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CLASSIFICATION OF LAGRANGIAN WILLMORE SUBMANIFOLDS OF THE NEARLY KAEHLER 6-SPHERE $S^6(1)$ WITH CONSTANT SCALAR CURVATURE†

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Abstract. In this paper, we classify 3-dimensional Lagrangian Willmore submanifolds of the nearly kaehler 6-sphere $S^6(1)$ with constant scalar curvature.

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1. Introduction. It is well known that a 6-dimensional sphere $S^6(1)$ admits an almost kaehler structure J by making use of the Cayley system. Many interesting theorems about the topology and the geometry of nearly kaehler manifolds have been proved (see[**2**, **4**, **7**]). There have been many results on geometry of submanifolds in a kaehler manifold. Especially, submanifolds (called Lagrangian submanifolds) for which J interchanges the tangent and normal spaces. The theory of Lagrangian submanifolds in a nearly kaehler manifold was studied by many authors (cf. e.g. N. Ejiri, B. Y. Chen, F. Dillen, L. Vrancken and L. Verstraelen etc.). About Lagrangian submanifolds of $S^6(1)$, in [**5**], the authors classified the compact Lagrangian submanifolds of $S^6(1)$ whose sectional curvatures satisfy $K \geq \frac{1}{16}$. In [**2**], the authors classified the Lagrangian submanifolds of $S^6(1)$ with constant scalar curvature that realize the Chen's inequality. In this paper, we classify Lagrangian Willmore submanifold of the nearly kaehler 6-sphere $S^6(1)$ with constant scalar curvature and obtain all possible values for the norm square of the second fundamental form S about these submanifolds. It is similar to Chern's conjecture which states that the set of all possible values for S of a compact minimal submanifold in the sphere with $S =$ constant is a limit set.

2. Preliminaries. We give a brief introduction to the standard nearly kaehler structure on $S^6(1)$. Let $e_0, e_1, \cdots, e_7$ be the standard basis of R^8. Then each point m of R^8 can be written in a unique way as $m = ae_0 + x$, where $a \in R$ and x is a linear combination of $e_1, e_2, \cdots, e_7$. m can be regarded as a Caylay number, and is called purely imaginary when $a = 0$. If x and y are purely imaginary, we defined the multiplication $\cdot$ as

$$x \cdot y = - < x, y > e_0 + x \times y,$$

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where $<,>$ is the standard inner product on R^8 and $x \times y$ is defined by the following multiplication table for $e_j \times e_k$:

Table 1. multiplication table for $e_j \times e_k$

$\times$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
e_2	$-e_3$	0	e_1	e_6	$-e_7$	$-e_4$	e_5
e_3	e_2	$-e_1$	0	$-e_7$	$-e_6$	e_5	e_4
e_4	$-e_5$	$-e_6$	e_7	0	e_1	e_2	$-e_3$
e_5	e_4	e_7	e_6	$-e_1$	0	$-e_3$	$-e_2$
e_6	$-e_7$	e_4	$-e_5$	$-e_2$	e_3	0	e_1
e_7	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	0

For two Cayley numbers $m = ae_0 + x$ and $n = be_0 + y$, the Cayley multiplication$\cdot$, which makes R^8 the Cayley algebra $\Im$, is defined by

$$m \cdot n = abe_0 + ay + bx + x \cdot y.$$

The set $\Im_+$ of all purely imaginary Cayley numbers clearly can be viewed as a 7-dimensional linear subspace R^7 of R^8. In $\Im_+$ we consider the unit hypersphere which is centered at the origin:

$$S^6(1) = \{x \in \Im_+ \mid < x, x > = 1\}.$$

Then the tangent space T_xS^6 of $S^6(1)$ at a point x may be identified with the affine subspace of $\Im_+$ which is orthogonal to x. The standard nearly kaehler structure on $S^6(1)$ is obtained as follows:

$$JA = x \times A, \quad x \in S^6(1), \quad A \in T_xS^6(1). \tag{2.1}$$

Let G be the (2,1)-tensor field on S^6 defined by

$$G(X, Y) = (\overline{\nabla}_X J)Y, \tag{2.2}$$

where $X, Y \in T(S^6)$ and $\overline{\nabla}$ is the Levi-Civita connection on S^6. This tensor field has the following properties (see[7])

$$G(X, X) = 0; \quad G(X, Y) + G(Y, X) = 0; \quad G(X, JY) + JG(X, Y) = 0. \tag{2.3}$$

It is clear that a Lagrangian submanifold M of $S^6(1)$ is 3-dimensional. In [7], Ejiri proved that M is minimal, orientable and that for tangent vector fields X and Y to M, $G(X, Y)$ is normal to M, i.e.

