ON DIFFERENCES OF UNITARILY EQUIVALENT SELF-ADJOINT OPERATORS†

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1. All operators considered in this paper are bounded operators on a Hilbert space. In case A and B are self-adjoint, certain conditions on A, B and their difference

$$H = A - B, \quad \text{...(1)}$$

assuring the unitary equivalence of A and B,

$$B = U^*AU, \quad \text{...(2)}$$

have recently been obtained by Rosenblum [**6**] and Kato [**2**]. The present paper will consider the problem of investigating consequences of an assumed relation of type (2) for some unitary U together with an additional hypothesis that the difference H of (1) be non-negative, so that

$$H = A - B \geqslant 0. \quad \text{...(3)}$$

First, it is easy to see that if only (2) and (3) are assumed, thereby allowing $H = 0$, relation (2) can hold for A arbitrary with $U = I$ (identity) and $B = A$. If $H = 0$ in (3) is not allowed, however (an impossible assumption in the finite dimensional case, incidentally, since then the trace of H is zero and hence $H = 0$), it will be shown, among other things, that any unitary operator U for which (2) and (3) hold must have a spectrum with a positive measure (as a consequence of (i) of Theorem 2 below). Moreover A (hence B) cannot differ from a completely continuous operator by a constant multiple of the identity (Theorem 1). In case 0 is not in the point spectrum of H, then U is even absolutely continuous (see (iv) of Theorem 2). In § 4, applications to semi-normal operators will be given.

Let U be any unitary operator with the spectral resolution

$$U = \int e^{i\lambda}\, dE(\lambda) \qquad \left(\int = \int_0^{2\pi}\right). \quad \text{..............................(4)}$$

Let $\{e^{i\lambda_n}\}$, $0 \leqslant \lambda_n < 2\pi$, denote the point spectrum (if any) of U and put

$$E_c(\lambda) = E(\lambda) - \sum_{\lambda_n < \lambda} \{E(\lambda_n + 0) - E(\lambda_n - 0)\}.$$

Then the $E_c(\lambda)$ are projections and one can write

$$U = \sum_n e^{i\lambda_n}[E(\lambda_n + 0) - E(\lambda_n - 0)] + \int e^{i\lambda}\, dE_c(\lambda),$$

where the integral (if present) represents the continuous component of U. In case this component is present and if $(E_c(\lambda)x, y)$ is absolutely continuous for all x, y, that is, if $\int_Z dE_c(\lambda) = 0$ for every zero set Z, then this component will be called absolutely continuous. The operator

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U itself will be called absolutely continuous if it has no point spectrum and if its continuous component is absolutely continuous.

Since $A - U^*AU$ can be expressed as $U(U^*A) - (U^*A)U$, the commutator of U and U^*A, relations (2) and (3), that is,

$$0 \leqslant H = A - U^*AU, \qquad (5)$$

imply, as was shown in [**3**], that

$$H \int_Z E(\lambda) = 0, \qquad (6)$$

where Z denotes an arbitrary zero set.

2. Relation (6) will be used to prove

THEOREM 1. *Suppose that the self-adjoint operators A and B satisfy* (2) *and* (3) *and let $\delta = \delta(A)$ denote the difference of the maximum and minimum points of the essential spectrum of A. Then*

$$\| H \| \leqslant \delta; \qquad (7)$$

in particular, if A differs from a completely continuous operator by a constant multiple of the identity, then $H = 0$.

Here, $\| C \|$ is defined by $\| C \| = \sup \| Cx \|$, where $\| x \| = 1$, and the essential spectrum of C is the set of cluster points, including points of the point spectrum of infinite multiplicity, of the spectrum of C. Incidentally, since, as was remarked above, $H \geqslant 0$ can hold for finite matrices only if $H = 0$, it can always be supposed that the basic Hilbert space is infinite dimensional, in which case any self-adjoint operator necessarily has a non-empty essential spectrum.

