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IDEMPOTENT MULTIPLIERS OF $H^1(T)$

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1. Introduction. Let as usual $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ be the circle, and H^1 the subspace of $L^1(\mathbf{T})$ of all f such that $\hat{f}(n) = 0$ for all integers $n < 0$. The norm

$$\|f\|_1 = \int_0^{2\pi} |f(t)|dt/2\pi, \quad f \in L^1,$$

restricted to H^1, makes it a Banach space. By a *multiplier* of H^1 we mean a bounded linear operator $m:H^1 \to H^1$ such that there is a sequence $\{c_n\}_{n=0}^{\infty}$ in $\mathbf{C}$ with

$$\widehat{m(f)}(n) = c_n\hat{f}(n) \quad \text{for all } n \geqq 0 \text{ and all } f \in H^1.$$

We use the notation

$$m * f = m(f) \quad \text{and} \quad \hat{m}(n) = c_n.$$

m is called *idempotent* if

$$\hat{m}(n) \in \{0, 1\} \quad \text{for all } n \geqq 0.$$

A measure $\mu \in M(\mathbf{T})$ is called idempotent if

$$\hat{\mu}(n) \in \{0, 1\} \quad \text{for all } n \in \mathbf{Z}.$$

Recall that the mapping $f \mapsto \mu * f =$ convolution of μ and f, $f \in L^1$, defines a multiplier, which restricts to a multiplier m of H^1 such that

$$\hat{m}(n) = \hat{\mu}(n), \quad n \geqq 0.$$

The support (abbreviated supp) of a sequence will mean the set of all indices at which the sequence is not 0. For idempotent measures we have the following characterization.

1.1 ([3]). *A set $E \subset \mathbf{Z}$ is of the form*

$$E = \text{supp } \hat{\mu}$$

for some idempotent $\mu \in M(\mathbf{T}) \Leftrightarrow$

$$(1) \quad E = \left(\bigcup_{i=1}^{N} a_i\mathbf{Z} + b_i\right)/F$$

for some $N \in \mathbf{N}$, a_i, $b_i \in \mathbf{Z}$, $1 \leqq i \leqq N$, and some finite set $F \subset \mathbf{Z}$.

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In this paper we will characterize the sets $E \subset \{n \in \mathbf{Z}: n \geqq 0\}$ of the form

$$E = \operatorname{supp} \hat{m}$$

for some idempotent multiplier m of H^1. We first note that the collection of such sets is closed under finite intersection and complementation in $\{n \geqq 0\}$, and that it includes the intersections of $\{n \geqq 0\}$ with all sets of the form (1) above. It also includes *lacunary sets*: $E = \{n_1 < n_2 < \dots\} \subset \mathbf{N}$ is called lacunary if there exists $q \in \mathbf{R}$, $q > 1$ such that

$$n_{k+1} \geqq q n_k \quad \text{for all } k \geqq 1.$$

This is a consequence of Paley's inequality [**8**]:

$$\left(\sum_{k=1}^{\infty} |\hat{f}(n_k)|^2 \right)^{1/2} \leqq c(q) \|f\|_1, \quad f \in H^1.$$

This means that there is a bounded linear operator $m: H^1 \to H^2 \subset H^1$ such that

$$\widehat{m(f)} = \chi_E \hat{f} \quad \text{for all } f \in H^1.$$

These remarks prove the easy direction ($\Leftarrow$) of the following conjecture of A. Pełczyński.

1.2 *A set* $E \subset \{n \in \mathbf{Z}: n \geqq 0\}$ *is of the form*

$$E = \operatorname{supp} \hat{m}$$

for some idempotent multiplier m of H^1 $\Leftrightarrow$ E is a finite Boolean combination of lacunary sets, finite sets, and sets of the form

$$(a\mathbf{Z} + b) \cap \{n \in \mathbf{Z}: n \geqq 0\}$$

(*i.e., arithmetic sequences*).