$$G(X, Y) \in T^{\perp}M.$$

We denote the Levi-Civita connection of M by ∇. The formulas of Gauss and Weingarten are then given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y); \quad \overline{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.4}$$

where X and Y are vector fields on M and ξ is a normal vector field on M. The second fundamental form h is related to A_ξ by

$$< h(X, Y), \xi > = < A_\xi X, Y > . \tag{2.5}$$

From (2.3) and (2.4), we find

$$D_X(JY) = G(X, Y) + J\nabla_X Y; \quad A_{JX}Y = -Jh(X, Y). \tag{2.6}$$

Since M is a Lagrangian submanifold of $S^6(1)$, $JT^\perp M = TM$ and $JTM = T^\perp M$. We can easily verify that the second formula of (2.6) is equivalent to

$$< h(X, Y), JZ > = < h(X, Z), JY > = < h(Y, Z), JX > . \tag{2.7}$$

Next, we give some lemmas

LEMMA 2.1. ([**15**]) *Let M be a 3-dimensional Lagrangian submanifold of (S^6, J). If p is every non totally geodesic point of M. Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of T_pM such that*

$$\begin{aligned} &h(e_1, e_1) = \lambda_1 Je_1, && h(e_2, e_2) = \lambda_2 Je_1 + \lambda_3 Je_2 + \lambda_4 Je_3,\\ &h(e_1, e_2) = \lambda_2 Je_2, && h(e_2, e_3) = \lambda_4 Je_2 - \lambda_3 Je_3,\\ &h(e_1, e_3) = -(\lambda_1 + \lambda_2)Je_3, && h(e_3, e_3) = -(\lambda_1 + \lambda_2)Je_1 - \lambda_3 Je_2 - \lambda_4 Je_3. \end{aligned} \tag{2.8}$$

where $\lambda_1 > 0$ and h is the second fundamental form of M.

REMARK 2.1. Lemma 1 means that $(h_{ij}^{k^*})$ can be expressed as

$$(h_{ij}^{1^*}) = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & -\lambda_1 - \lambda_2 \end{pmatrix}$$

$$(h_{ij}^{2^*}) = \begin{pmatrix} 0 & \lambda_2 & 0 \\ \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_4 & -\lambda_3 \end{pmatrix}$$

$$(h_{ij}^{3^*}) = \begin{pmatrix} 0 & 0 & -\lambda_1 - \lambda_2 \\ 0 & \lambda_4 & -\lambda_3 \\ -\lambda_1 - \lambda_2 & -\lambda_3 & -\lambda_4 \end{pmatrix}$$

where $k^* = k + 3$, $1 \le i, j, k, \cdots \le 3$ and $h_{ij}^{k^*}$ denotes the element of the second fundamental form of the immersion.

Recently, B. Y. Chen has given in [**1**] a best possible inequality between the sectional curvature K, the scalar curvature $\tau(p) = \sum_{i<j} K(e_i \wedge e_j)$ definded in terms of an

orthonormal basis $\{e_1, e_2, e_3\}$ of the tangent space T_pM to 3-dimensional submanifolds of $S^6(1)$, states

$$\delta_M(p) \leq \frac{9}{4}H^2(p) + 2$$

for each point $p \in M$, where H denotes the length of the mean curvature vector and $\delta_M(p)$ is the Riemannian invariant, definded by

$$\delta_M(p) = \tau(p) - (inf K)(p).$$

Here

$$(inf K)(p) = inf\{K(\pi) \mid \pi \text{ is a } 2 - \text{dimensional subspace of } T_pM\}.$$

Submanifolds realizing the equality are called submanifolds satisfying Chen's equality. For a Lagrangian submanifold of $S^6(1)$, M realizes Chen's equality if and only if $\delta_M = 2$. About those submanifolds, we have

LEMMA 2.2. (*Theorem* 2.2 *of* [**2**]). *Let M be a 3-dimensional Lagrangian submanifold of $S^6(1)$. Then $\delta_M \leq 2$ and equality holds at a point p of M if there exists a tangent basis $\{e_1, e_2, e_3\}$ of T_pM such that*

$$\begin{aligned} h(e_1, e_1) &= \lambda J e_1, & h(e_2, e_2) &= -\lambda J e_1,\\ h(e_1, e_2) &= -\lambda J e_2, & h(e_2, e_3) &= 0,\\ h(e_1, e_3) &= 0, & h(e_3, e_3) &= 0. \end{aligned}$$

where λ is a positive number satisfying $2\lambda^2 = 3 - \tau(p)$.

REMARK 2.2. If we replace e_2, e_3 by $e_3, -e_2$ respectively in Lemma 2.2, then we have: Let M be a 3-dimensional totally real submanifold of $S^6(1)$. Then $\delta_M \leq 2$ and equality holds at a point p of M if there exists a tangent basis $\{e_1, e_2, e_3\}$ of T_pM such that

$$\begin{aligned} h(e_1, e_1) &= \lambda J e_1, & h(e_3, e_3) &= -\lambda J e_1,\\ h(e_1, e_3) &= -\lambda J e_3, & h(e_2, e_3) &= 0,\\ h(e_1, e_2) &= 0, & h(e_2, e_2) &= 0. \end{aligned}$$

where λ is a positive number satisfying $2\lambda^2 = 3 - \tau(p)$.

