ON COVERING SYSTEMS

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1. Introduction. Differentiation of a set function μ with respect to another ν at a point x involves taking the limit of the ratio $\mu A/\nu A$ as A "converges" to x in some sense which requires that A belong to some family of sets $N(x)$ (e.g. spheres with centre x). For the development of a reasonable theory of differentiation certain restrictions must be placed on the families $N(x)$. The best-known restriction is that they form a Vitali system. However, other systems have been considered.

In this paper we study the relationships between three systems: Vitali systems, which we call V-systems; a modification of the systems having property (V) introduced by Sion in **(4)**, which we call S-systems; and a modification of the tile systems studied in Hahn and Rosenthal **(2)**, which we call T-systems. The main difference between systems having property (V) and S-systems is that sets in the latter are not required to be open. The main difference between tile systems and T-systems is that sets in the first are assumed to be measurable whereas in the latter they need not be.

The main results in Section 3 state that V-systems are always S-systems; under certain conditions, V-systems are T-systems; under more stringent conditions, S-systems are T-systems. We then show that the converses, in general, do not hold.

In Section 4, we prove that for T-systems, measurable functions are approximately continuous and apply this result to obtain a density theorem. This generalizes similar results for tile systems and parallels similar results for systems having property (V).

2. Notation and terminology. The following notation and terminology will be used throughout this paper:

(1) ω is the set of all integers greater than zero.

(2) $(A \sim B) = \{x : x \in A, x \notin B\}$.

(3) $\sigma F = \bigcup_{\alpha \in F} \alpha$.

(4) μ is a (outer) measure on X if and only if μ is a function defined on all subsets of X, $\mu 0 = 0$, and

$$0 \leqslant \mu A \leqslant \sum_{n \in \omega} \mu B_n \text{ whenever } A \subset \bigcup_{n \in \omega} B_n \subset X.$$

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(5) For μ a measure on X, a set A is μ-measurable if and only if, for every $B \subset X$, $\mu B = \mu(B \cap A) + \mu(B \sim A)$.

(6) $f^{-1}V = \{x : f(x) \in V\}$.

(7) For μ a measure on X, a function f on X to a topological space Y is μ-measurable if and only if, for every open $V \subset Y$, $f^{-1}V$ is a μ-measurable set.

3. Covering systems. In this section X is a topological space and μ is a measure on X. In the next set of definitions we introduce the three types of covering systems that will be compared.

DEFINITIONS 3.1.

3.1.1. N is an arbitrarily fine system if and only if N is a function on X such that, for every $x \in X$, $N(x)$ is a family of sets and for every neighbourhood U of x there exists $\alpha \in N(x)$ with $\alpha \subset U$.

3.1.2. For every $A \subset X$, $\bar{N}(A)$ is the collection of all families F such that

(i) $F \subset \bigcup_{x \in A} N(x)$,

(ii) for every $x \in A$ and neighbourhood U of x there exists $\alpha \in F \cap N(x)$ with $\alpha \subset U$.

3.1.3. N is a V-system for μ if and only if N is an arbitrarily fine system and for every $A \subset X$ and $F \in \bar{N}(A)$ there exists a countable disjoint family F' of μ-measurable sets such that $F' \subset F$ and $\mu(A \sim \sigma F') = 0$.

3.1.4. N is an S-system for μ with factor λ if and only if N is an arbitrarily fine system, $1 \leqslant \lambda < \infty$, and for every $A \subset X$ and $F \in \bar{N}(A)$ there exists a countable family $F' \subset F$ such that $\mu(A \sim \sigma F') = 0$ and for every $B \subset \sigma F'$

$$\sum_{\alpha \in F'} \mu(\alpha \cap B) \leqslant \lambda \cdot \mu B.$$

3.1.5. N is a T-system for μ if and only if N is an arbitrarily fine system and for every $A \subset X$, $F \in \bar{N}(A)$, and $\epsilon > 0$ there exists a countable family $F' \subset F$ such that $\mu(A \sim \sigma F') = 0$ and

$$\sum_{\alpha \in F'} \mu\alpha \leqslant \mu A + \epsilon.$$

The next set of definitions introduces two conditions on the measure μ. These will be used in the hypotheses of theorems later on.