2. Proof of 1.2 ($\Rightarrow$). Our first step is to remove the arithmetic sequences from $\operatorname{supp} \hat{m}$ using weak* limits. This idea has appeared before for measures; see for instance [**4**] and [**2**, Chapter 1]. We prove:

2.1 *For some idempotent measure μ, the multiplier m_0 defined by*

(2) $$m_0 * f = m * f - \mu * f, \quad f \in H^1$$

has the gap property: for all $y \geqq 0$ there is $x \geqq 0$ such that

$$[x, x + y] \cap \operatorname{supp} \hat{m}_0 = \emptyset.$$

Proof of 2.1. For each $n \geqq 0$ let K_n denote the Fejér kernel

$$K_n(t) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) e^{ijt}, \quad t \in \mathbf{T}.$$

Recall that $K_n \geqq 0$ and

$$\|K_n\|_1 = \int_0^{2\pi} K_n(t)dt/2\pi = 1$$

for all n. For $n \in \mathbf{Z}$ let γ_n denote the function

$$\gamma_n(t) = e^{int}.$$

Since the functions $\gamma_n K_n$ are in H^1, we may define functions g_n by

$$g_n = \gamma_{-n} m * (\gamma_n K_n), \quad n = 0, 1, \dots.$$

Then

$$\|g_n\|_1 \leqq \|m\| \, \|K_n\|_1 = \|m\| \quad \text{for all } n;$$

hence the sequence $\{g_n dt/2\pi\}_{n=0}^{\infty}$ has a weak* limit point ν in $M(\mathbf{T})$. This implies that, for some increasing sequence $\{n_k\}_{k=1}^{\infty}$ and for all $l \in \mathbf{Z}$,

$$\lim_{k\to\infty} \hat{g}_{n_k}(l) = \hat{\nu}(l).$$

Note that for $|l| \leqq n$ we have

$$\hat{g}_n(l) = \hat{K}_n(l)\hat{m}(n + l) = \left(1 - \frac{|l|}{n+1}\right)\hat{m}(n + l).$$

Now for fixed $l \in \mathbf{Z}$ we eventually have $|l| \leqq n_k$ so that

$$\lim_{k\to\infty} \hat{g}_{n_k}(l) = \lim_{k\to\infty} \left(1 - \frac{|l|}{n_k + 1}\right)\hat{m}(n_k + l) = \lim_{k\to\infty} \hat{m}(n_k + l).$$

Since $\hat{m}(n_k + l) \in \{0, 1\}$, this limit is 0 or 1; hence ν is idempotent. By 1.1, there exist $p \geqq 1$ and $l_0 \geqq 0$ such that

$$\hat{\nu}(l + p) = \hat{\nu}(l), \quad |l| \geqq l_0.$$

Consider the remainders of $\{n_k\}$ modulo p. There must be some r, $0 \leqq r \leqq p - 1$ such that $n_k \equiv r \bmod p$ for infinitely many n_k. Defining

$$d\mu(t) = \gamma_r d\nu(t)$$

satisfies 2.1, as will be verified:

Clearly

$$\hat{\mu}(n) = \hat{\nu}(n - r) \quad \text{for all } n \in \mathbf{Z},$$

and μ is idempotent. Let $y \geqq 0$ be given. For fixed l, $\hat{\nu}(l) = \hat{m}(n_k + l)$ eventually, and thus for all sufficiently large k we have

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(3) $\hat{\nu}(l) = \hat{m}(n_k + l), \quad l = l_0, l_0 + 1, \dots, l_0 + y.$

By the definition of r, there is also some $n_k \equiv r \bmod p$, $n_k \geqq r$ such that (3) holds. For this n_k we also have

$$\hat{\nu}(l) = \hat{\nu}(l + n_k - r) = \hat{\mu}(n_k + l), \quad \text{for all } l \geqq l_0.$$

Hence

$$\hat{m}_0(n) = \hat{m}(n) - \hat{\mu}(n) = 0 \quad \text{for all } n \in [n_k + l_0, n_k + l_0 + y],$$

so we can take $x = n_k + l_0$.

Now observe that by (2),

$$\text{supp } \hat{m} = (\text{supp } \hat{m}_0)\Delta(\{n \geqq 0\} \cap \text{supp } \hat{\mu})$$

where Δ denotes symmetric difference. So, to prove 1.2 ($\Rightarrow$), it remains to show supp $\hat{m}_0$ is a finite union of lacunary and finite sets. This follows by taking $m_1 = m_0$ in 2.2 below.