LEMMA 2.3. (*Main Theorem of* [**2**]). *Let $x : M^3 \to S^6(1)$ be a Lagrangian immersion. If M^3 has constant scalar curvature τ and $\delta_M = 2$ holds identically, then either x is totally geodesic, or locally congruent to φ_1 or φ_2, where φ_1 and φ_2 has been given in Section* 3.

From now on, we agree on the following index ranges:

$$1 \leq i, j, k, \cdots \leq 3; \quad i^* = 3 + i; \quad j^* = 3 + j; \cdots$$

Choose $\{e_1, e_2, e_3, e_{1^*}, e_{2^*}, e_{3^*}\}$ to be a local orthonormal frame field of the tangent bundle TS^6 such that e_i lies in TM and $e_{i^*} = Je_i$ lies in NM. Let

$\{\omega_1, \omega_2, \omega_3, \omega_{1^*}, \omega_{2^*}, \omega_{3^*}\}$ be the associated coframe field. Denote $(\omega_{i^*j^*})$ to be the associated Levi-Civita connection form. Then the structure equations of M are:

$$dx = \sum_i \omega_i e_i; \quad de_i = \sum_j \omega_{ij} e_j + \sum_{k,j} h_{ij}^{k^*} \omega_j e_{k^*} - \omega_i x, \tag{2.9}$$

$$de_{k^*} = -\sum_{i,j} h_{ij}^{k^*} \omega_j e_i + \sum_l \omega_{k^*l^*} e_{l^*}. \tag{2.10}$$

The Gauss equations are:

$$R_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_r (h_{ik}^{r^*} h_{jl}^{r^*} - h_{il}^{r^*} h_{jk}^{r^*}), \tag{2.11}$$

$$R_{ik} = \sum_l R_{illk} = 2\delta_{ik} - \sum_{r,j} h_{ij}^{r^*} h_{jk}^{r^*}; \quad 2\tau = 6 - S, \tag{2.12}$$

where $S = \sum_{l,i,j} (h_{ij}^{l^*})^2$ is the norm square of the second fundamental form.

The Codazzi equation is:

$$(\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z),$$

where $(\nabla h)(X, Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$.

Finally, we introduce Willmore submanifolds. $x : M^3 \to S^6(1)$ is called Willmore if it is an extremal submanifold of the following Willmore functional:

$$W(x) = \int_M (S - 3H^2)^{\frac{3}{2}} dv, \tag{2.13}$$

where $S = \sum_{i,j,k^*} (h_{ij}^{k^*})^2$ and H are respectively the norm square of the second fundamental form and the mean curvature of the immersion x, dv is the volume element of M. For more details about Willmore submanifolds we refer the reader to [**11**] and [**10**]. About Willmore submanifolds, we have:

LEMMA 2.4. *Let M be a Lagrangian submanifold in (S^6, J) with constant scalar curvature. Then M is a Willmore submanifold if and only if*

$$\rho \left(\sum_{i,j,k,l} h_{ij}^{t^*} h_{ik}^{l^*} h_{kj}^{l^*} \right) = 0, \qquad \forall t \text{ with } 1 \le t \le 3, \tag{2.14}$$

where $\rho^2 = S = \sum_{ijk} (h_{ij}^{k^*})^2$.

Proof. Since M is minimal and has constant scalar curvature, we can easily get our result by using of Theorem 1.1 of [**11**].

3. Examples. In this section, we give some examples of Lagrangian Willmore submanifolds of $S^6(1)$ with constant scalar curvature. In addition, we also give one example of Lagrangian submanifold of $S^6(1)$ with constant scalar curvature which is not a Willmore submanifold.

EXAMPLE 3.1. Define a map

$$f : S^3(1) = \left\{(x_1, x_2, x_3, x_4) \in R^4 \mid \sum_{i=1}^{4} x_i^2 = 1\right\} \longrightarrow$$
$$S^6(1) = \left\{(y_1, y_2, y_3, y_4, y_5, y_6, y_7) \in R^7 \mid \sum_{i=1}^{7} y_i^2 = 1\right\},$$

where

$$y_1 = x_1, \quad y_3 = x_2, \quad y_5 = x_3, \quad y_7 = x_4, \quad y_2 = y_4 = y_6 = 0.$$

It is clear that $f : S^3(1) \to S^6(1)$ is a Lagrangian totally geodesic immersion. That M is totally geodesic implies M is a Einstein submanifold. In [**8**], the authors proved that all n-dimensional minimal Einstein submanifolds in a sphere are Willmore submanifolds. So M^3 is a Lagrangian Willmore submanifold with constant scalar curvature.

