J. Aust. Math. Soc. **109** (2020), 250–261
doi:10.1017/S1446788719000272

DISTALITY OF CERTAIN ACTIONS ON p-ADIC SPHERES

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(Received 18 July 2018; accepted 30 April 2019; first published online 29 July 2019)

Communicated by B. Martin

Abstract

Consider the action of $\mathrm{GL}(n, \mathbb{Q}_p)$ on the p-adic unit sphere $\mathcal{S}_n$ arising from the linear action on $\mathbb{Q}_p^n \setminus \{0\}$. We show that for the action of a semigroup $\mathfrak{S}$ of $\mathrm{GL}(n, \mathbb{Q}_p)$ on $\mathcal{S}_n$, the following are equivalent: (1) $\mathfrak{S}$ acts distally on $\mathcal{S}_n$; (2) the closure of the image of $\mathfrak{S}$ in $\mathrm{PGL}(n, \mathbb{Q}_p)$ is a compact group. On $\mathcal{S}_n$, we consider the 'affine' maps $\overline{T}_a$ corresponding to T in $\mathrm{GL}(n, \mathbb{Q}_p)$ and a nonzero a in $\mathbb{Q}_p^n$ satisfying $\|T^{-1}(a)\|_p < 1$. We show that there exists a compact open subgroup V, which depends on T, such that $\overline{T}_a$ is distal for every nonzero $a \in V$ if and only if T acts distally on $\mathcal{S}_n$. The dynamics of 'affine' maps on p-adic unit spheres is quite different from that on the real unit spheres.

2010 *Mathematics subject classification*: primary 37B05, 22E35; secondary 20M20, 20G25.

Keywords and phrases: distal actions, affine maps, p-adic unit spheres.

1. Introduction

Let X be a (Hausdorff) topological space. Let $\mathfrak{S}$ be a semigroup of homeomorphisms of X. The action of $\mathfrak{S}$ is said to be *distal* if, for any pair of distinct elements $x, y \in X$, the closure of $\{(T(x), T(y)) \mid T \in \mathfrak{S}\}$ does not intersect the diagonal $\{(d, d) \mid d \in X\}$; (equivalently, we say that the $\mathfrak{S}$ acts *distally* on X). Let $T : X \to X$ be a homeomorphism. The map T is said to be *distal* if the group $\{T^n\}_{n\in\mathbb{Z}}$ acts distally on X. If X is compact, then T is distal if and only if the semigroup $\{T^n\}_{n\in\mathbb{N}}$ acts distally (cf. Berglund *et al.* [1]).

The notion of distality was introduced by Hilbert (cf. Moore [8]) and studied by many in different contexts (see Ellis [3], Furstenberg [4], Raja and Shah [10, 11] and Shah [12], and references cited therein). Note that a homeomorphism T of a topological space is distal if and only if T^n is so, for every $n \in \mathbb{Z}$.

For the p-adic field $\mathbb{Q}_p$ for a prime p, let $|\cdot|_p$ denote the p-adic absolute value on $\mathbb{Q}_p$ and for $x = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$, $n \in \mathbb{N}$, let $\|x\|_p = \max_{1\le i\le n} |x_i|_p$. This defines a p-adic

R. Shah would like to acknowledge the support of DST (SERB) through the MATRICS research grant. A. K. Yadav would like to acknowledge the support for a research assistantship from DST-PURSE grant in Jawaharlal Nehru University.

vector space norm on $\mathbb{Q}_p^n$. Let $\mathcal{S}_n = \{x \in \mathbb{Q}_p^n \mid \|x\|_p = 1\}$ be the p-adic unit sphere (in $\mathbb{Q}_p^n$). We refer the reader to Koblitz [7] for basic facts on p-adic vector spaces. We first define a canonical group action of $\mathrm{GL}(n, \mathbb{Q}_p)$ on $\mathcal{S}_n$ as follows. For $T \in \mathrm{GL}(n, \mathbb{Q}_p)$, let $\overline{T} : \mathcal{S}_n \to \mathcal{S}_n$ be defined as $\overline{T}(x) = \|T(x)\|_p T(x)$, $x \in \mathcal{S}_n$. This is a continuous group action. We show that a semigroup $\mathfrak{S}$ of $\mathrm{GL}(n, \mathbb{Q}_p)$ acts distally on $\mathcal{S}_n$ if and only if the closure of the image of $\mathfrak{S}$ in $\mathrm{PGL}(n, \mathbb{Q}_p) = \mathrm{GL}(n, \mathbb{Q}_p)/\mathcal{D}$ is a compact group, where $\mathcal{D}$ is the centre of $\mathrm{GL}(n, \mathbb{Q}_p)$ (see Theorem 2.4). This is a p-adic analogue of Shah and Yadav [13, Theorem 1] for the action of a semigroup in $\mathrm{GL}(n + 1, \mathbb{R})$ on the real unit sphere $\mathbb{S}^n$. In particular, we show for $T \in \mathrm{SL}(n, \mathbb{Q}_p)$ that T is distal if and only if $\overline{T}$ is distal (more generally, see Proposition 2.3, Corollary 2.5 and the subsequent remark). For $T \in \mathrm{GL}(n, \mathbb{Q}_p)$ and $a \in \mathbb{Q}_p^n \setminus \{0\}$, if $\|T^{-1}(a)\|_p < 1$, then the corresponding 'affine' map on $\mathcal{S}_n$, $\overline{T}_a(x) = \|a + T(x)\|_p (a + T(x))$, $x \in \mathcal{S}_n$, is a homeomorphism. If T or, more generally, $\overline{T}$ is distal, then there exists a neigbourhood V of 0 in $\mathbb{Q}_p^n$ such that, for every nonzero a in V, $\overline{T}_a$ is distal. If $\overline{T}$ is not distal, then every neighbourhood of 0 contains a nonzero a such that $\overline{T}_a$ is not distal; see Theorem 3.2 and Corollary 3.3. For such 'affine' actions on the real unit sphere $\mathbb{S}^n$, there are many examples where $T \in \mathrm{GL}(n + 1, \mathbb{R})$ is such that T and/or $\overline{T}$ are distal but $\overline{T}_a$ is not distal; see [13, Theorem 7, Corollaries 10 and 12]. This illustrates that the dynamics of such 'affine' actions on $\mathcal{S}_n$ is different from that on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.

For an invertible linear map T on a p-adic vector space $V \approx \mathbb{Q}_p^n$, let $C(T) = \{v \in V \mid T^m(v) \to 0 \text{ as } m \to \infty\}$ and $M(T) = \{v \in V \mid \{T^m(v)\}_{m\in\mathbb{Z}} \text{ is relatively compact}\}$. These are closed subspaces of V, $C(T)$ is known as the contraction space of T, and $V = C(T) \oplus M(T) \oplus C(T^{-1})$. It is easy to see that T is distal (on V) if and only if $C(T)$ and $C(T^{-1})$ are trivial. We refer the reader to Wang [14] for more details on the structure of p-adic contraction spaces. We will use the notion of contraction spaces below.