2.2 *Suppose the multiplier* $m_1:H^1 \to H^1$ *has the gap property* (*see* 2.1) *and*

$$|\hat{m}_1(n)| \geqq 1 \textit{ for all } n \in \text{supp } \hat{m}_1.$$

Then supp $\hat{m}_1$ *is a finite union of lacunary and finite sets.*

We need lower and upper bounds on certain 1-norms:

2.3 ([7]). *There exists* $c > 0$ *such that for any trigonometric polynomial* f *on* $\mathbf{T}$,

$$\|f\|_1 \geqq c \sum_{k=1}^{K} |\hat{f}(n_k)|/k,$$

where $\{n_k\}_{k=1}^K$ *are the elements of* supp $\hat{f}$ *in either strictly increasing or strictly decreasing order.*

2.4 *Suppose* f *is a trigonometric polynomial of the form*

$$f(t) = \sum_{k=1}^{N} c_k e^{ix_k t} K_{y-1}(t)$$

where $y \in \mathbf{N}$, K_n *is the Fejér kernel,* $\{c_k\}_{k=1}^N \subset \mathbf{C}$, *and the integers* $\{x_k\}_{k=1}^N$ *satisfy*

$$x_{k+1} \geqq x_k + y, \quad k = 1, 2, \dots, N - 1.$$

Then

$$\|f\|_1 \leqq \left(\sum_{k=1}^{N} |c_k|^2\right)^{1/2}.$$

Proof of 2.4. Since $K_{y-1} \geqq 0$, the Cauchy-Schwartz inequality gives

$$\left(\int_0^{2\pi} \left|\sum_{k=1}^{N} c_k e^{ix_k t} K_{y-1}(t)\right| dt/2\pi\right)^2$$

$$= \left(\int_0^{2\pi} \left|\sum_{k=1}^{N} c_k e^{ix_k t}\right| \sqrt{K_{y-1}(t)}\ \sqrt{K_{y-1}(t)} dt/2\pi\right)^2$$

$$\leqq \int_0^{2\pi} \left(\sum_{k=1}^{N} c_k e^{ix_k t}\right)\left(\sum_{l=1}^{N} \bar{c}_l e^{-ix_l t}\right) K_{y-1}(t) dt/2\pi.$$

Since

$$K_{y-1}(t) = \sum_{j=-y+1}^{y-1} \left(1 - \frac{|j|}{y}\right) e^{ijt},$$

and since $|j| \leqq y - 1, \quad x_{k+1} - x_k \geqq y$ imply

$$x_k - x_l + j = 0 \Leftrightarrow k = l, j = 0,$$

we see that the last integral equals

$$\sum_{k=1}^{N} |c_k|^2.$$

We will only make use of the case $c_1 = c_2 \ldots = c_N = 1$ of 2.4.

Proof of 2.2.

LEMMA. *There exists $c > 0$ such that for any multiplier $m:H^1 \to H^1$ satisfying*

$$|\hat{m}(n)| \geqq 1 \quad \textit{for all } n \in \operatorname{supp} \hat{m},$$

and for any pair of adjacent intervals in $\mathbf{N}$ *of the form*

$$I = [x, x + y), \quad I' = [x + y, x + 2y)$$

where $x, y \in \mathbf{N}$, $x \geqq y$, the cardinalities

$$A = |I \cap \operatorname{supp} \hat{m}|, \quad A' = |I' \cap \operatorname{supp} \hat{m}|$$

satisfy

$$\left|\log\left(\frac{1 + A}{1 + A'}\right)\right| \leqq c\|m\|. \tag{4}$$

Proof of the lemma. Since $x \geqq y$, the function V defined by

$$V(t) = (e^{ixt} + e^{i(x+y)t})K_{y-1}(t), \quad t \in \mathbf{T},$$

is in H^1. Also $\|V\|_1 \leqq 2$ and

$$\hat{V}(j) = \begin{cases} 1 \text{ for } j \in [x, x + y] \\ 0 \text{ for } j \in [x + 2y, \infty). \end{cases}$$

Therefore

$$(5) \qquad |\widehat{m * V}(j)| = |\hat{m}(j)| |\hat{V}(j)| = \begin{cases} |\hat{m}(j)| \geqq 1 & \text{for } j \in [x, x + y] \cap \text{supp } \hat{m} \\ 0 & \text{for } j \in [x + 2y, \infty). \end{cases}$$