EXAMPLE 3.2. Define a map

$$f : S^3(\tfrac{1}{16}) = \left\{(x_1, x_2, x_3, x_4) \in R^4 \mid \sum_{i=1}^{4} x_i^2 = 16\right\} \longrightarrow$$
$$S^6(1) = \left\{(y_1, y_2, y_3, y_4, y_5, y_6, y_7) \in R^7 \mid \sum_{i=1}^{7} y_i^2 = 1\right\},$$

where

$$y_1 = \sqrt{15}2^{-10}(x_1x_3 + x_2x_4)(x_1x_4 - x_2x_3)\left(x_1^2 + x_2^2 - x_3^2 - x_4^2\right)$$
$$y_2 = 2^{-12}\left[-\sum_i x_i^6 + 5\sum_{i<j}(x_ix_j)^2\left(x_i^2 + x_j^2\right) - 30\sum_{i<j<k}(x_ix_jx_k)^2\right]$$
$$y_3 = 2^{-10}\left[x_3x_4\left(x_3^2 - x_4^2\right)\left(x_3^2 + x_4^2 - 5x_1^2 - 5x_2^2\right) + x_1x_2\left(x_1^2 - x_2^2\right)\left(x_1^2 + x_2^2 - 5x_3^2 - 5x_4^2\right)\right]$$
$$y_4 = 2^{-12}\left[x_2x_4\left(x_2^4 + 3x_3^4 - x_4^4 - 3x_1^4\right) + x_1x_3\left(x_3^4 + 3x_2^4 - x_1^4 - 3x_4^4\right)\right.$$
$$\left. + 2(x_1x_3 - x_2x_4\left(x_1^2\left(x_2^2 + 4x_4^2\right) - x_3^2\left(x_4^2 + 4x_2^2\right)\right)\right]$$
$$y_5(x_1, x_2, x_3, x_4) = y_4(x_2, -x_1, x_3, x_4)$$
$$y_6 = \sqrt{6}2^{-12}\left[x_1x_3\left(x_1^4 + 5x_2^4 - x_3^4 - 5x_4^4\right) - x_2x_4\left(x_2^4 + 5x_1^4 - x_4^4 - 5x_3^4\right)\right.$$
$$\left. + 10(x_1x_3 - x_2x_4)((x_3x_4)^2 - (x_1x_2)^2)\right]$$
$$y_7(x_1, x_2, x_3, x_4) = y_6(x_2, -x_1, x_3, x_4).$$

In [**5**], the authors proved that $f : S^3(\frac{1}{16}) \to S^6(1)$ is a Lagrangian immersion with constant sectional curvature $\frac{1}{16}$. That M^3 is a constant sectional curvature submanifold implies M^3 is a Einstein submanifold. From [**7**], we know that M is minimal. In [**8**], the authors proved that all n-dimensional minimal Einstein submanifolds in a sphere are Willmore submanifolds. So M^3 is a Lagrangian Willmore submanifold with constant scalar curvature.

EXAMPLE 3.3. (*Example* 3.1 *of* [**2**]) Consider the unit sphere

$$S^3 = \left\{(y_1, y_2, y_3, y_4) \in R^4 \middle| y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1\right\}$$

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in R^4. Let X_1, X_2 and X_3 be the vector fields defined by

$$X_1(y_1, y_2, y_3, y_4) = (y_2, -y_1, y_4, -y_3); \quad X_2(y_1, y_2, y_3, y_4) = (y_3, -y_4, -y_1, y_2)$$
$$X_3(y_1, y_2, y_3, y_4) = (y_4, y_3, -y_2, -y_1).$$

Then X_1, X_2 and X_3 form a basis of tangent vector fields to S^3. Moreover, we have $[X_1, X_2] = 2X_3$, $[X_2, X_3] = 2X_1$ and $[X_3, X_1] = 2X_2$. In [**2**], the authors define a metric $<,>_1$ on S^3 such that X_1, X_2 and X_3 are orthogonal and such that $< X_1, X_1 >_1 = < X_2, X_2 >_1 = 6$ and $< X_3, X_3 >_1 = 36$. Then $E_1 = \frac{1}{\sqrt{6}}X_1$, $E_2 = \frac{1}{\sqrt{6}}X_2$ and $E_3 = \frac{1}{6}X_3$ form an orthonormal basis on S^3. We denote the Levi-Civita connection of $<,>_1$ by ∇, then $\nabla_{E_i}E_j$ and $R(E_i, E_j)E_k$ can be computed. We now define a symmetric bilinear form α on TS^3 in accordance with Theorem 2.2 of [**2**] by

$$\alpha(E_1, E_1) = \sqrt{\tfrac{5}{3}}E_1, \quad \alpha(E_3, E_1) = 0, \quad \alpha(E_1, E_2) = -\sqrt{\tfrac{5}{3}}E_2,$$
$$\alpha(E_3, E_2) = 0, \quad \alpha(E_2, E_2) = -\sqrt{\tfrac{5}{3}}E_1, \quad \alpha(E_3, E_3) = 0.$$

A straightforward computation shows that α satisfies the conditions of the existence theorem, i.e. Theorem 3.2 of [**2**]. Hence we obtain a Lagrangian isometric immersion

$$\varphi_1 : (S^3, < .,. >_1) \to S^6(1),$$

whose second fundamental form satisfies $h(X, Y) = J\alpha(X, Y)$. That is,

$$h(E_1, E_1) = \sqrt{\tfrac{5}{3}}JE_1, \quad h(E_3, E_1) = 0, \quad h(E_1, E_2) = -\sqrt{\tfrac{5}{3}}JE_2,$$
$$h(E_3, E_2) = 0, \quad h(E_2, E_2) = -\sqrt{\tfrac{5}{3}}JE_1, \quad h(E_3, E_3) = 0.$$