2. Distality of the semigroup actions on $\mathcal{S}_n$

Let $\mathbb{Q}_p^n$ be an n-dimensional p-adic vector space equipped with the p-adic norm defined as above. For $T \in \mathrm{GL}(n, \mathbb{Q}_p)$, let $\|T\|_p = \sup\{\|T(x)\|_p \mid x \in \mathbb{Q}_p, \|x\|_p = 1\}$. Observe that the norm of an element or a matrix, defined this way, is of the from p^m for some $m \in \mathbb{Z}$. The map $\mathrm{GL}(n, \mathbb{Q}_p) \times \mathbb{Q}_p^n \to \mathbb{Q}_p^n$ given by $(T, x) \mapsto T(x)$, $T \in \mathrm{GL}(n, \mathbb{Q}_p)$, $x \in \mathbb{Q}_p^n$, is continuous. We call $T \in \mathrm{GL}(n, \mathbb{Q}_p)$ an *isometry* if it preserves the norm, that is, if T keeps the p-adic unit sphere $\mathcal{S}_n$ invariant. Note that T is an isometry if and only if $\|T\|_p = 1 = \|T^{-1}\|_p$. For $x, y \in \mathbb{Q}_p^n$, $\|x + y\|_p \leq \max\{\|x\|_p, \|y\|_p\}$; the equality holds if $\|x\|_p \neq \|y\|_p$. We will use this fact extensively. We first consider the group action of $\mathrm{GL}(n, \mathbb{Q}_p)$ on $\mathcal{S}_n$ defined in the introduction. For semigroups of $\mathrm{GL}(n, \mathbb{Q}_p)$, we prove a result analogous to [13, Theorem 1] (see Theorem 2.4).

Recall that $T \in \mathrm{GL}(n, \mathbb{Q}_p)$ is said to be distal if $\{T^m\}_{m\in\mathbb{Z}}$ acts distally on $\mathbb{Q}_p^n$.

The following useful lemma may be known. We will give a short proof for the sake of completeness.

https://doi.org/10.1017/S1446788719000272 Published online by Cambridge University Press

Lemma 2.1. *Let $T \in \mathrm{GL}(n, \mathbb{Q}_p)$. The following statements are equivalent.*

(1) *T is distal.*

(2) *The closure of the group generated by T in $\mathrm{GL}(n, \mathbb{Q}_p)$ is compact.*

(3) *T^m is an isometry for some $m \in \mathbb{N}$.*

Proof. (3) $\Rightarrow$ (2) is obvious and (2) $\Rightarrow$ (1) follows as compact groups act distally. Now suppose T is distal, that is. $\{T^m\}_{m\in\mathbb{Z}}$ acts distally on $\mathbb{Q}_p^n$. Then the contraction spaces $C(T)$ and $C(T^{-1})$ are trivial. By [14, Lemma 3.4], we get that

$$\mathbb{Q}_p^n = M(T) = \{x \in \mathbb{Q}_p^n \mid \{T^m(x)\}_{m\in\mathbb{Z}} \text{ is relatively compact}\}.$$

By [14, Proposition 1.3], $\bigcup_{m\in\mathbb{Z}} T^m(\mathcal{S}_n)$ is relatively compact, that is, $\{\|T^m\|_p\}_{m\in\mathbb{Z}}$ is bounded and hence $\{T^m \mid m \in \mathbb{Z}\}$ is relatively compact in $\mathrm{GL}(n, \mathbb{Q}_p)$. This proves (1) $\Rightarrow$ (2). Now suppose T is contained in a compact group. Then $T^{\pm m_k} \to \mathrm{Id}$, for some $\{m_k\} \subset \mathbb{N}$ (cf. [6]). Therefore, $\|T^{\pm m_k}\|_p \to 1$, and as $\{\|T^m\|_p \mid m \in \mathbb{Z}\} \subset \{p^l \mid l \in \mathbb{Z}\}$ we get that, for all large k, $\|T^{\pm m_k}\|_p = 1$ and T^{m_k} is an isometry. Therefore, (2) $\Rightarrow$ (3). □

We say that a topological group G acts continuously on a topological space X by homeomorphisms if there is a homomorphism $\psi : G \to \mathrm{Homeo}(X)$ such that the corresponding map $G \times X \to X$ given by $(g, x) \mapsto \psi(g)(x)$, $g \in G$, $x \in X$, is continuous. We say that a semigroup H of G (respectively, $T \in G$) acts distally on X if $\psi(H)$ acts distally on X (respectively, $\psi(T)$ is distal). We state a useful lemma which is well known and can be proven easily.

Lemma 2.2. *Let X be a Hausdorff topological space and let G be a Hausdorff topological group which acts continuously on X by homeomorphisms. Let H be a semigroup and K be a compact subgroup of G such that all the elements of H normalize K. Then the semigroup HK acts distally on X if and only if H acts distally on X. In particular, if $T, S \in G$ are such that $TS = ST$ and S generates a relatively compact group in G, then T acts distally on X if and only if TS acts distally on X.*

We now recall the natural action of $\mathrm{GL}(n, \mathbb{Q}_p)$ on $\mathcal{S}_n$ defined earlier: for $T \in \mathrm{GL}(n, \mathbb{Q}_p)$ and $x \in \mathcal{S}_n$, $\overline{T}(x) = \|T(x)\|_p T(x)$. Here, $\overline{T}$ defines a homeomorphism of $\mathcal{S}_n$ and it is trivial if and only if $T = p^n \,\mathrm{Id}$ for some $n \in \mathbb{Z}$. The map $\mathrm{GL}(n, \mathbb{Q}_p) \to \mathrm{Homeo}(\mathcal{S}_n)$, given by $T \mapsto \overline{T}$ as above, is a homomorphism which factors through the discrete central subgroup $\{p^n \,\mathrm{Id} \mid n \in \mathbb{Z}\}$ of $\mathrm{GL}(n, \mathbb{Q}_p)$. The corresponding map $\mathrm{GL}(n, \mathbb{Q}_p) \times \mathcal{S}_n \to \mathcal{S}_n$, given by $(T, x) \mapsto \overline{T}(x)$, is continuous. Therefore, $\mathrm{GL}(n, \mathbb{Q}_p)$ acts continuously on $\mathcal{S}_n$ by homeomorphisms as above. Observe that $\mathcal{S}_1 = \mathbb{Z}_p^* = \{x \in \mathbb{Q}_p \mid |x|_p = 1\}$ and $\mathrm{GL}(1, \mathbb{Q}_p) = \mathbb{Q}_p \setminus \{0\}$ acts distally on $\mathcal{S}_1$ as $\overline{T} = \|T\|_p T \in \mathcal{S}_1$ for every $T \in \mathrm{GL}(1, \mathbb{Q}_p)$. The following will be useful in proving the main result of this section.