DEFINITIONS 3.2.

3.2.1. μ satisfies H_1 if and only if $\mu X < \infty$ and for every $\epsilon > 0$ and $A \subset X$ there exists an open set $A' \supset A$ with $\mu A' < \mu A + \epsilon$.

3.2.2. μ satisfies H_2 if and only if $\mu X < \infty$ and for every $A \subset X$ there exists a μ-measurable set $A' \supset A$ with $\mu A' = \mu A$.

The main relations between the various covering systems are listed below. The proofs and counter-examples follow the statement of the theorems.

Theorem 3.4 can be found in **(2**, p. 269**)** but is included here for the sake of completeness.

THEOREM 3.3. *If N is a* V-*system for μ, then N is an* S-*system for μ with factor* 1.

THEOREM 3.4. *If μ satisfies H_1 and N is a* V-*system for μ, then N is a* T-*system for μ.*

THEOREM 3.5. *If μ satisfies H_1 and N is an* S-*system for μ with factor λ and, for every $x \in X$, $N(x)$ consists of μ-measurable sets, then N is a* T-*system for μ.*

Remark 3.6. The converses of the above theorems do not hold in general, even when μ satisfies H_1, open sets are μ-measurable, and the $N(x)$ consist of open sets, as is shown by the examples 3.6.1. and 3.6.2. below.

Proofs and counterexamples. Theorem 3.3 follows immediately from the fact that for any countable disjoint family F of μ-measurable sets and any $B \subset \sigma F$, $\mu B = \sum_{\alpha \epsilon F} \mu(\alpha \cap B)$.

Proof of Theorem 3.4. Let N be a V-system for μ, $A \subset X$, $F \in \bar{N}(A)$, and $\epsilon > 0$. Then since μ satisfies H_1, there exists an open set $B \supset A$ such that $\mu B < \mu A + \epsilon$.

Let $F' = \{\alpha : \alpha \in F \text{ and } \alpha \subset B\}$. Then $F' \in \bar{N}(A)$ and there exists a disjoint countable family G of μ-measurable sets such that $G \subset F'$ and $\mu(A \sim \sigma G) = 0$. Since the elements of G are μ-measurable

$$\sum_{\alpha \epsilon G} \mu\alpha = \mu\sigma G \leqslant \mu B < \mu A + \epsilon.$$

Therefore N is a T-system for μ.

To prove Theorem 3.5 we need the following lemma.

LEMMA. *Let $A \subset X$, $1 \leqslant \lambda < \infty$ and $\mu X < \infty$. If F is a countable family of μ-measurable sets such that $\mu(A \sim \sigma F) = 0$, and for any $B \subset \sigma F$*

$$\sum_{\alpha \epsilon F} \mu(\alpha \cap B) \leqslant \lambda \cdot \mu B,$$

then for any $k > 1$ there exists a subfamily $G \subset F$ such that

$$\sum_{\alpha \epsilon G} \mu\alpha < \frac{k}{k-1} \mu\sigma G \quad \textit{and} \quad \mu\sigma G \geqslant \frac{\mu A}{k\lambda}.$$

Proof. Let the elements of F be ordered, i.e. let $F = \{\alpha_1, \alpha_2, \ldots\}$. Let $G = \{\alpha_{i_1}, \alpha_{i_2}, \ldots\} \subset F$ where the i_n are defined by recursion as follows: $i_1 = 1$ and for $n \in \omega$, i_{n+1} is the smallest $j \in \omega$, if any, such that $j > i_n$ and

$$\mu\left(\bigcup_{m=1}^{n} \alpha_{i_m} \cap \alpha_j\right) < \frac{\mu\alpha_j}{k}.$$

Then

$$\sigma G = \alpha_{i_1} \cup \bigcup_{n \in \omega} \left(\alpha_{i_{n+1}} \sim \bigcup_{j=1}^{n} \alpha_{i_j} \right)$$

and

$$\mu\sigma G > \mu\alpha_{i_1} + \sum_{n \in \omega} \frac{k-1}{k} \mu\alpha_{i_{n+1}} > \frac{k-1}{k} \sum_{n \in \omega} \mu\alpha_{i_n}.$$

