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Smooth Approximation of Lipschitz Projections

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Abstract. We show that any Lipschitz projection-valued function p on a connected closed Riemannian manifold can be approximated uniformly by smooth projection-valued functions q with Lipschitz constant close to that of p. This answers a question of Rieffel.

1 Introduction

The question of approximating continuous functions on Riemannian manifolds by smooth functions preserving geometric properties has a long history. In [1–3] Greene and Wu studied such questions for real-valued functions and geometric properties such as having Lipschitz constant bounded above by a fixed number, and used such results to obtain geometric applications [2].

In a previous version of [5] Rieffel asked the question whether, for any Lipschitz function p on a compact Riemannian manifold M with values in the projection set of a matrix algebra $M_n(\mathbb{C})$ and any $\varepsilon > 0$ there is a smooth function q on M also with values in the projection set of $M_n(\mathbb{C})$ with $\|p - q\|_\infty < \varepsilon$ and $L(q) < L(p) + \varepsilon$. Here $\|f\|_\infty$ denotes the supremum norm of f and

$$L(f) = \sup_{\substack{x,y\in \mathrm{M},\\ x\neq y}} \frac{\|f(x) - f(y)\|}{\rho(x, y)} \tag{1.1}$$

denotes the Lipschitz constant of f, for ρ denoting the geodesic distance on M. An affirmative answer to this question has direct application to obtaining lower bounds of the Lipschitz constants for projection-valued functions on M representing a fixed vector bundle on M.

In this note we answer Rieffel's question affirmatively. In fact, we shall deal more generally with functions with values in the projection set of any C^*-algebra. Following [5], by a *real C^*-subring* we mean a norm-closed $*$-subring of a C^*-algebra that is closed under multiplication by scalars in $\mathbb{R}$. For example, $M_n(\mathbb{R})$ is a real C^*-subring contained in $M_n(\mathbb{C})$. For a compact manifold M and a real C^*-subring A, denote by $C(\mathrm{M}, A)$ the real C^*-subring of all continuous A-valued functions on M, and denote by $C^\infty(\mathrm{M}, A)$ the subalgebra of all smooth A-valued functions on M.

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Theorem 1.1 *Let* M *be a connected closed Riemannian manifold. Let A be a real C^*-subring. For any projection $p \in C(\mathrm{M}, A)$ and any $\varepsilon > 0$, there exists a projection $q \in C^\infty(\mathrm{M}, A)$ with $\|p - q\|_\infty < \varepsilon$ and $L(q) < L(p) + \varepsilon$.*

As an example, Rieffel [5, Proposition 12.1] obtained lower bounds of the Lipschitz constants for projections representing certain complex line bundles on the two-torus $\mathbb{T}^2$ by obtaining such bounds for smooth projections first and then extending the bounds to nonsmooth projections using Theorem 1.1.

The proof of Theorem 1.1 has two ingredients. The first is that part of Greene and Wu's results hold generally for functions valued in any Banach space. We recall these results in Section 2. The second is a fine estimate of the Lipschitz seminorm of the projection in a C^*-algebra obtained in the usual way from an "almost-projection". We give this estimate and prove Theorem 1.1 in Section 3.

2 Smooth Approximation of Continuous Functions

In [1] Greene and Wu introduced *the Riemannian convolution smoothing process* for approximating continuous real-valued functions on a Riemannian manifold by smoothing functions preserving geometric properties. They used this technique to obtain certain geometric application [2]. In fact, much of Greene and Wu's results work for functions valued in a Banach space. Let us recall their construction on [1, pp. 646–647].

Fix a positive integer n. Let $\kappa\colon \mathbb{R} \to \mathbb{R}$ be a non-negative smooth function which has support contained in the interval $[-1, 1]$, is constant in a neighborhood of 0, and satisfies $\int_{v\in\mathbb{R}^n} \kappa(\|v\|) = 1$, where $\mathbb{R}^n$ is equipped with the standard Lebesgue measure. Let V be a real Banach space. Let M be a Riemannian manifold of dimension n without boundary, and let K be a compact subset of M. Then there is an $\varepsilon_K > 0$ such that, for every x in some neighborhood of K and every v in the tangent space TM_x of M at x with $\|v\| < \varepsilon_K$, the exponential $\exp_x v$ is defined. For any continuous function $f\colon \mathrm{M} \to V$ and any $0 < \varepsilon < \varepsilon_K$ define a V-valued function f_ε on a neighborhood of K by

$$f_\varepsilon(x) = \frac{1}{\varepsilon^n} \int_{v\in\mathrm{TM}_x} \kappa\Big(\frac{\|v\|}{\varepsilon}\Big) f(\exp_x v),$$

where TM_x is equipped with the standard Lebesgue measure via a linear isometry between TM_x and the Euclidean space $\mathbb{R}^n$.