Proposition 2.3. *Let $T \in \mathrm{GL}(n, \mathbb{Q}_p)$. If bT is distal for some $b \in \mathbb{Q}_p$, then $\overline{T}$ is distal. Conversely, if $\overline{T}$ is distal, then, for some $m \in \mathbb{N}$ and $l \in \mathbb{Z}$, $p^l T^m$ is distal. If $|\det T|_p = 1$ and $\overline{T}$ is distal, then T is distal.*

https://doi.org/10.1017/S1446788719000272 Published online by Cambridge University Press

Proof. Observe that bT is distal if and only if $|b|_p^{-1}T$ is so. As $\overline{T} = \overline{p^m T}$ for any $m \in \mathbb{Z}$, we may replace T by $|b|_p^{-1}T$ and assume that T is distal. By Lemma 2.1, T generates a relatively compact group, and hence $\overline{T}$ is distal. Conversely, suppose $\overline{T}$ is distal. By [14, 3.3], there exists $m \in \mathbb{N}$ such that $T^m = AUC$, where C is a diagonal matrix, U is unipotent, A is semisimple, A, U and C commute with each other and A as well as U generate a relatively compact group. Now by Lemma 2.2, we have that $\overline{C}$ is distal. Here, $C = DD' = D'D$ for some diagonal matrices D and D' such that the diagonal entries of D (respectively, D') are of the form p^{l_k}, $l_k \in \mathbb{Z}$, $k = 1, \dots n$ (respectively, in $\mathbb{Z}_p^*$). Since D' also generates a relatively compact group and it commutes with D, by Lemma 2.2, $\overline{D}$ is distal. It is enough to show that $D = p^l\,\mathrm{Id}$, as in this case, D would be central in $\mathrm{GL}(n, \mathbb{Q}_p)$ and this would imply that AU and D' commute, and hence, $p^{-l}T^m = AUD'$ would generate a relatively compact group which in turn would imply that it is distal. If possible, suppose p^l and p^{l_1} are two entries in D such that $l < l_1$. As $\overline{D} = \overline{p^{-l}D}$, we get for $D_1 = p^{-l}D$ that $\overline{D}_1 = \overline{D}$ is distal, 1 is an eigenvalue of D_1 and D_1 has another eigenvalue p^{l_1-l} which has p-adic absolute value less than 1. Then the contraction space of D_1, $C(D_1) \neq \{0\}$, as we can take a nonzero $y \in \mathbb{Q}_p^n$ satisfying $D_1(y) = p^k y$ for $k = l_1 - l \in \mathbb{N}$; and it follows that $y \in C(D_1)$. Let $x \in \mathcal{S}_n$ be such that $D_1(x) = x$ and let y be as above such that $0 < \|y\|_p < 1$. Then $\overline{D}_1(x) = x$ and $x + y \in \mathcal{S}_n$. Now $D_1^i(x + y) = (x + p^{ki}y) \to x \in \mathcal{S}_n$ as $i \to \infty$. Therefore, $\overline{D}_1^i(x + y) \to x$ and it leads to a contradiction as $\overline{D}_1$ is distal. Therefore, $D = p^l\,\mathrm{Id}$ and $p^{-l}T^m$ is distal.

Suppose $|\det T|_p = 1$. Then $|\det(T^m)|_p = |\det T|_p^m = 1$. As $\overline{T}$ is distal, $T^m = p^l S$ for some $l \in \mathbb{Z}$, where S generates a relatively compact group. Therefore, $|\det S|_p = 1$, and hence $l = 0$ and $T^m = S$. This implies that T also generates a relatively compact group, and by Lemma 2.1, it is distal. □

From now on, $\mathcal{D} = \{b\,\mathrm{Id} \mid b \in \mathbb{Q}_p\}$, the centre of $\mathrm{GL}(n, \mathbb{Q}_p)$. The following theorem characterizes distal actions of semigroups on $\mathcal{S}_n$. Recall that $\mathrm{PGL}(n, \mathbb{Q}_p) = \mathrm{GL}(n, \mathbb{Q}_p)/\mathcal{D}$.

Theorem 2.4. *Let $\mathfrak{S} \subset \mathrm{GL}(n, \mathbb{Q}_p)$ be a semigroup. Then the following are equivalent.*

(i) *$\mathfrak{S}$ acts distally on $\mathcal{S}_n$.*
(ii) *The group generated by $\mathfrak{S}$ acts distally on $\mathcal{S}_n$.*
(iii) *The closure of $(\mathfrak{S}\mathcal{D})/\mathcal{D}$ in $\mathrm{PGL}(n, \mathbb{Q}_p)$ is a compact group.*

Proof. Suppose (i) holds. First suppose $\mathfrak{S} \subset \mathrm{SL}(n, \mathbb{Q}_p)$. As the closure $\overline{\mathfrak{S}}$ of $\mathfrak{S}$ is a semigroup in $\mathrm{SL}(n, \mathbb{Q}_p)$ and it also acts distally on $\mathcal{S}_n$, we may assume that $\mathfrak{S}$ is closed. By Proposition 2.3 and Lemma 2.1, each element in $\mathfrak{S}$ generates a relatively compact group (in $\mathfrak{S}$). In particular, each element of $\mathfrak{S}$ has an inverse in $\mathfrak{S}$ and $\mathfrak{S}$ is a group. Now by [5, Lemma 3.3], $\mathfrak{S}$ is contained in a compact extension of a unipotent subgroup in $\mathrm{GL}(n, \mathbb{Q}_p)$ which is normalized by $\mathfrak{S}$. Now suppose $\mathfrak{S} \not\subset \mathrm{SL}(n, \mathbb{Q}_p)$. We will first show that $\mathfrak{S}$ is contained in a compact extension of a nilpotent group in $\mathrm{GL}(n, \mathbb{Q}_p)$ and the latter is isomorphic to a direct product of $\mathcal{D}$ and a unipotent subgroup normalized by elements of $\mathfrak{S}$.