To apply 2.3 to $m * V$, write

$$\sup \widehat{m * V} = \{n_1 > n_2 > \ldots > n_K\},$$

and observe that $n_k \in [x, x + y)$ for $A' < k \leqq A' + A$. Then 2.3 gives

$$\begin{aligned} \|m * V\|_1 &\geqq c \sum_{k=1}^{K} |\widehat{m * V}(n_k)|/k \\ &\geqq c \sum_{k=A'+1}^{A'+A} |\widehat{m * V}(n_k)|/k \\ &= c \sum_{k=A'+1}^{A'+A} |\hat{m}(n_k)|/k \quad \text{(by (5))} \\ &\geqq c \sum_{k=A'+1}^{A'+A} 1/k \\ &\geqq c \log\left(\frac{1 + A' + A}{1 + A'}\right) \\ &\geqq c \log\left(\frac{1 + A}{1 + A'}\right). \end{aligned}$$

Therefore

$$\|m\| \geqq \|m * V\|_1/\|V\|_1 \geqq (c/2) \log\left(\frac{1 + A}{1 + A'}\right).$$

For the case $A < A'$ there is a similar argument using, instead of V, the function $W \in H^1$ defined by

$$W(t) = (e^{i(x+y)t} + e^{i(x+2y)t})K_{y-1}(t), \quad t \in \mathbf{T}.$$

The only change is that we enumerate supp $\widehat{m * W}$ from left to right;

$$\text{supp } \widehat{m * W} = \{n_1 < n_2 < \ldots < n_K\},$$

when applying 2.3.

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Now, from the conclusion (4) of the lemma, we can deduce that there is an integer $p \geqq 2$, depending only on $\|m\|$, such that

(6) $$\frac{1}{p}A' \leqq A \leqq pA' \quad \text{provided } \max(A, A') \geqq p.$$

We let p be this constant for the multiplier $m = m_1$ in what follows. The conclusion of 2.2 is clearly equivalent to the estimate

(7) $$\sup_{y \in \mathbf{N}} |[3y, 6y) \cap \operatorname{supp} \hat{m}_1| < \infty.$$

To obtain (7), fix, if possible, some $y \in \mathbf{N}$ such that

(8) $$|[3y, 6y) \cap \operatorname{supp} \hat{m}_1| \geqq 3p,$$

and let

$$S = |[3y, 6y) \cap \operatorname{supp} \hat{m}_1|.$$

Define $N \in \mathbf{N}$ by

$$3p^N \leqq S < 3p^{N+1}.$$

We claim that there is a sequence

$$\{x_k\}_{k=1}^N \subset \mathbf{N}, \quad \text{with } 3y \leqq x_1 < x_2 < \ldots < x_N,$$

satisfying

(9) $$x_{k+1} \geqq x_k + 3y, \quad k = 1, 2, \ldots, N - 1, \quad \text{and}$$

(10) $$|[x_k, x_k + 3y) \cap \operatorname{supp} \hat{m}_1| = 3p^{N-k+1}, \quad k = 1, 2, \ldots, N.$$

To prove this, let x_1 be the first integer $\geqq 3y$ satisfying (10) with $k = 1$. It clearly exists, since on the one hand, by the gap property, there exists $x \geqq 3y$ with

$$|[x, x + 3y) \cap \operatorname{supp} \hat{m}_1| = 0,$$

and on the other hand

$$|[3y, 6y) \cap \operatorname{supp} \hat{m}| = S \geqq 3p^N,$$

by definition. Inductively, suppose that $1 \leqq n \leqq N - 1$ and that $x_1 < \ldots < x_n$ have been defined and satisfy (9) for $1 \leqq k \leqq n - 1$ and (10) for $1 \leqq k \leqq n$. Consider the adjacent intervals

$$I = [x_n, x_n + 3y), \quad I' = [x_n + 3y, x_n + 6y),$$

and note that by (10) we have

$$|I \cap \operatorname{supp} \hat{m}_1| = 3p^{N-n+1} \geqq 3p^2 \geqq p.$$

Thus (6) applies and gives

$$(11)\quad |I' \cap \operatorname{supp} \hat{m}_1| = A' \geqq \frac{1}{p}A = \frac{1}{p}|I \cap \operatorname{supp} \hat{m}_1| = 3p^{N-(n+1)+1}.$$

So define x_{n+1} to be the first integer $\geqq x_n + 3y$ satisfying (10) with $k = n + 1$. The gap property and (11) again show that x_{n+1} exists, and, by definition we now have (9) for $1 \leqq k \leqq n$ and (10) for $1 \leqq k \leqq n + 1$. By induction, the claim is true.