Therefore

$$\sum_{\alpha \in G} \mu\alpha < \frac{k}{k-1} \mu\sigma G.$$

Also, since $\sigma G \subset \sigma F$,

$$\sum_{\alpha \in G} \mu\alpha + \sum_{\alpha \in F \sim G} \mu(\alpha \cap \sigma G) = \sum_{\alpha \in F} \mu(\alpha \cap \sigma G) \leqslant \lambda\mu\sigma G.$$

And, since

$$\mu\sigma G \leqslant \sum_{\alpha \in G} \mu\alpha,$$

$$\sum_{\alpha \in F \sim G} \mu(\alpha \cap \sigma G) \leqslant (\lambda - 1)\mu\sigma G.$$

But, for $\alpha \in F \sim G$, $\mu(\alpha \cap \sigma G) \geqslant \mu\alpha/k$. Therefore

$$\sum_{\alpha \in F \sim G} \mu\alpha \leqslant \sum_{\alpha \in F \sim G} k\mu(\alpha \cap \sigma G) \leqslant k(\lambda - 1)\mu\sigma G.$$

Thus

$$\mu A \leqslant \mu\sigma F \leqslant \sum_{\alpha \in F \sim G} \mu\alpha + \mu\sigma G \leqslant [k(\lambda - 1) + 1]\mu\sigma G \leqslant k\lambda\mu\sigma G,$$

and

$$\mu\sigma G \geqslant \mu A/k\lambda.$$

Proof of Theorem 3.5. Let μ and N satisfy the hypotheses of 3.5. Let $C \subset X$, $\mu C > 0$, and $F \in \bar{N}(C)$. For any $\epsilon > 0$ choose $k > 3$, $k > \epsilon$ such that $\mu C/(k-1) < \epsilon/10$, and define A_n, δ_n, A_n', F_n, and G_n by recursion so that $A_1 = C$ and for any $n \in \omega$:

$$\delta_n = \min\left(\frac{\epsilon}{10 \cdot 2^n}, \frac{\mu A_n}{2k\lambda}\right);$$

A_n' is an open set, $A_n \subset A_n'$, $\mu A_n' < \mu A_n + \delta_n$;

$F_n \subset F$, F_n is countable, $\sigma F_n \subset A_n'$, $\mu(A_n \sim \sigma F_n) = 0$,

and for any $B \subset \sigma F_n$,

$$\sum_{\alpha \in F_n} \mu(\alpha \cap B) \leqslant \lambda\mu B$$

(F_n exists since N is an S-system);

$$G_n \subset F_n, \quad \mu\sigma G_n \geqslant \frac{\mu A_n}{k\lambda}, \quad \text{and} \quad \sum_{\alpha \in G_n} \mu\alpha \leqslant \frac{k}{k-1} \mu\sigma G_n,$$

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(G_n exists by the previous lemma);

$$A_{n+1} = A_n \sim \sigma G_n;$$
$$A'_{n+1} \subset A_n'.$$

Then

$$\mu A_{n+1} \leqslant \mu(A_n' \sim \sigma G_n) = \mu A_n' - \mu\sigma G_n$$
$$\leqslant \left(1 + \frac{1}{2k\lambda}\right)\mu A_n - \frac{\mu A_n}{k\lambda} = \left(1 - \frac{1}{2k\lambda}\right)\mu A_n.$$

By induction,

$$\mu A_{n+1} \leqslant \left(1 - \frac{1}{2k\lambda}\right)^n \mu A_1.$$

Let M be so large that

$$\left(1 - \frac{1}{2k\lambda}\right)^{M-1} \mu A_1 < \frac{\epsilon}{3k\lambda}.$$

Then $\mu A_M < \epsilon/3k\lambda$ and

$$\mu\sigma F_M \leqslant \mu A_M' < \left(1 + \frac{1}{2k\lambda}\right)\frac{\epsilon}{3k\lambda} < \frac{\epsilon}{k\lambda}.$$