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Let $C = \{p^n \,\mathrm{Id} \mid n \in \mathbb{Z}\}$ and let $\mathcal{Z} = \{z \,\mathrm{Id} \mid z \in \mathbb{Z}_p^*\}$. Then C and $\mathcal{Z}$ are closed subgroups of $\mathcal{D}$, C is discrete, $\mathcal{Z}$ is compact and $\mathcal{D} = C \times \mathcal{Z}$. As the actions of both $\mathfrak{S}$ and $\mathfrak{S}C$ on $\mathcal{S}_n$ are the same and the latter is also a semigroup, without loss of any generality, we may replace $\mathfrak{S}$ by $\mathfrak{S}C$ and assume that $C \subset \mathfrak{S}$. As noted earlier, we may also assume that $\mathfrak{S}$ is closed. Moreover, as $\mathcal{Z}$ is compact and central, $\mathfrak{S}\mathcal{Z}$ is a closed semigroup and by Lemma 2.2, $\mathfrak{S}\mathcal{Z}$ acts distally on $\mathcal{S}_n$. Therefore, we may replace $\mathfrak{S}$ by $\mathfrak{S}\mathcal{Z}$ and assume that $\mathcal{Z} \subset \mathfrak{S}$. Now we have $\mathcal{D} \subset \mathfrak{S}$ and $\mathfrak{S}\mathcal{D} = \mathfrak{S}$. Let $T \in \mathfrak{S}$. By Proposition 2.3, $T^m = p^l S$ for some $l \in \mathbb{Z}$ and $S \in \mathrm{GL}(n, \mathbb{Q}_p)$ such that S generates a relatively compact group. Moreover, as $\mathcal{D} \subset \mathfrak{S}$, we have that $S \in \mathfrak{S}$. Therefore, the closure of the semigroup generated by S in $\mathfrak{S}$ is compact and hence a group. In particular, S is invertible in $\mathfrak{S}$ and we have that T^m is invertible in $\mathfrak{S}$, and hence $T^{m-1}(T^m)^{-1}$ is the inverse of T in $\mathfrak{S}$. Therefore, we may assume that $\mathfrak{S}$ is a closed group. Let $\mathbb{T}$ be a maximal torus in $\mathrm{GL}(n, \mathbb{Q}_p)$. Let $\mathbb{T}_a$ (respectively, $\mathbb{T}_d$) be the anisotropic (respectively, split) torus in $\mathbb{T}$. Then $\mathbb{T}_a$ is compact and $\mathbb{T}_a\mathbb{T}_d$ is an almost direct product. Moreover, since all maximal tori are conjugate to each other, there exists $m \in \mathbb{N}$ such that, for any element $T \in \mathfrak{S} \subset \mathrm{GL}(n, \mathbb{Q}_p)$, we have $T^m = \tau_a\tau_d\tau_u$, where τ_u is unipotent, $\tau_s = \tau_a\tau_d \in \mathbb{T}$ is semisimple, $\tau_a \in \mathbb{T}_a$ which is a compact group, $\tau_d \in \mathbb{T}_d$ and τ_a, τ_d and τ_u commute with each other. Note that $\mathbb{T}$ depends on T, but m is independent of the choice of T. We know that τ_u, being unipotent, generates a relatively compact group. As $\overline{T}$ is distal, arguing as above in the proof of Proposition 2.3, we get that $\tau_d = (t_{ij})_{n\times n}$ is such that $t_{ij} = 0$ if $i \neq j$ and $|t_{ii}|_p$ is the same for all i. We have $T^m = CS$, where $C \in \mathcal{D}$ and S generates a relatively compact group. Let $\pi : \mathrm{GL}(n, \mathbb{Q}_p) \to \mathrm{GL}(n, \mathbb{Q}_p)/\mathcal{D}$ be the natural projection. Note that $\mathrm{GL}(n, \mathbb{Q}_p)/\mathcal{D}$ is an algebraic group and is linear, that is, it can be realized as a subgroup of $\mathrm{GL}(V)$ for some p-adic vector space V. Now $\pi(\mathfrak{S})$ is a (closed) subgroup of $\mathrm{GL}(n, \mathbb{Q}_p)/\mathcal{D}$ and every element of $\pi(\mathfrak{S})$ generates a relatively compact group. By [5, Lemma 3.3], $\pi(\mathfrak{S})$ is contained in a compact extension of a unipotent group, that is, $\pi(\mathfrak{S}) \subset K \ltimes \mathcal{U}$, a semi-direct product, where $K, \mathcal{U} \subset \mathrm{GL}(V)$, K is a compact group and $\mathcal{U}$ is unipotent. We can choose K such that $\overline{\pi(\mathfrak{S})\mathcal{U}}/\mathcal{U}$ is isomorphic to K. Let $H = K \ltimes \mathcal{U} = \overline{\pi(\mathfrak{S})\mathcal{U}}$. Let $\mathcal{G}$ be the smallest algebraic subgroup of $\mathrm{GL}(n, \mathbb{Q}_p)/\mathcal{D}$ containing $\pi(\mathfrak{S})$. Here, since $\mathcal{U}$ is unipotent and is normalized by $\pi(\mathfrak{S})$, it is also normalized by $\mathcal{G}$. Then $\mathcal{G}\mathcal{U}$ is an algebraic group, and hence closed. As $H \subset \mathcal{G}\mathcal{U}$ and $H = (\mathcal{G} \cap H)\mathcal{U}$, it follows that $K = H/\mathcal{U}$ is isomorphic to $(\mathcal{G} \cap H)/(\mathcal{G} \cap \mathcal{U})$. Let $H_0 = \mathcal{G} \cap H$ and let $\mathcal{U}_0 = \mathcal{G} \cap \mathcal{U}$. Then $\pi(\mathfrak{S}) \subset H_0$ and $\pi(\mathfrak{S})$ normalizes $\mathcal{U}_0$. Here, $\pi^{-1}(\mathcal{U}_0) = \mathcal{D} \times \mathcal{U}'$ where $\mathcal{U}'$ is the unipotent radical of $\pi^{-1}(\mathcal{U}_0)$; $\mathcal{U}'$ is isomorphic to $\pi(\mathcal{U}') = \mathcal{U}_0$ and is normalized by $\mathfrak{S}$. Therefore $\mathfrak{S} \subset H' = \pi^{-1}(H_0)$. Also, $\mathcal{D} \times \mathcal{U}'$ is a co-compact nilpotent normal subgroup in H' and is also algebraic; see [2] and [9] for details on algebraic groups.

By Kolchin's theorem, there exists a flag $\{0\} = V_0 \subset \cdots \subset V_k = \mathbb{Q}_p^n$ of maximal $\mathcal{U}'$-invariant subspaces such that $\mathcal{U}'$ acts trivially on V_j/V_{j-1}, $j = 1, \ldots, k$. Note that each V_j is maximal in the sense that for any subspace W containing V_j such that $W \neq V_j$, $\mathcal{U}'$ does not act trivially on W/V_{j-1}. It is easy to see that each V_j is $\mathfrak{S}$-invariant as $\mathfrak{S}$ normalizes $\mathcal{U}'$. Now suppose $\mathfrak{S}/\mathcal{D}$ is not compact. Then there exists

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a sequence $\{T_i\} \subset \mathfrak{S}$ such that $\{\pi(T_i)\}$ is unbounded. Now we have $T_i = K_i D_i U_i$, $i \in \mathbb{N}$, where $D_i \in \mathcal{D}$, $U_i \in \mathcal{U}'$ and $K_i \in \pi^{-1}(K)$ such that $\{K_i\}$ is relatively compact and $\{U_i\}$ is unbounded. Note that, for each j, as $\mathfrak{S}$, $\mathcal{U}'$ and $\mathcal{D}$ keep V_j invariant, $K_i(V_j) = V_j$ for all i. Passing to a subsequence if necessary, we get that there exists $w \in \mathcal{S}_n$ such that $\|U_i(w)\|_p \to \infty$, $\overline{T}_i(w) \to w' \in \mathcal{S}_n$ and $K_i \to K_0$. For every $v \in V_1$, $T_i(v) = D_i K_i(v) \in V_1$, and hence $\{\|D_i^{-1} T_i(v)\|_p\}$ is bounded. Let $v \in V_1 \setminus \{0\}$ be such that $\|v\|_p < 1$. Then $v + w \in \mathcal{S}_n$. As $\{\|K_i U_i(v + w)\|_p\}$ is unbounded and $K_i U_i(v) = K_i(v) \to K_0 v$, we get that $\overline{T}_i(v + w) = \overline{K_i U_i}(v + w) \to w'$. This contradicts (i). Therefore, $\mathfrak{S}/\mathcal{D}$ is compact, that is, (i) $\Rightarrow$ (iii).