Hence

$$\Sigma_{i,j,k,l} h_{ij}^{1^*} h_{ik}^{l^*} h_{kj}^{l^*} = \left(\sqrt{\frac{5}{3}}\right)^3 - \left(\sqrt{\frac{5}{3}}\right)^3 + \sqrt{\frac{5}{3}}\frac{5}{3} - \sqrt{\frac{5}{3}}\frac{5}{3} = 0,$$
$$\Sigma_{i,j,k,l} h_{ij}^{2^*} h_{ik}^{l^*} h_{kj}^{l^*} = 0 + 0 = 0 = \Sigma_{i,j,k,l} h_{ij}^{3^*} h_{ik}^{l^*} h_{kj}^{l^*},$$
$$S = \sum_{l,i,j} (h_{ij}^{l^*})^2 = \frac{20}{3}, \qquad \tau = -\frac{1}{3}.$$

From Lemma 2.4, we know that M is a Lagrangian Willmore submanifold with constant scalar curvature.

Example 3.4. (*Example* 3.2 *of* [**2**]) We also consider the unit sphere S^3 in R^4. Let X_1, X_2 and X_3 be the vector fields defined in the previous example. In [**2**], the authors define a metric $< .,. >_2$ on S^3 such that X_1, X_2 and X_3 are orthogonal and such that $< X_1, X_1 >_2 = < X_2, X_2 >_2 = 2$ and $< X_3, X_3 >_2 = 4$. Then $E_1 = \frac{1}{\sqrt{2}}X_1$, $E_2 = \frac{1}{\sqrt{2}}X_2$ and $E_3 = -\frac{1}{2}X_3$ form an orthonormal basis on S^3. We denote the Levi-Civita connection of $<,>_2$ by ∇, then $\nabla_{E_i}E_j$ and $R(E_i, E_j)E_k$ can be computed. We now define a symmetric bilinear form α on TS^3 in accordance with Theorem 2.2 of [**2**] by

$$\alpha(E_1, E_1) = E_1, \quad \alpha(E_3, E_1) = 0, \quad \alpha(E_1, E_2) = -E_2,$$
$$\alpha(E_3, E_2) = 0, \quad \alpha(E_2, E_2) = -E_1, \quad \alpha(E_3, E_3) = 0.$$

A straightforward computation shows that α satisfies the conditions of the existence theorem, i.e. Theorem 3.2 of [**2**]. Hence we obtain a Lagrangian isometric immersion

$$\varphi_2 : (S^3, <.,.>_2) \to S^6(1),$$

whose second fundamental form satisfies $h(X, Y) = J\alpha(X, Y)$. That is,

$$\begin{aligned} &h(E_1, E_1) = JE_1, \quad h(E_3, E_1) = 0, \quad h(E_1, E_2) = -JE_2, \\ &h(E_3, E_2) = 0, \quad h(E_2, E_2) = -JE_1, \quad h(E_3, E_3) = 0. \end{aligned}$$

Hence

$$\Sigma_{i,j,k,l} h^{1^*}_{ij} h^{l^*}_{ik} h^{l^*}_{kj} = 1 - 1 + 1 - 1 = 0,$$

$$\Sigma_{i,j,k,l} h^{2^*}_{ij} h^{l^*}_{ik} h^{l^*}_{kj} = 0 = \Sigma_{i,j,k,l} h^{3^*}_{ij} h^{l^*}_{ik} h^{l^*}_{kj},$$

$$S = \sum_{l,i,j} (h^{l^*}_{ij})^2 = 4, \quad \tau = 1.$$

From Lemma 2.4, we know that M is a Lagrangian Willmore submanifold with constant scalar curvature.

EXAMPLE 3.5. ([**5**]) Define a map

$$\varphi_3 : S^3(1) = \left\{(x_1, x_2, x_3, x_4) \in R^4 \middle| \sum_{i=1}^4 x_i^2 = 1\right\}$$
$$\longrightarrow S^6(1) = \left\{(y_1, y_2, y_3, y_4, y_5, y_6, y_7) \in R^7 \mid \sum_{i=1}^7 y_i^2 = 1\right\}$$

where

$$\begin{aligned} &y_1 = \tfrac{1}{9}\left(5x_1^2 + 5x_2^2 - 5x_3^2 - 5x_4^2 + 4x_1\right); \quad y_2 = -\tfrac{2}{3}x_2 \\ &y_3 = \tfrac{2\sqrt{5}}{9}\left(x_1^2 + x_2^2 - x_3^2 - x_4^2 - x_1\right); \quad y_4 = \tfrac{\sqrt{3}}{9\sqrt{2}}(-10x_3x_1 - 2x_3 - 10x_2x_4) \\ &y_5 = \tfrac{\sqrt{3}\sqrt{5}}{9\sqrt{2}}(2x_1x_4 - 2x_4 - 2x_2x_3); \quad y_6 = \tfrac{\sqrt{3}\sqrt{5}}{9\sqrt{2}}(2x_1x_3 - 2x_3 + 2x_2x_4) \\ &y_7 = -\tfrac{\sqrt{3}}{9\sqrt{2}}(10x_1x_4 + 2x_4 - 10x_2x_3) \end{aligned}$$

