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# KANTOROVITCH POLYNOMIALS DIMINISH GENERALIZED LENGTH

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The Kantorovitch polynomials of a summable function $s$, defined on [0, 1], are

$$K_n s(x) = \sum_{r=0}^{n} I_{n,r} p_{nr}(x)$$

where

$$n = 1, 2, 3, \ldots; x \in [0, 1],$$

$$I_{n,r} = (n+1) \int_{r/(n+1)}^{(r+1)/(n+1)} s(t)\, dt$$

and

$$p_{nr}(x) = \binom{n}{r} x^r (1-x)^{n-r}.$$

They are the analogue for summable functions of the Bernstein polynomials $B_n f(x)$, and they possess similar properties [1].

In [2], Goffman defined a generalized variation $\varphi(s)$ for $s \in L_1[0, 1]$ as follows: Consider the space $P$ of polygonal functions on [0, 1], with norm given by $\|p\|_1 = \int_0^1 |p|$, and denote the ordinary total variation of $p$ by $Vp$. By the Frechet process, we may extend $V$ to a unique lower-semicontinuous functional $\varphi$ on the completion of $P$, the space $L_1 = L_1[0, 1]$ of equivalence classes of summable functions, such that for every $s \in L_1$ there exists a sequence $p_n \in P$ with $\|p_n - s\|_1 \to 0$ and $V(p_n) \to \varphi(s)$. We show that $VK_n s \le \varphi(s)$ for all $s \in L_1$ and all $n$.

In [3], Hughs developed a generalized length $L$ based on Goffman's variation $\varphi$. $L$ is defined on the space $A$ of equivalence classes of parametric generalized curves $S(t) = (s_1(t), s_2(t), s_3(t))$, $t \in [0, 1]$, $s_i \in L_1$ with $\|S\|_A = \sum_{i=1}^{3} \int_0^1 |s_i|$. $L$ is obtained by completing the space $B$ of polygonal triples $(p_1, p_2, p_3)$ with the above norm and using the Frechet process to extend the elementary length $l$ to a unique lower semicontinuous functional $L$ on the completion of $B$, the space $A$. Set $K_n S \equiv (K_n s_1, K_n s_2, K_n s_3)$. We show that $lK_n S \le LS$ for all $n$ and all $S \in A$.

We need the following facts:

(A) $\|s - K_n s\|_1 \to 0$ if $s \in L_1$. [1]

(B) If $s \in L_1$, $\varphi(s) = V_E(s)$ where $V_E$ is the total variation computed over the set $E$ of points of approximate continuity of $s$. Thus $\varphi(s) \le V(s)$ and $\varphi(s) = V(s)$ if $s$ is continuous. [2]

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(C) (Cauchy–Steinhaus Formula) See [3]. Let $B$ be the surface of the unit sphere in $E^3$, $z=(a, b, c)\in B$, $S=(s_1, s_2, s_3)\in A$, and $L(S)<\infty$, then

$$L(S) = \frac{1}{2\pi}\iint_B \varphi(z\cdot S)\, d\sigma$$

where $\varphi(z\cdot S)$ is the generalized variation of the scalar function $z\cdot S$.

THEOREM 1. *For $s\in L_1$ and all $n$, $VK_n s\le\varphi(s)$.*

**Proof.**

$$\begin{aligned} VK_n s &= \int_0^1 |K_n' s(x)|\,dx = n\int_0^1 \left|\sum_{r=0}^{n-1}(I_{n,r+1}-I_{n,r})p_{n-1,r}(x)\right| dx \\ &\le \sum_{r=0}^{n-1}|I_{n,r+1}-I_{n,r}|, \end{aligned}$$

using properties of Euler's $B$ function. By rewriting $|I_{n,r+1}-I_{n,r}|+|I_{n,r+2}-I_{n,r+1}|$ as $|I_{n,r+2}-I_{n,r}|$ whenever possible, the last sum becomes

$$\sum_{j=0}^{n}|I_{n,r_{j+1}}-I_{n,r_j}|;\qquad m\le n-1,\quad r_0=0,\quad r_{m+1}=n,$$

and for each $j$, $I_{n,r_{j+2}}-I_{n,r_{j+1}}$ is of opposite sign from $I_{n,r_{j+1}}-I_{n,r_j}$.

Let $E\subset[0, 1]$ be the set of points of approximate continuity of $s(x)$. $E$ is of measure one since $s$ is summable. Now if $I_{n,r_1}>I_{n,r_0}$, pick $x_0$ in $E\cap(0, 1/(n+1))$ such that $s(x_0)\le I_{n,0}$, and $x_1$ in $E\cap(r_1/(n+1), (r_1+1)/(n+1))$ such that $s(x_1)\ge I_{n,r_1}$. Thus $|s(x_1)-s(x_0)|\ge|I_{n,r_1}-I_{n,r_0}|$. Continuing, we can pick $x_2\in E\cap(r_2/(n+1), (r_2+1)/(n+1))$ such that $s(x_2)\le I_{n,r_2}$, and since $I_{n,r_2}<I_{n,r_1}$,

$$|s(x_2)-s(x_1)|\ \ge\ |I_{n,r_2}-I_{n,r_1}|,$$

etc. In case $I_{n,r_1}<I_{n,r_0}$, we proceed analogously. In either case,

$$VK_n f\le\sum_{j=0}^{m}|I_{n,r_{j+1}}-I_{n,r_j}|\le\sum_{j=0}^{m}|s(x_{j+1})-s(x_j)|\le V_E s=\varphi(s).$$