Suppose $\overline{\mathfrak{S}\mathcal{D}}/\mathcal{D}$ is a compact group. Then it contains $G\mathcal{D}/\mathcal{D}$, where G is the group generated by $\mathfrak{S}$. Therefore, $G\mathcal{D}/\mathcal{D}$ is relatively compact. Using this fact, we want to show that G acts distally on $\mathcal{S}_n$. Let C and $\mathcal{Z}$ be closed subgroups of $\mathcal{D}$ as above. Since the actions of both G and GC on $\mathcal{S}_n$ are the same, without loss of any generality we may replace G by GC and assume that $C \subset G$. We may also assume that G is closed. Now, using the facts that $\mathcal{D} = C \times \mathcal{Z}$ and $\mathcal{Z}$ is compact, we get that $G\mathcal{D} = G\mathcal{Z}$ is closed. Now $G\mathcal{D}/\mathcal{D}$ is compact and is isomorphic to $G/[(G \cap \mathcal{Z}) \times C]$. Therefore, G/C is compact, as $G \cap \mathcal{Z}$ is compact. Since the action of G on $\mathcal{S}_n$ factors through C, the preceding assertion implies that G acts distally on $\mathcal{S}_n$. Hence (iii) $\Rightarrow$ (ii). It is obvious that (ii) $\Rightarrow$ (i). □

The following corollary is a consequence of the above theorem.

Corollary 2.5. *A semigroup $\mathfrak{S}$ of $\mathrm{SL}(n, \mathbb{Q}_p)$ acts distally on $\mathcal{S}_n$ if and only if the closure of $\mathfrak{S}$ is a compact group.*

Proof. Let $\pi : \mathrm{GL}(n, \mathbb{Q}_p) \to \mathrm{GL}(n, \mathbb{Q}_p)/\mathcal{D}$ be as above. By Theorem 2.4, $\mathfrak{S}$ acts distally on $\mathcal{S}_n$ if and only if $\overline{\pi(\mathfrak{S})}$ is a compact group. Note that $\pi(\overline{\mathfrak{S}}) \subset \overline{\pi(\mathfrak{S})}$. As $\mathrm{SL}(n, \mathbb{Q}_p)\mathcal{D}$ is closed and $\mathrm{SL}(n, \mathbb{Q}_p) \cap \mathcal{D}$ is finite, we get that $\overline{\pi(\mathfrak{S})}$ is compact if and only if $\overline{\mathfrak{S}}$ is compact; the latter statement is equivalent to $\overline{\mathfrak{S}}$ being a compact group (cf. [6]). Now the assertion follows from the above. □

Remark. Note that the above corollary is valid for a semigroup $\mathfrak{S} \subset \mathrm{GL}(n, \mathbb{Q}_p)$ satisfying the condition that $|\det(T)|_p = 1$ for all $T \in \mathfrak{S}$, as the elements of $\mathrm{GL}(n, \mathbb{Q}_p)$ satisfying the condition form a closed subgroup (say) G such that $G\mathcal{D}$ is closed and $G \cap \mathcal{D}$ is a compact group (isomorphic to $\mathbb{Z}_p^*$).

In the real case, [13, Corollary 5] showed that a semigroup $\mathfrak{S} \subset \mathrm{GL}(n + 1, \mathbb{R})$ acts distally on the (real) unit sphere $\mathbb{S}^n$ if (and only if) every cyclic subsemigroup of $\mathfrak{S}$ acts distally on $\mathbb{S}^n$. The corresponding statement does not hold in the p-adic case, as there exists a class of closed noncompact subgroups of $\mathrm{SL}(n, \mathbb{Q}_p)$, every cyclic subgroup of which is relatively compact but which do not act distally on $\mathcal{S}_n$ as they are not compact; for example, the group of strictly upper triangular matrices in $\mathrm{SL}(n, \mathbb{Q}_p)$, $n \geq 2$.

3. Distality of 'affine' actions on $\mathcal{S}_n$

In this section we discuss the 'affine' maps on the p-adic unit sphere $\mathcal{S}_n$. Consider the affine action on $\mathbb{Q}_p^n$ given by $T_a(x) = a + T(x)$, $x \in \mathbb{Q}_p^n$, where $T \in \mathrm{GL}(n, \mathbb{Q}_p)$, and $a \in \mathbb{Q}_p^n$. We first consider the corresponding 'affine' map $\overline{T}_a$ on $\mathcal{S}_n$ which is defined for any nonzero a satisfying $\|T^{-1}(a)\|_p \neq 1$ as follows: $\overline{T}_a(x) = \|T_a(x)\|_p T_a(x)$, $x \in \mathcal{S}_n$. (For $a = 0$, $\overline{T}_a = \overline{T}$, which is studied in Section 2.) Observe that $T_a(x) = 0$ for some $x \in \mathcal{S}_n$ if and only if $T^{-1}(a)$ has norm 1. Therefore, $\overline{T}_a(\mathcal{S}_n) \subset \mathcal{S}_n$ if $\|T^{-1}(a)\|_p \neq 1$. The map $\overline{T}_a$ is a homeomorphism for any nonzero a satisfying $\|T^{-1}(a)\|_p < 1$ (see Lemma 3.1 below). In this section we study the distality of such homeomorphisms $\overline{T}_a$.

Lemma 3.1. *Let $T \in \mathrm{GL}(n, \mathbb{Q}_p)$ and let $a \in \mathbb{Q}_p^n \setminus \{0\}$ be such that $\|T^{-1}(a)\|_p \neq 1$. Then the map $\overline{T}_a$ on $\mathcal{S}_n$ is continuous and injective. $\overline{T}_a$ is a homeomorphism if and only if $\|T^{-1}(a)\|_p < 1$.*

Proof. Suppose $\|T^{-1}(a)\|_p \neq 1$. From the definition of $\overline{T}_a$, it is obvious that it is continuous. Suppose $x, y \in \mathcal{S}_n$ such that $\overline{T}_a(x) = \overline{T}_a(y)$. Then

$$\|a + T(x)\|_p (a + T(x)) = \|a + T(y)\|_p (a + T(y))$$

or $(\beta - 1)T^{-1}(a) = y - \beta x$, where $\beta = \|a + T(x)\|_p / \|a + T(y)\|_p = p^m$ for some $m \in \mathbb{Z}$. If possible, suppose $\beta \neq 1$. Interchanging y and x if necessary, we may assume that $\beta > 1$ or equivalently, that $m \in \mathbb{N}$. This implies that $\|\beta x\|_p = |\beta|_p = p^{-m} < 1$, and we get

$$\|T^{-1}(a)\|_p = |\beta - 1|_p \|T^{-1}(a)\|_p = \|y - \beta x\|_p = 1,$$

a contradiction. Hence, $\beta = 1$ and $x = y$. Therefore, $\overline{T}_a$ is injective.