By direct computation, we have

$$\begin{aligned} &h(e_1, e_1) = \tfrac{\sqrt{5}}{2}Je_1, \quad h(e_2, e_2) = -\tfrac{\sqrt{5}}{4}Je_1, \quad h(e_3, e_3) = -\tfrac{\sqrt{5}}{4}Je_1, \\ &h(e_1, e_2) = -\tfrac{\sqrt{5}}{4}Je_2, \quad h(e_2, e_3) = 0, \quad h(e_1, e_3) = -\tfrac{\sqrt{5}}{4}Je_3. \end{aligned}$$

Then we know that $\varphi_3 : S^3 \to S^6(1)$ is a Lagrangian immersion with constant scalar curvature $\frac{23}{16}$. On the other hand, we have

$$\Sigma_{i,j,k,l} h^{1^*}_{ij} h^{l^*}_{ik} h^{l^*}_{kj} = \frac{45\sqrt{5}}{64} \neq 0.$$

From Lemma 2.4, we obtain that $\varphi_3 : S^3 \to S^6(1)$ is not a Willmore submanifold.

4. Theorem and the proof. First of all, we give this paper's main theorem.

THEOREM 4.1. *Let $\varphi : M^3 \to S^6(1)$ be a Lagrangian Willmore immersion with constant scalar curvature. Then locally one of the following four possibilities occurs:*
(1) *φ is congruent with a totally geodesic immersion;*
(2) *φ is congruent with a constant sectional curvature $\frac{1}{16}$ immersion;*
(3) *φ is congruent with φ_1;*
(4) *φ is congruent with φ_2;*
Here φ_1 and φ_2 are as in Section 3.

Proof. From Gauss equations (2.12) and $\tau =$ constant, we get

$$S = \sum_{i,j,k} \left(h_{ij}^{k^*}\right)^2 = 6 - 2\tau = C = \text{const.} \tag{4.1}$$

If $C = 0$, then M is totally geodesic and φ is congruent with a totally geodesic immersion.

If $C \neq 0$, then every point is not totally geodesic point. From Lemma 2.1, we can choose an orthonormal basis $\{e_1, e_2, e_3\}$ of T_pM such that

$$\begin{aligned} h(e_1, e_1) &= \lambda_1 Je_1, & h(e_2, e_2) &= \lambda_2 Je_1 + \lambda_3 Je_2 + \lambda_4 Je_3,\\ h(e_1, e_2) &= \lambda_2 Je_2, & h(e_2, e_3) &= \lambda_4 Je_2 - \lambda_3 Je_3,\\ h(e_1, e_3) &= -(\lambda_1 + \lambda_2)Je_3, & h(e_3, e_3) &= -(\lambda_1 + \lambda_2)Je_1 - \lambda_3 Je_2 - \lambda_4 Je_3. \end{aligned}$$

where $\lambda_1 > 0$.

By direct calculation, we obtain

$$\begin{aligned} S &= 4\lambda_1^2 + 6\lambda_2^2 + 6\lambda_1\lambda_2 + 4\lambda_3^2 + 4\lambda_4^2,\\ \Sigma_{i,j,k,l} h_{ij}^{1^*} h_{ik}^{l^*} h_{kj}^{l^*} &= -4\lambda_1^2\lambda_2 - 4\lambda_1\lambda_2^2 - 2\lambda_1\lambda_3^2 - 2\lambda_1\lambda_4^2,\\ \Sigma_{i,j,k,l} h_{ij}^{2^*} h_{ik}^{l^*} h_{kj}^{l^*} &= -2\lambda_1^2\lambda_3 - 2\lambda_1\lambda_2\lambda_3 + 4\lambda_2^2\lambda_3,\\ \Sigma_{i,j,k,l} h_{ij}^{3^*} h_{ik}^{l^*} h_{kj}^{l^*} &= -4\lambda_1^2\lambda_4 - 10\lambda_1\lambda_2\lambda_4 - 4\lambda_2^2\lambda_4. \end{aligned}$$

Then, by using of Lemma 2.4, $\lambda_1 > 0$ and $S =$ constant, we can deduce that

$$\begin{cases} 4\lambda_1^2 + 6\lambda_2^2 + 6\lambda_1\lambda_2 + 4\lambda_3^2 + 4\lambda_4^2 = C = \text{const},\\ 2\lambda_1\lambda_2 + 2\lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 0,\\ \lambda_3\left(\lambda_1^2 + \lambda_1\lambda_2 - 2\lambda_2^2\right) = 0,\\ \lambda_4\left(2\lambda_1^2 + 5\lambda_1\lambda_2 + 2\lambda_2^2\right) = 0. \end{cases} \tag{4.2}$$

In order to solve these equations, we consider the following cases.