In [1, p. 647] and [2, Lemma 8] Greene and Wu proved the following theorem for the case $V = \mathbb{R}$. Their proof works for any Banach space V.

Theorem 2.1 (Greene-Wu) *Let $f\colon \mathrm{M} \to V$ be continuous. Let K be a compact subset of* M. *When $\varepsilon > 0$ is small enough (depending only on* M *and K), f_ε is a smooth function on an open neighborhood of K. The function f_ε converges to f uniformly on K as $\varepsilon \to 0$. If f is smooth on a neighborhood of K, then f_ε converges to f in the smooth topology on K as $\varepsilon \to 0$. If* M *is connected and f has Lipschitz constant D on a neighborhood W of K,* i.e.,

$$\sup_{\substack{x,y\in W,\\ x\neq y}} \frac{\|f(x) - f(y)\|}{\rho(x, y)} = D,$$

where ρ denotes the geodesic distance on M, *then for any $\delta > 0$, when ε is small enough one has $\|\partial_v(f_\varepsilon)\| < D + \delta$ for all $x \in K$ and unit vector $v \in \mathrm{TM}_x$.*

One direct consequence of the above approximation theorem is the following.

Corollary 2.2 *Let* M *be a connected closed Riemannian manifold. For any continuous map $f\colon \mathrm{M} \to V$ and any $\delta > 0$, when $\varepsilon > 0$ is small enough, f_ε is a smooth function on* M *with $\|f - f_\varepsilon\|_\infty < \delta$ and $L(f_\varepsilon) < L(f) + \delta$, where $L(f_\varepsilon)$ and $L(f)$ are defined by* (1.1).

Using Theorem 2.1 for the case $V = \mathbb{R}$, Greene and Wu [3, Proposition 2.1] actually showed that for any connected (possibly noncompact) Riemannian manifold M without boundary, any Lipschitz function $f\colon \mathrm{M} \to \mathbb{R}$, any continuous function $g\colon \mathrm{M} \to \mathbb{R}_{>0}$, and any $\delta > 0$, there is some smooth function $h\colon \mathrm{M} \to \mathbb{R}$ such that $|f - h| < g$ on M and $L(h) < L(f) + \delta$. It would be interesting to see whether this result extends to every Banach space V.

3 Estimate of Lipschitz Constants for Projections

The key in our approach is the following estimate of the Lipschitz seminorm of the projection obtained from an "almost-projection" in the usual way.

Proposition 3.1 *Let A be a unital C^*-algebra, and let $a \in A$ be self-adjoint whose spectrum has empty intersection with the interval $(\delta, 1-\delta)$ in $\mathbb{R}$ for some $0 < \delta < 1/2$. Let L be a (possibly $+\infty$-valued) seminorm on A. Suppose that L is lower semi-continuous, vanishes on 1_A, and $L(b^{-1}) \le \|b^{-1}\|^2 L(b)$ for every invertible $b \in A$. Let f denote the characteristic function of the interval $[1-\delta, +\infty)$ on $\mathbb{R}$. Then*

$$L(f(a)) \le L(a)/(1-2\delta),$$

where $f(a)$ denotes the continuous-functional calculus of a under f.

Proof Take $R > 0$ such that the spectrum of a is contained in the union of the intervals $(-\infty, \delta]$ and $[1-\delta, 1+R]$ in $\mathbb{R}$.

Denote by g the characteristic function of $\{z \in \mathbb{C} : \mathrm{Re}\, z \ge (\delta + 1/2)/2\}$ on $\mathbb{C}$. Note that g is holomorphic on $\{z \in \mathbb{C} : \mathrm{Re}\, z \ne (\delta + 1/2)/2\}$. For each $s > R$, denote by γ_s the rectangle with vertices $1/2 + si$, $1/2 - si$, $(1+s) - si$, and $(1+s) + si$, parameterized as a piecewise smooth simple closed curve in the anti-clockwise direction. Then

$$\begin{aligned}
L(f(a)) = L(g(a)) &= L\Big(\frac{1}{2\pi i}\int_{\gamma_s} g(z)(z-a)^{-1}\, dz\Big) \\
&\le \frac{1}{2\pi}\int_{\gamma_s} L(g(z)(z-a)^{-1})\, d|z| = \frac{1}{2\pi}\int_{\gamma_s} L((z-a)^{-1})\, d|z| \\
&\le \frac{1}{2\pi}\int_{\gamma_s} \|(z-a)^{-1}\|^2 L(z-a)\, d|z| \\
&= \frac{L(a)}{2\pi}\int_{\gamma_s} \|(z-a)^{-1}\|^2\, d|z|,
\end{aligned}$$