Now suppose $\|T^{-1}(a)\|_p < 1$. It is enough to show that $\overline{T}_a$ is surjective, as any continuous bijection on a compact Hausdorff space is a homeomorphism.

Let $y \in \mathcal{S}_n$. Let $z = T^{-1}(y)$ and let $x = \|z\|_p z - T^{-1}(a)$. Since the norm of $\|z\|_p z$ is 1 and $\|T^{-1}(a)\|_p < 1$, we have that $\|x\|_p = 1$. Moreover, as $\|y\|_p = 1$, we have that $\|z\|_p^{-1} = \|a + T(x)\|_p$. Therefore, $\overline{T}_a(x) = y$. Hence $\overline{T}_a$ is surjective.

Conversely, Suppose $\overline{T}_a$ is surjective. Then there exists $x \in \mathcal{S}_n$ such that $\overline{T}_a(x) = \|a\|_p a$. We get that $x = (p^m - 1)T^{-1}(a)$, where $p^m = \|a + T(x)\|_p^{-1} \|a\|_p$ for some $m \in \mathbb{Z}$; here $m \neq 0$ since $x \neq 0$. Now $1 = \|x\|_p = |p^m - 1|_p \|T^{-1}(a)\|_p \geq \|T^{-1}(a)\|_p$ since $|p^m - 1|_p \geq 1$ for every $m \in \mathbb{Z} \setminus \{0\}$. As $\|T^{-1}(a)\|_p \neq 1$, we have that $\|T^{-1}(a)\|_p < 1$. □

In [13], we have studied 'affine' maps $\overline{T}_a$ on the real unit sphere $\mathbb{S}^n$. The following result shows that in the p-adic case, $\overline{T}_a$ is distal for every nonzero a in a certain neighbourhood of 0 in $\mathbb{Q}_p^n$ if and only if $\overline{T}$ is distal. This illustrates that the behaviour of such maps in the p-adic case is very different from that in the real case.

Theorem 3.2. *Suppose $T \in \mathrm{GL}(n, \mathbb{Q}_p)$. For $a \in \mathbb{Q}_p^n \setminus \{0\}$ with $\|T^{-1}(a)\|_p \neq 1$, let $\overline{T}_a : \mathcal{S}_n \to \mathcal{S}_n$ be defined as $\overline{T}_a(x) = \|a + T(x)\|_p (a + T(x))$, $x \in \mathcal{S}_n$. There exists a compact open subgroup $V \subset \mathbb{Q}_p^n$ such that, for all $a \in V \setminus \{0\}$, we have that $\|T^{-1}(a)\|_p < 1$, $\overline{T}_a$ is a homeomorphism and the following statements hold.*

https://doi.org/10.1017/S1446788719000272 Published online by Cambridge University Press

(I) *If $\overline{T}$ is distal, then $\overline{T}_a$ is distal for all nonzero $a \in V$.*

(II) *If $\overline{T}$ is not distal, then for every neighbourhood U of 0 contained in V, there exists a nonzero $a \in U$ such that $\overline{T}_a$ is not distal.*

Proof. By [14, 3.3], we get that there exist D and S which commute with T and $m \in \mathbb{N}$ such that $T^m = SD = DS$, where D is a diagonal matrix with the diagonal entries in $\{p^i \mid i \in \mathbb{Z}\}$ and S generates a relatively compact group. Therefore, S^k is an isometry for some $k \in \mathbb{N}$. Replacing m by km, we may assume that S itself is an isometry. Let $c_0 = \min\{(1/\|T^{-j}\|_p) \mid 1 \leq j \leq m-1\}$ and $c_1 = \max\{\|T^j\|_p \mid 1 \leq j \leq m-1\}$. As $\mathcal{S}_n$ is compact, $0 < c_0 \leq c_1 < \infty$. Also, $c_0 \leq \|T^j(x)\|_p \leq c_1$ for all $x \in \mathcal{S}_n$ and $1 \leq j \leq m-1$. Since $\|T^j\|_p \in \{p^i \mid i \in \mathbb{Z}\}$, we get that $\{\|T^j(x)\|_p \mid x \in \mathcal{S}_n, 1 \leq j \leq m-1\}$ is finite.

Let V be a compact open S-invariant subgroup in $\mathbb{Q}_p^n$ such that $V \cup c_0V \cup c_0^2V \cup c_0^2c_1^{-2}V \subset W = \{w \in \mathbb{Q}_p^n \mid \|w\|_p < 1\}$. Then $\|v\|_p < \min\{1, c_0, c_0^2, c_0^2c_1^{-2}\} \leq 1$ for all $v \in V$ and $c_0c_1^{-1}V \subset c_0^2c_1^{-2}V \subset W$. Moreover, $\|T^{-1}(v)\|_p < c_0^{-1}c_0 = 1$ for every $v \in V$.

Let p^l be the smallest nonzero entry in the diagonal matrix D and let

$$H = \{x \in \mathbb{Q}_p^n \mid D(x) = p^l x\}.$$

This is a nontrivial closed subspace of $\mathbb{Q}_p^n$. As S and T commute with D, they keep H invariant and, as S is an isometry, $\|T^m(x)\|_p = p^{-l}\|x\|_p$ for all $x \in H$.

Let $a \in V \setminus \{0\}$. As noted above, $\|T^{-1}(a)\|_p < 1$, and hence, by Lemma 3.1, $\overline{T}_a$ is a homeomorphism. Take any $x \in \mathcal{S}_n$. Since $\|a\|_p < c_0$ and $\|T(x)\|_p \geq c_0$, we have $\|T_a(x)\|_p = \|a + T(x)\|_p = \|(T(x)\|_p$ and

$$\overline{T}_a(x) = \|T_a(x)\|_p T_a(x) = \|T(x)\|_p(a + T(x)).$$

Let $\alpha_1(x) = \|T_a(x)\|_p = \|T(x)\|_p = \beta_{1,x}$. Let $\alpha_j(x) = \|T_a(\overline{T}_a^{j-1}(x))\|_p = \|a + T(\overline{T}_a^{j-1}(x))\|_p$ and let $\beta_{j,x} = \alpha_1(x) \cdots \alpha_j(x)$ for all $j \in \mathbb{N}$ ($j \geq 2$). Take $\beta_{0,x} = 1$ and $\phi^0 = \mathrm{Id}$ for any map ϕ. From the above, we have that $\alpha_j(x) = \|T(\overline{T}_a^{j-1}(x))\|_p$ for all $j \in \mathbb{N}$. It is easy to show by induction that, for every $j \in \mathbb{N}$,

$$\overline{T}_a^j(x) = \beta_{j,x}T^j(x) + \beta_{j,x}\sum_{i=1}^{j}\beta_{j-i,x}^{-1}T^{i-1}(a). \tag{3.1}$$