Proof of Theorem 1. Let λ_0 denote the maximum point in the essential spectrum of A and denote the eigenvalues of A (if any) greater than λ_0 by $\lambda_1 > \lambda_2 > \ldots$. If x_1 is any eigenfunction of U^*AU belonging to λ_1 then, by (5),

$$0 \leqslant (Hx_1, x_1) = (Ax_1, x_1) - \lambda_1(x_1, x_1) \leqslant 0$$

and so $(Ax_1, x_1) = \lambda_1(x_1, x_1)$. Hence $0 = (\lambda_1 I - A)^{\frac{1}{2}}x_1 = (\lambda_1 I - A)x_1$ and so x_1 is an eigenfunction of A belonging to λ_1. Since λ_1 belongs to the spectra of A and U^*AU with the same (finite) multiplicity, it follows that the eigenfunctions of A and U^*AU belonging to λ_1 are identical. On treating successively $\lambda_2, \lambda_3, \ldots$ in a similar manner, it follows that the eigenfunctions of A and U^*AU for each of the numbers λ_n are identical.

Let $\mu_1 < \mu_2 < \ldots$ denote the eigenvalues of A (if any) less than the least point μ_0 of the essential spectrum of A. If y_1 is an eigenfunction of A belonging to μ_1, then one has

$$0 \leqslant (Hy_1, y_1) = \mu_1(y_1, y_1) - (U^*AUy_1, y_1) \leqslant 0;$$

hence $(U^*AUy_1, y_1) = \mu_1(y_1, y_1)$, and so y_1 must be an eigenfunction of U^*AU belonging to μ_1. As before, it follows that the eigenfunctions of A and UA^*U belonging to eigenvalues μ_n less than μ_0 are identical.

It is now easy to complete the proof of the theorem. For if x is any element of Hilbert space, it can be written as $x = z + w$, where z is the projection of x on the space spanned by the eigenfunctions of A belonging to eigenvalues outside the interval $\mu_0 \leqslant \lambda \leqslant \lambda_0$ and w is in the orthogonal complement. Clearly $Hz = 0$ and hence

$$(Hx, x) = (Hw, w) = (Aw, w) - (U^*AUw, w) \leqslant (\lambda_0 - \mu_0) \| w \|^2 \leqslant (\lambda_0 - \mu_0) \| x \|^2.$$

Relation (7) follows and the proof of Theorem 1 is complete.

3. Theorem 2. *Suppose that the self-adjoint operators A and B satisfy* (2) *and* (3) *and let N denote the multiplicity of the eigenvalue* 0 *of H* $(0 \leqslant N \leqslant \infty)$. *Then :* (i) *If $H \neq 0$, and if U has the spectral resolution* (4), *then $\int_Z dE(\lambda) < I$ for every zero set Z.* (ii) *The point spectrum of U has no more than N values (counting multiplicities).* (iii) *If $N < \infty$, then the continuous component of U is absolutely continuous.* (iv) *If $N = 0$, then U is absolutely continuous.* (v) *If $N = 0$, the maximum and minimum points of the spectrum of A cannot belong to the point spectrum of A (and hence must belong to the essential spectrum of A).*

Proof of Theorem 2. Assertion (i) is an immediate consequence of (6); cf. [**3**]. Let x be an eigenfunction of U; then, by (5), one has

$$0 \leqslant (Hx, x) = (Ax, x) - (Ax, x) = 0;$$

hence $0 = H^{\frac{1}{2}}x = Hx$. This proves (ii). In order to prove (iii) note that, by (ii), U has at most a finite number of points in its point spectrum and so its continuous component is present. But if this component were not absolutely continuous, there would exist a zero set Z and an element x such that $\int dE_c(\lambda)x \neq 0$. Clearly Z can be written as $Z = \Sigma Z_n$ where $Z_1, Z_2, \ldots$ denotes an infinite sequence of non-overlapping zero sets for which $x_n = \int_{Z_n} dE_c(\lambda)x \neq 0$. Thus the x_n are orthogonal and, by (6), each is an eigenfunction of H belonging to 0. Thus $N = \infty$, a contradiction, and (iii) is proved. Assertion (iv) is a consequence of (ii) and (iii). Assertion (v) follows from (5). For if the maximum point λ_M of the spectrum of A were in the point spectrum of A, hence of U^*AU, then for a corresponding eigenfunction x of U^*AU one would have

$$0 < (Hx, x) = (Ax, x) - \lambda_M(x, x) \leqslant 0,$$

a contradiction. Similarly the minimum point λ_m cannot be in the point spectrum and the proof of (v) is complete.

