internat. J. Math.* **12** (2001), 867–876.

[5] R. Caddeo, S. Montaldo and P. Piu, ‘Biharmonic maps’, *Contemp. Math.* **288** (2001), 286–290.

[6] E. Cartan, *Leçons sur la géométrie des espaces de Riemann*, 2nd edn (Gauthier-Villards, Paris, 1946).

[7] B. Y. Chen and S. Ishikawa, ‘Biharmonic surfaces in pseudo-Euclidean spaces’, *Mem. Fac. Kyushu Univ. Ser. A* **45**(2) (1991), 323–347.

[8] J. T. Cho, 'Geometry of contact strongly pseudo-convex CR-maniflods', *J. Korean Math. Soc.* **43**(5) (2006), 1019–1045.

[9] J. T. Cho, J. Inoguchi and J.-E. Lee, 'On slant curves in Sasakian 3-manifolds', *Bull. Austral. Math. Soc.* **74**(3) (2006), 359–367.

[10] ———, 'Biharmonic curves in 3-dimensional Sasakian space form', *Ann. Mat. Pura Appl.* **186** (2007), 685–701.

[11] J. Inoguchi, 'Submanifolds with harmonic mean curvature in contact 3-manifold', *Colloq. Math.* **100**(6) (2004), 163–179.

[12] S. Kobayashi, *Transformation Groups in Differential Geometry*, Ergebnisse der Mathematik und Ihere Grenzgebiete, 70 (Springer, Berlin, 1972).

[13] M. Tamura, 'Gauss maps of surfaces in contact space forms', *Comment. Math. Univ. St. Pauli* **52** (2003), 117–123.

[14] N. Tanaka, 'On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections', *Japan J. Math.* **2** (1976), 131–190.

[15] S. Tanno, 'Variantional problems on contact Riemannian manifolds', *Trans. Amer. Math.Soc.* **314** (1989), 349–379.

[16] G. Vranceanu, *Leçons de géométrie différentielle*, Éditions de l'Académie de la République Populaire Roumaine, Bucharest (1947).

[17] S. M. Webster, 'Pseudohermitian structures on a real hypersurface', *J. Differential Geom.* **13** (1978), 25–41.

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