Observe that, as $a \in V$, $\|T^k(a)\|_p \leq c_1\|a\|_p < c_1(c_0c_1^{-1}) = c_0$, and for any $x \in \mathcal{S}_n$, $\|T^k(x)\|_p \geq c_0$, $1 \leq k \leq m-1$. Therefore, for $j \in \mathbb{N}$ and $1 \leq k \leq m-2$,

$$\begin{aligned}\|T^k(\overline{T}_a^j(x))\|_p &= [\alpha_j(x)]^{-1}\|T^k(a) + T^{k+1}(\overline{T}_a^{j-1}(x))\|_p \\ &= [\alpha_j(x)]^{-1}\|T^{k+1}(\overline{T}_a^{j-1}(x))\|_p.\end{aligned}$$

Applying the above equation successively, we get that, for $1 \leq j \leq m-1$,

$$\alpha_j(x) = \|T(\overline{T}_a^{j-1}(x))\|_p = [\alpha_{j-1}(x)\cdots\alpha_1(x)]^{-1}\|T^j(x)\|_p,$$

that is, $\beta_{j,x} = \|T^j(x)\|_p$. Hence, $c_0 \le \beta_{j,x} \le c_1$ for all $x \in \mathcal{S}_n$ and $1 \le j \le m-1$. Moreover, applying the same equation again successively, we get for $j \ge m$ that

$$\alpha_j(x) = [\alpha_{j-1}(x) \cdots \alpha_{j-m+1}(x)]^{-1} \|T^{m-1}(a) + T^m(\overline{T}_a^{j-m}(x))\|_p. \tag{3.2}$$

Now we take $a \in V \cap H \setminus \{0\}$. Then $\overline{T}_a(\mathcal{S}_n \cap H) = \mathcal{S}_n \cap H$. Let $x \in \mathcal{S}_n \cap H$. Then $\|T^{-1}(a)\|_p < c_0^{-1}c_0 = 1$, and hence $\|T^{-1}(a) + \overline{T}_a^{j-m}(x)\|_p = 1$. This implies that

$$\|T^{m-1}(a) + T^m(\overline{T}_a^{j-m}(x))\|_p = \|T^m(T^{-1}(a) + \overline{T}_a^{j-m}(x))\|_p = p^{-l}. \tag{3.3}$$

Using Equations (3.2) and (3.3), we get $\alpha_j(x) = [\alpha_{j-1}(x) \cdots \alpha_{j-m+1}(x)]^{-1} p^{-l}$, and hence $\beta_{j,x} = p^{-l}\beta_{j-m,x}$ for all $j \ge m$. In particular, $\beta_{m,x} = p^{-l} = \|T^m(x)\|_p$. This implies that $\beta_{km+j,x} = p^{-kl}\beta_{j,x} = p^{-kl}\|T^j(x)\|_p$, $k, j \in \mathbb{N}$. Therefore, $\beta_{j,x} = \|T^j(x)\|_p$, $j \in \mathbb{N}$. Moreover, for all $j, k \in \mathbb{Z}$ and $x \in H$, $T^{km+j}(x) = p^{kl}S^kT^j(x) = p^{kl}T^jS^k(x)$ and $\|T^{km+j}(x)\|_p = p^{-kl}\|T^j(x)\|_p$, as S is an isometry. In particular, $\beta_{km+j,x} = p^{-kl}\|T^j(x)\|_p$ for all $k, j \in \mathbb{Z}$ such that $km + j \ge 0$. Using the above facts together with Equation (3.1), we get, for $k \in \mathbb{N}$,

$$\begin{aligned} \overline{T}_a^{km}(x) &= \beta_{km,x}T^{km}(x) + \beta_{km,x} \sum_{j=1}^{km} \beta_{km-j,x}^{-1} T^{j-1}(a) \\ &= S^k(x) + \sum_{j=1}^{km} \|T^{-j}(x)\|_p^{-1} T^{j-1}(a) \\ &= S^k(x) + \sum_{i=1}^{k} \sum_{j=1}^{m} \|T^{-j}(x)\|_p^{-1} T^{j-1}(S^{i-1}(a)) \\ &= S^k(x) + \sum_{j=1}^{m} \gamma_{j,x}^{-1} T^{j-1}(a_k), \end{aligned}$$

where $a_k = \sum_{i=1}^{k} S^{i-1}(a) \in V \cap H$, $k \in \mathbb{N}$, $\gamma_{j,x} = \|T^{-j}(x)\|_p = p^l\beta_{m-j,x}$ and $c_1^{-1} \le \gamma_{j,x} \le c_0^{-1}$, $1 \le j \le m-1$, and $\gamma_{m,x} = p^l$. From the above, we get that, for any $k \in \mathbb{N}$ and $x, y \in \mathcal{S}_n \cap H$,

$$\overline{T}_a^{km}(x) - \overline{T}_a^{km}(y) = S^k(x-y) + \sum_{j=1}^{m-1} [\gamma_{j,x}^{-1} - \gamma_{j,y}^{-1}]\, T^{j-1}(a_k). \tag{3.4}$$

Let $x, y \in \mathcal{S}_n \cap H$ such that $\|x - y\|_p < c_0c_1^{-1}$. As T is linear, $T^j(x) = T^j(y) + T^j(x-y)$, $j \in \mathbb{N}$. For $1 \le j \le m-1$, as $\|T^j(x-y)\|_p \le c_1\|x-y\|_p < c_0$ and $\|T^j(y)\|_p \ge c_0$, we get that $\beta_{j,x} = \|T^j(x)\|_p = \|T^j(y)\|_p = \beta_{j,y}$, and hence $\gamma_{j,x} = \gamma_{j,y}$. Therefore,

$$\|\overline{T}_a^{km}(x) - \overline{T}_a^{km}(y)\|_p = \|S^k(x) - S^k(y)\|_p = \|x-y\|_p, \quad k \in \mathbb{N}.$$

Now suppose $\|x - y\|_p \ge c_0c_1^{-1}$. Observe that $|\gamma_{j,x}^{-1} - \gamma_{j,y}^{-1}|_p \le c_0^{-1}$, $\|T^j(a_k)\|_p \le c_1\|a_k\|_p$, $1 \le j \le m-1$ and $a_k \in V$, $\|a_k\|_p < c_0^2c_1^{-2}$. Now Equation (3.4) implies that

$$\overline{T}_a^{km}(x) - \overline{T}_a^{km}(y) \in S^k(x-y) + c_0^{-1}c_1W.$$

Since $\|S^k(x-y)\|_p = \|x-y\|_p \geq c_0c_1^{-1}$, we get that $\|\overline{T}_a^{km}(x) - \overline{T}_a^{km}(y)\|_p = \|x-y\|_p$. This shows that $\overline{T}_a^m|_{\mathcal{S}_n\cap H}$ preserves the distance and is distal, and hence $\overline{T}_a|_{\mathcal{S}_n\cap H}$ is distal, where $a \in V \cap H$.

If $\overline{T}$ is distal, then so is $\overline{T}^m$, and hence its image in $\mathrm{GL}(n, \mathbb{Q}_p)/\mathcal{D}$ generates a relatively compact group. This implies that $D = p^l\,\mathrm{Id}$, $H = \mathbb{Q}_p^n$, $\mathcal{S}_n \cap H = \mathcal{S}_n$ and $V \cap H = V$. Therefore, (I) holds.