It can be remarked that if 0 is not in the point spectrum of H, then the proof of Theorem 1 is an immediate consequence of (v) of Theorem 2. For obviously

$$(Hx, x) = (Ax, x) - (U^*AUx, x) \leqslant (\lambda_M - \lambda_m)\|x\|^2.$$

4. **Applications to semi-normal operators.** Let D be an arbitrary (bounded) operator and consider

$$H = DD^* - D^*D. \quad \ldots\ldots\ldots(8)$$

If H is semi-definite (in which case, only $H \geqslant 0$ will be supposed), D is called semi-normal. In case D is non-singular, it has a polar decomposition $D = PU$ where P is positive self-adjoint and U is unitary. Then $DD^* = P^2$, $D^*D = U^*P^2U$ and (8) can be written as $H = P^2 - U^*P^2U$, so that P^2 can be identified with the A considered above. Of course, it is quite possible that $D^*D = U^*(DD^*)U$ holds for some unitary U even if D is singular.

It was shown in [4] that the spectra of the real and imaginary parts of a semi-normal, but not normal, operator D (in fact, the spectra of $\frac{1}{2}(e^{-i\theta}D + e^{i\theta}D^*)$ for θ arbitrary and real) are of positive measure. In case D is non-singular with the polar decomposition $D = PU$ then, as a consequence of (i) of Theorem 2, it follows that U also has a spectrum of positive measure. However, a similar claim cannot be made for the positive operator P. In fact, as is shown by

Theorem 3 below and the example following, P must have at least two points in its essential spectrum, and may possibly have only (these) two points in its spectrum.

As a corollary of Theorem 1, one has

THEOREM 3. *If H defined by* (8) *satisfies $H \geqslant 0$, and if DD^* and D^*D are unitarily equivalent, then* (7) *holds, where $\delta = \delta(DD^*)$ is the difference of the maximum and minimum points of the essential spectrum of DD^*. Thus, if in addition, $H \neq 0$, then $\delta(DD^*) > 0$ and DD^* (hence D^*D) cannot differ from a completely continuous operator by a multiple of the identity.*

It is easy to show that the inequality (7) occurring in Theorems 1 and 3 may become an equality and that A may have only two points in its spectrum. One need only choose $A = (a_{ij})$ and $B = (b_{ij})$, where $i, j = 0, \pm 1, \pm 2, \ldots$, to be doubly infinite matrices for which $a_{ii} = 1$ if $i = 0, 1, 2, \ldots$ and $a_{ij} = 0$ otherwise, and $b_{ii} = 1$ if $i = 1, 2, \ldots$ and $b_{ij} = 0$ otherwise. Then the spectra of both A and B consist of 0 and 1, each of infinite multiplicity. Consequently $B = U^*AU$ for a unitary U and moreover $A - B = H = (h_{ij})$, where $h_{00} = 1$ and $h_{ij} = 0$ otherwise. Clearly $\|H\| = 1$ and $\delta(A) = 1 - 0 = 1$, where $\delta(A)$ is defined in Theorem 1. The particular matrices A, B thus constructed are singular. However, it is clear that they can be replaced by, say, the non-singular positive matrices $A + I$ and $B + I$.

Furthermore, whenever (2) and (3) hold with an operator $A \geqslant 0$ (as, for example, in the preceding paragraph) one can take the unique non-negative self-adjoint square root P of A and form the operator $D = PU$. Then

$$H = A - B = A - U^*AU = DD^* - D^*D,$$

so that D is semi-normal. It should be noted however that D need not be non-singular.