Therefore

$$\sum_{\alpha \epsilon F_M} \mu\alpha \leqslant \lambda\mu\sigma F_M < \frac{\epsilon}{k}.$$

Now, since for each $n \in \omega$

$$\mu A_n' \geqslant \mu A_{n+1} + \mu\sigma G_n$$

and $\mu A_n \geqslant \mu A_n' - \delta_n$, we have

$$\mu C \geqslant \mu A_1' - \delta_1 \geqslant \mu A_2 + \mu\sigma G_1 - \delta_1$$
$$\geqslant \mu A_3 + \mu\sigma G_2 + \mu\sigma G_1 - (\delta_1 + \delta_2).$$

By induction,

$$\mu C \geqslant \mu A_{n+1} + \sum_{i=1}^{n} \mu\sigma G_i - \sum_{i=1}^{n} \delta_i,$$

and as $n \to \infty$,

$$\mu A_{n+1} \leqslant \left(1 - \frac{1}{2k\lambda}\right)^n \mu A_1 \to 0,$$

and we have

$$\mu C \geqslant \sum_{i \epsilon \omega} \mu\sigma G_i - \sum_{i \epsilon \omega} \delta_i \geqslant \sum_{i \epsilon \omega} \mu\sigma G_i - \frac{\epsilon}{10}.$$

Let

$$H = \bigcup_{i \epsilon \omega} G_i \cup F_M.$$

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Then H is countable, $H \subset F$, and $\mu(C \sim \sigma H) = 0$. Furthermore,

$$\sum_{\alpha \epsilon H} \mu\alpha = \sum_{i \epsilon \omega} \sum_{\alpha \epsilon G_i} \mu\alpha + \sum_{\alpha \epsilon F_M} \mu\alpha \leqslant \sum_{i \epsilon \omega} \frac{k}{k-1} \mu\sigma G_i + \frac{\epsilon}{k}$$
$$\leqslant \frac{k}{k-1}\left(\mu C + \frac{\epsilon}{10}\right) + \frac{\epsilon}{k}$$
$$= \mu C + \frac{\mu C}{k-1} + \frac{k\epsilon}{10(k-1)} + \frac{\epsilon}{k}$$
$$< \mu C + \frac{\epsilon}{10} + \frac{\epsilon}{5} + \frac{\epsilon}{3} < \mu C + \epsilon.$$

Thus, N is a T-system for μ.

The following example shows that the converse of Theorem 3.2 does not hold in general although μ satisfies both H_1 and H_2.

Example 3.6.1. Let X be the interval (0, 1) with the usual topology. Let the rationals in (0, 1) be ordered $r_1, r_2, \ldots$, and let μ be such that $\mu\{r_i\} = 1/2^i$ and

$$\mu\big((0, 1) \sim \{r_1, r_2, \ldots\}\big) = 0.$$

For $x \neq 1/2$, let $N(x)$ be the family of all open intervals which have irrational end-points, contain $\{x\}$, and do not contain $\{1/2\}$. Let $N(1/2)$ consist of all open intervals in X which contain $\{1/2\}$ and have rational end-points. Then

(1) N is an S-system for μ with factor 2.

Proof. Let $A \subset X$, $F \in \bar{N}(A)$, and $C = A \sim \{1/2\}$. Since the intervals in $N(x)$, $x \neq 1/2$, have irrational end-points, we can define, by recursion, disjoint elements $\alpha_1, \alpha_2, \ldots$ of F such that

$$\mu\Big(C \sim \bigcup_{i \epsilon \omega} \alpha_i\Big) = 0.$$

If $1/2 \in A$, take $\alpha_0 \in F$ with $1/2 \in \alpha_0$, and if $1/2 \notin A$, take $\alpha_0 = \alpha_1$. Let $F' = \{\alpha_0, \alpha_1, \ldots\}$. Then $\mu(A \sim \sigma F') = 0$, and for any $B \subset \sigma F'$

$$\sum_{\beta \epsilon F'} \mu(B \cap \beta) = \sum_{i \epsilon \omega} \mu(B \cap \alpha_i) + \mu(B \cap \alpha_0) \leqslant 2\mu B.$$

(2) N is not a V-system for μ.