Now suppose $\overline{T}$ is not distal. Then $\overline{T}^m$ is not distal and hence $D \neq p^l\,\mathrm{Id}$. Let $l_1 > l$ be such that $H_1 = \{x \in \mathbb{Q}_p^n \mid D(x) = p^{l_1}x\}$ is nonzero. Then H_1 is a vector subspace and is invariant under D, S and T. Let $a \in V \cap H \setminus \{0\}$ as above. It is easy to see that $\overline{T}_a(\mathcal{S}_n \cap (H \oplus H_1)) = \mathcal{S}_n \cap (H \oplus H_1)$. We show that the restriction of $\overline{T}_a$ to $\mathcal{S}_n \cap (H \oplus H_1)$ is not distal. This would imply that (II) holds.

Take $y = x + z \in \mathcal{S}_n$, where $x \in \mathcal{S}_n \cap H$ and $z \in H_1$ such that $\|T^j(z)\|_p < \|T^j(x)\|_p$, $j \in \mathbb{N}$. It is possible to choose such a z; we can take $z \in H_1$ with the property that $\|T^j(z)\|_p < \|T^j(x)\|_p$ for all $0 \leq j \leq m-1$, then as S is an isometry, $\|T^{km+j}(z)\|_p = p^{-kl_1}\|T^j(z)\|_p < p^{-kl}\|T^j(x)\|_p = \|T^{km+j}(x)\|_p$, $k \in \mathbb{N}$. Now $\|T^j(y)\|_p = \|T^j(x)\|_p = \beta_{j,x}$ for all $j \in \mathbb{N}$. Here,

$$\overline{T}^{km}(y) - \overline{T}^{km}(x) = p^{-kl}[S^k(p^{kl}x + p^{kl_1}z)] - S^k(x) = p^{k(l_1-l)}S^k(z) \to 0$$

as $k \to \infty$, since S is an isometry and since $l_1 > l$. We now show for all $k \in \mathbb{N}$ that $\overline{T}_a^{km}(y) - \overline{T}_a^{km}(x) = \overline{T}^{km}(y) - \overline{T}^{km}(x)$. (From the above, the latter is equal to $\beta_{km,x}T^{km}(z)$.) This in turn would imply that $\overline{T}_a$ is not distal.

From Equation (3.1), it is enough to show for all $j \in \mathbb{N} \cup \{0\}$ that $\beta_{j,y} = \beta_{j,x}$ or, equivalently, $\beta_{j,y} = \|T^j(y)\|_p$ as the latter is equal to $\|T^j(x)\|_p$ which is the same as $\beta_{j,x}$. This is trivially true for $j = 0$. As shown earlier, for $1 \leq j < m-1$, $\beta_{j,u} = \|T^j(u)\|_p$ for all $u \in \mathcal{S}_n$, and hence $\beta_{j,y} = \beta_{j,x}$; that is, the above statement holds for $1 \leq j < m$, and we get that

$$\overline{T}_a^j(y) = \beta_{j,y}T^j(y) + \beta_{j,y}\sum_{i=1}^{j}\beta_{j-i,y}^{-1}T^{i-1}(a) = \overline{T}_a^j(x) + \beta_{j,x}T^j(z). \tag{3.5}$$

We prove by induction on k that $\beta_{j,y} = \beta_{j,x} = \|T^j(x)\|_p$ and Equation (3.5) is satisfied for all $1 \leq j < km$, $k \in \mathbb{N}$. We have already proven these for $k = 1$. Suppose, for some $k \in \mathbb{N}$, that these hold for all j such that $(k-1)m \leq j < km$. Let $km \leq j < (k+1)m$. Recall that, for all $j \in \mathbb{N}$, $\alpha_j(u) = \|T(\overline{T}_a^{j-1}(u))\|_p$, $u \in \mathcal{S}_n$, and Equation (3.2) holds for any $x \in \mathcal{S}_n$ and $j \geq m$. As $\beta_{j,y}\beta_{j-m,y}^{-1} = \alpha_j(y)\ldots\alpha_{j-m+1}(y)$, from Equation (3.2), and also Equation (3.5) which is assumed to hold for $(k-1)m \leq j < km$ by the induction hypothesis, we get for x, y, z as above and $km \leq j < (k+1)m$ that

$$\begin{aligned}\beta_{j,y}\beta_{j-m,y}^{-1} &= \|T^{m-1}(a) + T^m(\overline{T}_a^{j-m}(y))\|_p \\ &= \|T^m[T^{-1}(a) + \overline{T}_a^{j-m}(x) + \beta_{j-m,x}T^{j-m}(z)]\|_p.\end{aligned}$$

https://doi.org/10.1017/S1446788719000272 Published online by Cambridge University Press

Now using this, we get that

$$\beta_{j,y}\beta_{j-m,y}^{-1} = \|S[p^l(T^{-1}(a) + \overline{T}_a^{j-m}(x)) + p^{l_1}\beta_{j-m,x}T^{j-m}(z)]\|_p = p^{-l},$$

as S is an isometry, $l_1 > l$ and $\|\beta_{j-m,x}T^{j-m}(z)\|_p < 1$ (see also Equation (3.3)). Since $(k-1)m \leq j-m < km$, $\beta_{j,y} = p^{-l}\beta_{j-m,y} = \|p^l T^{j-m}(x)\|_p = \|T^j(x)\|_p$. Hence Equation (3.5) holds for $km \leq j < (k+1)m$. Now by induction for all $j \in \mathbb{N}$, $\beta_{j,x} = \beta_{j,y}$ and Equation (3.5) holds. Therefore, $\overline{T}_a$ is not distal. (Note that Equation (3.5) also directly shows that $\overline{T}_a^{km}(y) - \overline{T}_a^{km}(x) = p^{k(l_1-l)}S^k(z) \to 0$ as $k \to \infty$.) Now if $U \subset V$ is a neighbourhood of 0, then $U \cap H \neq \{0\}$ and hence (II) holds. □

Observe that if $\overline{T}$ is not distal, then from Theorem 3.2(II) we get that every neighbourhood of 0 in $\mathbb{Q}_p$ contains a nonzero a such that $\|T^{-1}(a)\|_p < 1$ and $\overline{T}_a$ is not distal. Now the following corollary is an easy consequence of Theorem 3.2.

Corollary 3.3. *For $T \in \mathrm{GL}(n, \mathbb{Q}_p)$, $\overline{T}$ is distal if and only if there exists a neighbourhood V of 0 in $\mathbb{Q}_p^n$ such that, for every $a \in V \setminus \{0\}$, $\|T^{-1}(a)\|_p < 1$ and $\overline{T}_a$ on $\mathcal{S}_n$ is distal.*

If T is distal, then $\overline{T}$ is also distal and Theorem 3.2(I) and Corollary 3.3 hold for T. If $\overline{T}$ is distal, then, for some $m \in \mathbb{N}$ and $l \in \mathbb{Z}$, $p^l T^m$ is distal.

Acknowledgements

R. Shah would like to thank S. G. Dani and C. R. E. Raja for discussions on p-adic algebraic groups. We thank the referee for valuable comments and suggestions.

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