Case 1: $\lambda_2 = 0$.

Then (4.2) becomes

$$\begin{cases} 4\lambda_1^2 + 4\lambda_3^2 + 4\lambda_4^2 = C\\ \lambda_3^2 + \lambda_4^2 = 0\\ \lambda_1^2\lambda_3 = 0\\ \lambda_1^2\lambda_4 = 0 \end{cases}$$

Therefore, we have

$$\lambda_1 = \frac{\sqrt{C}}{2}, \quad \lambda_2 = 0, \quad \lambda_3 = 0, \quad \lambda_4 = 0. \tag{4.3}$$

Case 2: $\lambda_2 \neq 0, \quad \lambda_3 = 0, \quad \lambda_4 = 0.$
In this case, (4.2) becomes

$$\begin{cases} 4\lambda_1^2 + 6\lambda_2^2 + 6\lambda_1\lambda_2 = C \\ \lambda_1 + \lambda_2 = 0 \end{cases}$$

Then we have

$$\lambda_1 = \frac{\sqrt{C}}{2}, \quad \lambda_2 = -\frac{\sqrt{C}}{2}, \quad \lambda_3 = \lambda_4 = 0. \tag{4.4}$$

Case 3: $\lambda_2 \neq 0, \quad \lambda_3 = 0, \quad \lambda_4 \neq 0.$
In this case, (4.2) becomes

$$\begin{cases} 4\lambda_1^2 + 6\lambda_2^2 + 6\lambda_1\lambda_2 + 4\lambda_4^2 = C \\ 2\lambda_1\lambda_2 + 2\lambda_2^2 + \lambda_4^2 = 0 \\ 2\lambda_1^2 + 5\lambda_1\lambda_2 + 2\lambda_2^2 = 0 \end{cases}$$

From $2\lambda_1\lambda_2 = -2\lambda_2^2 - \lambda_4^2 < 0$, we deduce that $\lambda_2 < 0$; From $2\lambda_1^2 + 5\lambda_1\lambda_2 + 2\lambda_2^2 = 0$, it then follows that either $\lambda_1 = -2\lambda_2$ or $\lambda_1 = -\frac{1}{2}\lambda_2$. If $\lambda_1 = -\frac{1}{2}\lambda_2$, then we obtain $\lambda_2^2 + \lambda_4^2 = 0$. It is a contradiction. Hence $\lambda_1 = -2\lambda_2$. After a straightforward calculation one has

$$\lambda_1 = \frac{\sqrt{2C}}{3}, \quad \lambda_2 = -\frac{\sqrt{2C}}{6}, \quad \lambda_3 = 0, \quad \lambda_4^2 = \frac{C}{9}. \tag{4.5}$$

Case 4: $\lambda_2 \neq 0, \lambda_3 \neq 0, \lambda_4 = 0.$
In this case, (4.2) becomes

$$\begin{cases} 4\lambda_1^2 + 6\lambda_2^2 + 6\lambda_1\lambda_2 + 4\lambda_3^2 = C \\ 2\lambda_1\lambda_2 + 2\lambda_2^2 + \lambda_3^2 = 0 \\ \lambda_1^2 + \lambda_1\lambda_2 - 2\lambda_2^2 = 0 \end{cases}$$

From $2\lambda_1\lambda_2 = -2\lambda_2^2 - \lambda_3^2$, we see that $\lambda_2 < 0$; From $\lambda_1^2 + \lambda_1\lambda_2 - 2\lambda_2^2 = 0$, it follows that $\lambda_1 = \lambda_2$ or $\lambda_1 = -2\lambda_2$. Since $\lambda_1 > 0$ and $\lambda_2 < 0$, we deduce $\lambda_1 = -2\lambda_2$. Then we obtain

$$\lambda_1 = \frac{\sqrt{2C}}{3}, \quad \lambda_2 = -\frac{\sqrt{2C}}{6}, \quad \lambda_3^2 = \frac{C}{9}, \quad \lambda_4 = 0. \tag{4.6}$$

Case 5: $\lambda_2 \neq 0, \lambda_3 \neq 0, \lambda_4 \neq 0$.
In this case, (4.2) becomes

$$\begin{cases} 4\lambda_1^2 + 6\lambda_2^2 + 6\lambda_1\lambda_2 + 4\lambda_3^2 + 4\lambda_4^2 = C \\ 2\lambda_1\lambda_2 + 2\lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 0 \\ \lambda_1^2 + \lambda_1\lambda_2 - 2\lambda_2^2 = 0 \\ 2\lambda_1^2 + 5\lambda_1\lambda_2 + 2\lambda_2^2 = 0 \end{cases}$$