Proof. Let $A = X$, $F = \bigcup_{x \epsilon X} N(x)$. Then any countable covering $F' \subset F$ of the rationals must contain some $\alpha \in N(1/2)$. Since the end-points of α are rational, they must be covered by elements of F'. However, this implies that the elements of F' are not disjoint.

The next example shows that the converse of Theorem 3.5 does not hold in general.

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Example 3.6.2. The construction in this example was used by Banach **(1)** to show that the family of all open rectangles is not a Vitali system for Lebesgue measure on the plane. It will be used here to show that this class of sets does not form an S-system for Lebesgue measure on the plane, although it is known to be a tile-system **(2)**.

Let Q be the open unit square $(0, 1) \times (0, 1)$, and μ be Lebesgue measure on Q.

We first observe that if for $\nu \in \omega$, $C_\nu > 0$, $m \in \omega$, and $a = (a_1, a_2) \in Q$, we let

$$W_m{}^\nu(a) = \{(x, y) : 0 \leqslant x - a_1 \leqslant 1/m, 0 \leqslant y - a_2 \leqslant 1/m,$$
$$= (x - a_1)(y - a_2) \leqslant e^{-C_\nu}/m^2\},$$
$$F_\nu = \{\alpha : \alpha = W_m{}^\nu(a) \subset Q \text{ for some } a \in Q, m \geqslant \nu\},$$

then we can easily check **(**see **1)** that

$$\mu W_m{}^\nu(a) = \{e^{-C_\nu}(C_\nu + 1)\}/m^2$$

so that F_ν satisfies the hypotheses of the Vitali Covering Theorem **(2**, p. 265**)** in the plane. Hence there exists a countable disjoint subfamily $F' \subset F$ such that

$$\mu(Q \sim \sigma F_\nu') = 0.$$

For each $x \in Q$, let $N(x)$ be the family of open rectangles α such that $x \in \alpha \subset Q$. It is known that N is a T-system for μ **(2**, p. 284**)**. We shall show that it is not an S-system for μ for any factor λ.

Suppose N is an S-system for μ with factor λ. Let $n \in \omega$ and for each $\nu \in \omega$ let $C_\nu > 0$ and such that

$$\sum_{\nu \in \omega} \frac{4}{C_\nu + 1} < \frac{1}{n} \leqslant \frac{1}{\lambda}.$$

Taking $W_m{}^\nu(a)$, F_ν, F_ν' as above, let

$$P = \bigcap_{\nu \in \omega} \sigma F_\nu'.$$

Then $P \subset Q$ and $\mu(Q \sim P) = 0$, so that $\mu P = 1$.

Given $x \in P$, $\nu \in \omega$, there exists exactly one $\alpha \in F_\nu'$ with $x \in \alpha$. Let $p_\nu(x) \in Q$ and $m_\nu(x) \in \omega$, $m_\nu(x) \geqslant \nu$ be such that $\alpha = W_{m_\nu(x)}\,(p_\nu(x))$. Let $B_\nu(x)$ be the open rectangle with centre x and $p_\nu(x)$ as a vertex, and let

$$G = \{\alpha : \alpha = B_\nu(x) \text{ for some } x \in P \text{ and } \nu \in \omega\}.$$

Then $G \in \bar{N}(P)$ so that there exists a countable $G' \subset G$ such that $\mu(P \sim \sigma G') = 0$ and for any $B \subset \sigma G'$,

$$\sum_{\alpha \in G'} \mu(\alpha \cap B) \leqslant \lambda \mu B.$$

Recalling that $n \in \omega$ and $n \geqslant \lambda$, we see that for a fixed $\nu \in \omega$, no more than