THEOREM 4. *If H of* (8) *satisfies $H \geqslant 0$ and $H \neq 0$, if DD^* differs from a completely continuous operator by a multiple of the identity and if $z = |z| e^{i\theta}$ satisfies $|z| < \|H\| / \delta$, where $\delta = \delta(D(\theta))$ denotes the difference of the maximum and minimum points of the essential spectrum of $D(\theta) = e^{-i\theta}D + e^{i\theta}D^*$, then $D_z D_z^*$ and $D_z^* D_z$, where $D_z = D - zI$, cannot be unitarily equivalent.*

Proof of Theorem 4. First, note that (8) holds if D is replaced by D_z so that

$$H = D_z D_z^* - D_z^* D_z.$$

Now if $D_z D_z^*$ and $D_z^* D_z$ are unitarily equivalent, then, by Theorem 3, $\|H\| \leqslant \delta(D_z D_z^*)$. Since

$$D_z D_z^* = DD^* + |z|^2 I - \bar{z}D - zD^*$$

and since, by hypothesis, $DD^* = tI + C$, where C is completely continuous, it follows from Weyl's theorem [7] that the essential spectrum of $D_z D_z^*$ is identical with that of

$$(|z|^2 + t)I - \bar{z}D - zD^*.$$

But the essential spectrum of this operator is simply that of $-\bar{z}D - zD^* = -|z| D(\theta)$ displaced by the amount $|z|^2 + t$ and the proof of Theorem 4 is now complete.

A corollary of Theorem 4 is

THEOREM 5. *If H of* (8) *satisfies $H \geqslant 0$ and $H \neq 0$, if DD^* differs from a completely continuous operator by a multiple of the identity and if $|z| \leqslant \frac{1}{4} \|H\| / \|D\|$, then z is in the spectrum of D.*

Proof of Theorem 5. Since not only the essential spectrum but even the spectrum of any self-adjoint operator G is contained in an interval of length $2\|G\|$, it follows that

$$\delta(D(\theta)) \leqslant 2 \| D(\theta) \| \leqslant 4 \| D \|.$$

Hence, if $| z | < \frac{1}{4} \| H \| / \| D \|$, then $D_z D_z^*$ and $D_z^* D_z$ are not unitarily equivalent and so z must surely be in the spectrum of D. The sign $\leqslant$ occurring in the theorem, rather than just $<$, follows from the fact that the spectrum is a closed set.

If V is an isometric but not unitary operator, so that $H = V^*V - VV^* \geqslant 0, H \neq 0$, where $V^*V = I$, Theorem 5 implies (with $D = V^*$) that the disk $| z | \leqslant \frac{1}{4}$ is in the spectrum of V^* (hence of V). Actually it is easy to show that the entire disk $| z | \leqslant 1$ is in the spectrum; cf. [4, p. 1650].

5. Remarks. It will remain undecided whether the hypothesis $| z | \leqslant \frac{1}{4} \| H \| / \| D \|$ in Theorem 5 can, as in the isometric non-unitary case, be weakened to $| z | \leqslant \| H \| / \| D \|$. An analogous situation exists for the real part $\frac{1}{2}(D + D^*)$ of a semi-normal operator for which it is known [4] that, if $H \geqslant 0$ in (8),

$$\| H \| \leqslant 2 \| D \| s, \qquad (9)$$

where s denotes the measure of the spectrum of $\frac{1}{2}(D + D^*) \equiv J$, and for which it is undecided whether $\| H \| \leqslant \frac{1}{2} \| D \| s$ can also be claimed. (In the isometric operator example mentioned one has $\| H \| = \frac{1}{2} \| D \| s$; cf. [4, p. 1651].)

Actually the inequality $\| H \| \leqslant 4 \| D \| s$, rather than (9), was stated in [4] but it is clear from the proof as given in [3] and applied to the case at hand, that the refinement (9) holds. In fact, it follows from (8) that $\frac{1}{2}H = DJ - JD$. Hence, if $J = \int \lambda \, dE(\lambda)$, then, proceeding as in [3], one obtains

$$\tfrac{1}{2} \Delta E \, H \Delta E = \Delta E \, D \int_\Delta (\lambda - \lambda_0) \, dE - \int_\Delta (\lambda - \lambda_0) \, dE \, D \Delta E,$$

where Δ denotes a real interval and λ_0 is any point of Δ. If λ_0 is chosen to be the mid-point of Δ, the argument of [3] then yields the desired inequality (9). It can be remarked here that the 4 in both Theorem 2 and Corollary 3 of [4] can be replaced by 2.

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