One more property of the $\{x_k\}$ will be needed. Fix k, $1 \leqq k \leqq N$, and consider the 3 adjacent intervals

$$I = [x_k, x_k + y), \quad I' = [x_k + y, x_k + 2y), \quad I'' = [x_k + 2y, x_k + 3y),$$

whose union is $[x_k, x_k + 3y)$. By (10), we have

$$A + A' + A'' = 3p^{N-k+1}.$$

Suppose $A' < p^{N-k+1}$. Then either

$$A \geqq p^{N-k+1} \quad \text{or} \quad A'' \geqq p^{N-k+1},$$

so by (6) applied to either the pair A, A' or the pair A', A'' we get

$$A' \geqq \frac{1}{p}p^{N-k+1} = p^{N-k}.$$

Therefore

$$(12)\quad |[x_k + y, x_k + 2y) \cap \operatorname{supp} \hat{m}_1| \geqq p^{N-k}, \quad k = 1, 2, \dots, N.$$

To finish the proof of (7), define $f \in H^1$ by

$$f(t) = \sum_{l=1}^{N} (e^{i(x_l+y)t} + e^{i(x_l+2y)t})K_{y-1}(t), \quad t \in \mathbf{T}.$$

By (9) and 2.4, we have $\|f\|_1 \leqq \sqrt{2N}$. As in the proof of the lemma, write

$$\operatorname{supp} \widehat{m_1 * f} = \{n_1 > n_2 > \dots > n_K\} = \bigcup_{l=1}^{N} (x_l, x_l + 3y) \cap \operatorname{supp} \hat{m}_1,$$

and observe that if $n_k \in (x_l, x_l + 3y)$ then

$$k \leqq \sum_{n=l}^{N} |(x_n, x_n + 3y) \cap \operatorname{supp} \hat{m}_1|$$

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$$\leqq \sum_{n=l}^{N} 3p^{N-n+1} \quad \text{(by (10))}$$

$$\leqq 3p^{N-l+2} \quad (\text{since } p \geqq 2).$$

So 2.3 gives

$$\|m_1 * f\|_1 \geqq c \sum_{k=1}^{K} |\widehat{m_1 * f}(n_k)|/k$$

$$\geqq c \sum_{l=1}^{N} \sum_{n_k \in [x_l+y, x_l+2y)} |\widehat{m_1 * f}(n_k)|/k$$

$$= c \sum_{l=1}^{N} \sum_{n_k \in [x_l+y, x_l+2y)} |\hat{m}_1(n_k)|/k$$

(since $\hat{f} = 1$ on $[x_l + y, x_l + 2y]$)

$$\geqq c \sum_{l=1}^{N} |[x_l + y, x_l + 2y) \cap \operatorname{supp} \hat{m}_1|/3p^{N-l+2}$$

$(|\hat{m}_1(n_k)| \geqq 1,\ k \leqq 3p^{N-l+2})$

$$\geqq c \sum_{l=1}^{N} p^{N-l}/3p^{N-l+2} \quad \text{(by (12))}$$

$$= cN/3p^2.$$

Therefore

(13) $$\|m_1\| \geqq \|m_1 * f\|_1/\|f\|_1 \geqq (cN/3p^2)/\sqrt{2N}$$
$$= (c/3\sqrt{2}p^2)\sqrt{N} \geqq c(p)(\log S)^{1/2}.$$

In particular, S is bounded independently of y and this proves (7). Thus the proofs of 2.2 and 1.2 ($\Rightarrow$) are complete.

The sequence (10) was motivated to an extent by a certain "geometric gap theorem" for measures, and by its proof [**1**, Theorem 6]. Since the average length of a gap in $[x_k, x_k + 3y)$ is $\approx 3y/3p^{N-k+1} = yp^{k-N-1}$, the gaps grow geometrically in this sense.