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n elements $B_\nu(x)$ all having the same vertex $p_\nu(x)$ can be in G', for if there were, the intersection D of $n+1$ of them would be open, $\mu D > 0$ and

$$\sum_{\alpha\in G'} \mu(\alpha \cap D) \geqslant (n+1)\mu D > \lambda\mu D.$$

Also one can check that

$$\mu B_\nu(x) \leqslant \frac{4e^{-C_\nu}}{(m_\nu(x))^2} = \frac{4}{C_\nu + 1}\mu W_{m_\nu(x)}(p_\nu(x)).$$

Let

$$H_\nu = \{\alpha : \alpha = B_\nu(x) \in G' \text{ for some } x \in P\}.$$

Then

$$\sum_{\alpha\in H_\nu} \mu\alpha \leqslant \frac{4n}{C_\nu + 1}\sum_{\alpha\in F_\nu'} \mu\alpha = \frac{4n}{C_\nu + 1},$$

and

$$\sum_{\alpha\in G'} \mu\alpha = \sum_{\nu\in\omega}\ \sum_{\alpha\in H\nu} \mu\alpha \leqslant \sum_{\nu\in\omega} \frac{4n}{C_\nu + 1} < 1.$$

Therefore $\mu(P \sim \sigma G') > 0$, contradicting the choice of G'. Thus, N cannot be an S-system for any λ.

4. Approximate continuity and density. Throughout this section we assume X and Y are topological spaces, f is a function on X to Y, N is an arbitrarily fine system for X, and μ is a measure on X.

DEFINITION 4.1. *f is (μ, N)-continuous at x if and only if for every $\epsilon > 0$ and neighbourhood V of $f(x)$ there exists a neighbourhood U of x such that for every $W \in N(x)$ with $W \subset U$ we have $\mu(W \sim f^{-1}V) \leqslant \epsilon\cdot\mu W$.*

It has been shown that if f is a μ-measurable function and the range of f has a countable base, then for μ almost all x, f is (μ, N)-continuous at x provided either

(i) N is a tile system for μ and μ satisfies H_1 and H_2, and f is real-valued **(2**, p, 288**)**, or

(ii) N is a system having property (V) for μ and μ satisfies H_1 **(4**, Theorem 3.8**)**.

The proof under assumption (i) given in **(2)** makes use of density theorems. We follow the direct method used in **(4)** and get the same conclusion provided

(iii) N is a T-system for μ and μ satisfies H_1 and H_2.

For the proof of this theorem we need the following two results.

THEOREM 4.2. *If μ satisfies H_1, f is μ-measurable, $\epsilon > 0$, and Y has a countable base, then there exists $C \subset X$ such that $\mu(X \sim C) < \epsilon$ and f is continuous on C.*

Proof. See **(4**, Theorem 3.5**)**.

LEMMA 4.3. *If μ satisfies H_2, F is countable, $A \subset X$, $\epsilon > 0$, $\mu(A \sim \sigma F) = 0$, and*

$$\sum_{W \in F} \mu W \leqslant \mu A + \epsilon,$$

then for every $B \subset X$

$$\sum_{W \in F} \mu(B \cap W) \leqslant \mu B + \epsilon.$$

Proof. Let A, ϵ, and F satisfy the hypothesis of the lemma. For $B \subset X$ let B' be a μ-measurable set such that $B \subset B'$ and $\mu B' = \mu B$. If

$$\sum_{W \in F} \mu(W \cap B') > \mu B' + \epsilon,$$

then

$$\begin{aligned}\sum_{W \in F} \mu W &= \sum_{W \in F} \mu(W \cap B') + \sum_{W \in F} \mu(W \sim B') \\ &> \mu B' + \epsilon + \mu(\sigma F \sim B') \\ &\geqslant \mu(\sigma F \cap B') + \mu(\sigma F \sim B') + \epsilon \\ &= \mu \sigma F + \epsilon \geqslant \mu A + \epsilon,\end{aligned}$$

which contradicts the assumptions. Thus,

$$\begin{aligned}\sum_{W \in F} \mu(B \cap W) &\leqslant \sum_{W \in F} \mu(B' \cap W) \leqslant \mu B' + \epsilon \\ &= \mu B + \epsilon.\end{aligned}$$