By facts (B) and (A) and the lower semi-continuity of $\varphi$, we get

COROLLARY. *$VK_n s\le Vs$ and $\lim_{n\to\infty} VK_n s=\varphi(s)$ for $s\in L_1$.*

THEOREM 2. *If $s\in A$, $l(K_nS)\le L(S)$ for all $n$.*

**Proof.** We may assume $L(S)<\infty$. Consider the scalar functions $z\cdot K_n$ and $z\cdot S$ for each fixed $z\in B$. We apply the theorem to get

$$\begin{aligned} V(z\cdot K_nS) &= V(aK_ns_1+bK_ns_2+cK_ns_3) \\ &= V(K_n(as_1+bs_2+cs_3))\le\varphi(as_1+bs_2+cs_3)=\varphi(z\cdot s), \end{aligned}$$

and thus by fact (C),

$$l(K_nS) = L(K_nS) = \iint_B \varphi(z \cdot K_nS)\, d\sigma = \iint_B V(z \cdot K_nS)\, d\sigma$$

$$\leq \iint_B \varphi(z \cdot S)\, d\sigma = L(S).$$

COROLLARY. *$lK_nS \leq lS_n$ and $\lim_{n\to\infty} l(K_nS) = L(S)$ for $S \in A$.*

**Proof.** $L \leq l$ always, and $L$ is lower semicontinuous with respect to $L_1$ convergence [3].

REMARKS. Let $f$ be an arbitrary finite-valued function on [0, 1], and let

$$B_nf(x) = \sum_{r=0}^{n} f\left(\frac{r}{n}\right) p_{nr}(x), \quad x \in [0, 1],$$

be the $n$th Bernstein polynomial of $f$.

It is known that $VB_nf \leq Vf$ for arbitrary $f$, [1]. The Cauchy–Steinhaus formula with $V$ replacing $\varphi$ shows that $lB_nF \leq lF$ for arbitrary triples $F=(f_1, f_2, f_3)$. However the Bernstein polynomials behave erratically for discontinuous functions and in particular $VB_nf \not\to Vf$ for most discontinuous functions. There exist badly discontinuous functions $f$ such that $Vf=\infty$ but such that $VB_nf \to 0$ [4].

Even for continuous functions of bounded variation we can have either

$$\text{(a) } VK_nf < VB_nf \quad \text{or} \quad \text{(b) } VB_nf < VK_nf.$$

For (a), define $f_\epsilon$ to be a continuous "spike" function such that $f_\epsilon \equiv 0$ on

$$[0, \tfrac{1}{2}-\epsilon] \cup [\tfrac{1}{2}+\epsilon, 1], \quad f_\epsilon(\tfrac{1}{2}) = 1,$$

and linear on $[\frac{1}{2}-\epsilon, \frac{1}{2}]$ and $[\frac{1}{2}, \frac{1}{2}+\epsilon]$. Let $n=2$. Since $B_2f_\epsilon$ depends only on $x=0, \frac{1}{2}$, and 1, the width of the spike does not affect $VB_2f_\epsilon$. On the other hand, the coefficients of $K_2f_\epsilon$ are integral means, so that $K_2f_\epsilon$ can be made uniformly small by making $\epsilon \to 0$. Since $K_2f_\epsilon$ is always a quadratic, this means $VK_2f_\epsilon < VB_2f_\epsilon$ for some $\epsilon$.

For (b), define a spike function by $f(0)=0$, $f(\frac{1}{6})=1$, $f$ linear on $[0, \frac{1}{6}]$, and $[\frac{1}{6}, \frac{1}{3}]$, and $f\equiv 0$ on $[\frac{1}{3}, 1]$. $B_2f\equiv 0$, hence $VB_2f = 0$ but since $K_2f(0) = \int_0^{1/3} f > 0$, and $K_2f(1) = \int_{2/3}^1 f = 0$ we have $VK_2f > 0$.

If $f$ is of bounded variation, Lorentz [1] showed that $\lim VB_nf = V_0f$ where $V_0f$ is the variation computed over the points of continuity of $f$. For such functions, $V_0f = V_Ef = \varphi(f)$ since discontinuities of the first kind cannot be points of approximate continuity. Thus by Theorem 1, $\lim VK_nf = \varphi(f) = V_0f = \lim VB_nf$ whenever $f$ is of bounded variation. If $f$ is continuous but $Vf=\varphi(f) = +\infty$, both $VB_nf$ and $VK_nf$ tend to $+\infty$ by lower-semicontinuity.

Finally we remark that all of the above results hold in the plane, and in particular for the nonparametric case $y=f(x)$, by using the Cauchy–Steinhaus formula in $E^2$, i.e. as an integral over the unit circle.

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