From $2\lambda_1\lambda_2 = -2\lambda_2^2 - \lambda_3^2 - \lambda_4^2$, we know that $\lambda_2 < 0$; Since $\lambda_1^2 + \lambda_1\lambda_2 - 2\lambda_2^2 = 0$, we deduce that $\lambda_1 = \lambda_2$ or $\lambda_1 = -2\lambda_2$. Since $\lambda_1 > 0$ and $\lambda_2 < 0$, we find $\lambda_1 = -2\lambda_2$. Then we obtain

$$\lambda_1 = \frac{\sqrt{2C}}{3}, \quad \lambda_2 = -\frac{\sqrt{2C}}{6}, \quad \lambda_3^2 + \lambda_4^2 = \frac{C}{9}. \tag{4.7}$$

Firstly, we consider Case 3, Case 4 and Case 5. Let $a_1 = \lambda_1$, $a_2 = \lambda_2$ and $a_3 = -(\lambda_1 + \lambda_2)$, from (2.7) and Gauss equations (2.11), we have

$$\begin{aligned} R_{1jkl} &= -(\delta_{1k}\delta_{jl} - \delta_{1l}\delta_{jk}) - \textstyle\sum_r \left(h_{1k}^{r^*}h_{jl}^{r^*} - h_{1l}^{r^*}h_{jk}^{r^*}\right) \\ &= -(\delta_{1k}\delta_{jl} - \delta_{1l}\delta_{jk}) - \textstyle\sum_r \left(a_k\delta_{kr}h_{jl}^{r^*} - a_l\delta_{lr}h_{jk}^{r^*}\right) \\ &= -(\delta_{1k}\delta_{jl} - \delta_{1l}\delta_{jk}) - \left(a_k h_{jl}^{k^*} - a_l h_{jk}^{l^*}\right) \\ &= -(\delta_{1k}\delta_{jl} - \delta_{1l}\delta_{jk}) - (a_k - a_l)h_{jk}^{l^*}, \end{aligned}$$

$$R_{1j1l} = -(\delta_{11}\delta_{jl} - \delta_{1l}\delta_{j1}) - (a_1 - a_l)h_{j1}^{l^*},$$

$$R_{1212} = -1 - (a_1 - a_2)h_{21}^{2^*} = -1 - (\lambda_1 - \lambda_2)\lambda_2 = -1 + 3\lambda_2^2 = -\left(1 - \frac{C}{6}\right),$$

$$R_{1313} = -1 - (a_1 - a_3)h_{31}^{3^*} = -1 + (2\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2) = -1 + 3\lambda_2^2 = -\left(1 - \frac{C}{6}\right),$$

$$R_{1223} = -(\delta_{12}\delta_{23} - \delta_{13}\delta_{22}) - (a_2 - a_3)h_{22}^{3^*} = -(\lambda_1 + 2\lambda_2)\lambda_4 = 0,$$

$$R_{1323} = -(\delta_{12}\delta_{33} - \delta_{13}\delta_{23}) - (a_2 - a_3)h_{32}^{3^*} = (\lambda_1 + 2\lambda_2)\lambda_3 = 0,$$

$$\begin{aligned} R_{2323} &= -1 - \textstyle\sum_r \left(h_{22}^{r^*}h_{33}^{r^*} - h_{23}^{r^*}h_{23}^{r^*}\right) \\ &= -1 - \left[\lambda_2(-\lambda_1 - \lambda_2) - 2\left(\lambda_3^2 + \lambda_4^2\right)\right] \\ &= -1 - \left(\lambda_2^2 - \tfrac{2C}{9}\right) = -\left(1 - \tfrac{C}{6}\right). \end{aligned}$$

Hence we have

$$R_{ijkl} = -\left(1 - \frac{C}{6}\right)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).$$

That is, M is a submanifold with constant sectional curvature c. In [**7**], Ejiri proved that if M is a submanifold with constant sectional curvature c , then $c = 1$ (and M is totally geodesic) or $c = \frac{1}{16}$. In these cases, $C \neq 0$ (and M is not totally geodesic). We deduce that $c = \frac{1}{16}$. It follows that $1 - \frac{C}{6} = \frac{1}{16}$. Therefore $C = S = \frac{45}{8}$.

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Secondly, we consider Case 1:
In this case, we have

$$\begin{aligned} h(e_1, e_1) &= \frac{\sqrt{C}}{2} J e_1, \quad h(e_3, e_3) = -\frac{\sqrt{C}}{2} J e_1, \; h(e_1, e_2) = 0 \\ h(e_1, e_3) &= -\frac{\sqrt{C}}{2} J e_3, \; h(e_2, e_3) = 0, \qquad\quad h(e_2, e_2) = 0 \end{aligned} \tag{4.8}$$

We see from Remark 2.2 that $\delta_M = 2$. It follows from Lemma 2.3 that M is congruent with φ_1 or φ_2.

Thirdly, we consider Case 2. Applying the similar argument as in Case 1, we can obtain that φ is also congruent with φ_1 or φ_2. Theorem 4.1 is proved.

COROLLARY 4.1. *The values for the norm square of the second fundamental form S of Lagrangian Willmore submanifold with $S =$ constant in $(S^6(1), J)$ are* 0, 4, $\frac{45}{8}$, $\frac{20}{3}$.

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