THEOREM 4.4. *If μ satisfies H_1 and H_2, N is a* T-*system for μ, Y has a countable base, and f is a μ-measurable function, then f is (μ, N)-continuous at x for μ almost all x.*

Proof. For each $n \in \omega$, let $A_n = \{x \in X :$ there exists a neighbourhood V of $f(x)$ such that for any open U with $x \in U$ there exists $W \in N(x)$ with $W \subset U$ and $\mu(W \sim f^{-1}V) > (1/n)\mu W\}$. Then $\{x \in X : f$ is not (μ, N)-continuous at $x\} = \bigcup_{n \in \omega} A_n$ and we need only show that $\mu A_n = 0$ for all $n \in \omega$.

Given $n \in \omega$ and $\epsilon > 0$, using 4.2, let $C \subset X$, $\mu(X \sim C) < \epsilon$, f be continuous on C, and $A' = A_n \cap C$. For each $x \in A'$, let V_x be a neighbourhood of $f(x)$ such that for any open U with $x \in U$ there exists $W \in N(x)$ with $W \subset U$ and $\mu(W \sim f^{-1}V_x) > (1/n)\mu W$. Since f is continuous on C, let U_x be a neighbourhood such that $C \cap U_x \subset f^{-1}V_x$ and let

$$F = \{W : \text{for some } x \in A', W \in N(x), W \subset U_x, \text{and } \mu(W \sim f^{-1}V_x) > (1/n)\mu W\}.$$

Then $F \in \bar{N}(A')$.

Since N is a T-system, there exists a countable $F' \subset F$ such that $\mu(A' \sim \sigma F') = 0$ and $\sum_{W \in F'} \mu W < \mu A' + \epsilon$. Let $D = \sigma F' \sim C$. Then $\mu D < \epsilon$ and for every $W \in F'$ there is an $x \in A'$ with $W \subset U_x$ so that

$$D \cap W = W \sim C = W \sim (C \cap U_x) \supset W \sim f^{-1}V_x.$$

Therefore, $\mu(D \cap W) \geqslant \mu(W \sim f^{-1}V_x) > (1/n)\mu W$, and by Lemma 4.3

$$2\epsilon \geqslant \mu D + \epsilon \geqslant \sum_{W \in F'} \mu(D \cap W) > \frac{1}{n} \sum_{W \in F'} \mu W$$

$$\geqslant \frac{1}{n} \mu\sigma F' \geqslant \frac{1}{n} \mu A'.$$

Therefore,

$$\mu A_n \leqslant \mu(A_n \cap C) + \mu(A_n \sim C) \leqslant 2n\epsilon + \epsilon.$$

Since ϵ was arbitrary, $\mu A_n = 0$.

THEOREM 4.5. *If N is a T-system for μ, μ satisfies H_1 and H_2, and A is a μ-measurable set, then for μ almost all $x \in X \sim A$,*

$$\lim_{W \in N(x)} \frac{\mu(A \cap W)}{\mu W} = 0.$$

Proof. Let f be the characteristic function of A. By Theorem 4.4 f is (μ, N)-continuous at x for μ almost all $x \in X$. Let

$$B = \{x \in X \sim A : f \text{ is } (\mu, N)\text{-continuous at } x\}.$$

Let $\epsilon > 0$ and V be a neighbourhood of 0 which excludes 1. Then for every $x \in B$ there exists a neighbourhood U of x such that for every $W \in N(x)$ with $W \subset U$,

$$\mu(W \cap A) = \mu(W \sim f^{-1}V) \leqslant \epsilon \cdot \mu W.$$

Since N is a T-system,

$$\mu\{x \in X : \mu W = 0 \text{ for some } W \in N(x)\} = 0.$$

Therefore, for μ almost all $x \in B$, and therefore for μ almost all $x \in X \sim A$,

$$\frac{\mu(W \cap A)}{\mu W} \leqslant \epsilon.$$

Since $\epsilon > 0$ is arbitrary, for μ almost all $x \in X \sim A$,

$$\lim_{W \in N(x)} \frac{\mu(A \cap W)}{\mu W} = 0.$$

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