where the first inequality follows from the lower semi-continuity of L. For z in the line segment from $1/2 - si$ ($(1+s)+si$, resp.) to $(1+s) - si$ ($1/2 + si$, resp.), one has $\|(z-a)^{-1}\|^2 \leq s^{-2}$, since a is self-adjoint. For $z = 1/2 + ti$ with $t \in \mathbb{R}$, one has $\|(z-a)^{-1}\|^2 \leq ((1/2-\delta)^2 + t^2)^{-1}$. For $z = (1+s) + ti$ with $t \in \mathbb{R}$, one has $\|(z-a)^{-1}\|^2 \leq ((s-R)^2 + t^2)^{-1}$. Therefore

$$\begin{aligned} L(f(a)) &\leq \frac{L(a)}{2\pi} \int_{\gamma_s} \|(z-a)^{-1}\|^2 \, d|z| \\ &\leq \frac{L(a)}{2\pi} \Big(2s^{-2}(1/2+s) + \int_{-\infty}^{+\infty} ((1/2-\delta)^2 + t^2)^{-1} \, dt \\ &\qquad + \int_{-\infty}^{+\infty} ((s-R)^2 + t^2)^{-1} \, dt\Big) \\ &= \frac{L(a)}{2\pi} \Big(s^{-2}(1+2s) + \frac{1}{1/2-\delta} \int_{-\infty}^{+\infty} (1+t^2)^{-1} \, dt \\ &\qquad + \frac{1}{s-R} \int_{-\infty}^{+\infty} (1+t^2)^{-1} \, dt\Big) \\ &= \frac{L(a)}{2\pi} \Big(s^{-2}(1+2s) + \frac{\pi}{1/2-\delta} + \frac{\pi}{s-R} \Big). \end{aligned}$$

Letting $s \to \infty$, we obtain $L(f(a)) \leq L(a)/(1-2\delta)$ as desired. ■

Using Proposition 3.1, Rieffel improved various estimates in a previous version of Sections 3, 4, 6 and 10 of [5].

In general the estimate in Proposition 3.1 is the best possible, as the following example shows.

Example 3.2 Let $X = \{x_1, x_2\}$ be a 2-point metric space with the metric $\rho(x_1, x_2) = 1$. Define L on $C(X)$ via (1.1). By [5, Proposition 2.2] the seminorm L satisfies the conditions in Proposition 3.1. For $0 < \delta < 1/2$ define $a \in C(X)$ by $a(x_1) = \delta$ and $a(x_2) = 1 - \delta$. Then $(f(a))(x_1) = 0$ and $(f(a))(x_2) = 1$, where f is as in Proposition 3.1. Thus, $L(a) = 1 - 2\delta$ and $L(f(a)) = 1 = L(a)/(1-2\delta)$.

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1 Let $0 < \delta < 1/2$. By Corollary 2.2 applied to V equal to the self-adjoint part of A and f equal to p, we can find some self-adjoint $p_1 \in C^\infty(\mathrm{M}, A)$ with $\|p - p_1\|_\infty < \delta$ and $L(p_1) < L(p) + \varepsilon/2$. Since p is a projection, by [5, Lemma 3.2] the spectrum of p_1 is contained in the union of the intervals $[-\delta, \delta]$ and $[1-\delta, 1+\delta]$ in $\mathbb{R}$. Let f denote the characteristic function of the interval $[1-\delta, +\infty)$ on $\mathbb{R}$. Set $q = f(p_1)$.

Suppose A is a real C^*-subring contained in a C^*-algebra B. Since $C^\infty(\mathrm{M}, B)$ is closed under the holomorphic functional calculus, we have $q \in C^\infty(\mathrm{M}, B)$. Using polynomial approximations it is easy to see that the self-adjoint part of $C(\mathrm{M}, A)$ is closed under continuous functional calculus for real-valued continuous functions on $\mathbb{R}$ vanishing at 0 (see, for example, the proof of [5, Proposition 2.4]). Therefore $q \in C^\infty(\mathrm{M}, B) \cap C(\mathrm{M}, A) = C^\infty(\mathrm{M}, A)$.

One has $\|q - p\|_\infty \leq \|q - p_1\|_\infty + \|p_1 - p\|_\infty < \delta + \delta = 2\delta$. It is readily checked that the Lipschitz seminorm L on the C^*-algebra $C(\mathrm{M}, B)$ defined by (1.1) satisfies the conditions in Proposition 3.1 (see, for example, the proof of [5, Proposition 2.2]). By Proposition 3.1 one has

$$L(q) \leq L(p_1)/(1 - 2\delta) < (L(p) + \varepsilon/2)/(1 - 2\delta).$$

Thus, when $\delta > 0$ is small enough, we have $\|q - p\|_\infty < \varepsilon$ and $L(q) < L(p) + \varepsilon$ as desired. ■

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