3. Some refinements. Let

$$\bar{H}_0^1 = \{\bar{f}: f \in H^1,\ \hat{f}(0) = 0\}.$$

The result 1.2 also holds for idempotent multipliers

$$m: H^1 \to L^1/\bar{H}_0^1.$$

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In fact all the steps in the proof of 1.2 ($\Rightarrow$) can be adapted to this weaker assumption on m: In the proof of 2.1, change g_n to

$$g_n = \gamma_{-n}(m * (\gamma_n K_n) + h_n),$$

where $h_n \in \bar{H}_0^1$ is such that

$$\|m * (\gamma_n K_n) + h_n\|_1 \leqq 1 + \|m\|_{(H^1, L^1/\bar{H}_0^1)}.$$

This does not affect μ since for each $l \in \mathbf{Z}$,

$$\lim_{n\to\infty} \widehat{\gamma_{-n}h_n}(l) = 0.$$

In the lemma and the proof of 2.2, we only need to check that the lower bounds for $\|m * V\|_1$, $\|m * W\|_1$, and $\|m_1 * f\|_1$ also hold for the norm $\| \ \|_{L^1/\bar{H}_0^1}$. This is clear for $m * V$ and $m_1 * f$, since the $\{n_k\}$ were taken from right to left when applying 2.3. For $\|m * W\|_{L^1/\bar{H}_0^1}$ we use a well-known trick: for any $h \in \bar{H}_0^1$ we can write

$$m * W = V_0 * (m * W + h),$$

where

$$V_0 = (\gamma_x + \gamma_{x+y} + \gamma_{x+2y} + \gamma_{x+3y})K_{y-1}.$$

Therefore,

$$\|m * W\|_1 \leqq \|V_0\|_1 \|m * W\|_{L^1/\bar{H}_0^1} \leqq 4\|m * W\|_{L^1/\bar{H}_0^1}.$$

It may be of interest to remark that 1.1 has a similar refinement: the so-called semi-idempotent theorem **[4]**. One way to state this theorem is that if

$$m: L^1 \to L^1/\bar{H}_0^1$$

is an idempotent multiplier, then

$$\text{supp } \hat{m} = \{n \geqq 0\} \cap E$$

where E is of the form (1).

Our final point is this: To obtain a sequence with properties similar to (9), (10) and (12), one does not really need the lemma or (6). A purely combinatorial argument exists **[6]** for the following fact:

Given any $E \subset \mathbf{N}$, and any pair of intervals of the form

$$I = [x, x + y), \quad I^* = [x^*, x^* + y)$$

where $x, x^*, y \in \mathbf{N}$, $x \geqq 2y$, $x^* \geqq x + y$, let

$$A = |I \cap E|, \quad A^* = |I^* \cap E|,$$

and suppose $A > A^*$. Then there is a sequence of integers $x - y \leqq x_1 < x_2 < \ldots < x_N \leqq x^* - y$ such that:

$$x_{l+1} \geqq x_l + y \quad l = 1, 2, \dots, N - 1,$$

$$N \leqq c_1 \log\left(\frac{1 + A}{1 + A^*}\right) + c_2,$$

and such that, if

$$F = \bigcup_{l=1}^{N} (x_l - y, x_l + 2y) \quad \text{and}$$

$$F \cap E = \{n_1 > n_2 > \dots > n_K\},$$

then

$$\sum_{l=1}^{N} \sum_{n_k \in [x_l, x_l + y)} 1/k \geqq c_3 \log\left(\frac{1 + A}{1 + A^*}\right) + c_4,$$

where $c_1 > 0$, $c_3 > 0$, c_2, c_4 *are absolute constants.*

Applying this with $E = \text{supp } \hat{m}$, where the multiplier $m:H^1 \to H^1$ satisfies $|\hat{m}(n)| \geqq 1$, $n \in \text{supp } \hat{m}$, and then considering a test function

$$f = \sum_{l=1}^{N} (\gamma_{x_l} + \gamma_{x_l + y}) K_{y-1}$$

as before, one gets

$$(13)^* \quad \|m\| \geqq c_5\left(\log\left(\frac{1 + A}{1 + A^*}\right)\right)^{1/2}.$$

If m has the gap property, one can choose I^* such that $A^* = 0$ and thus retrieve (13), with the improvement that c_5 is an absolute constant, whereas $c(p)$ depends on $\|m\|$ (via the lemma and (6)). In view of the Littlewood conjecture, one may ask whether the 1/2 can be removed or improved when $A^* = 0$; this is left open.

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