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Variations of Integrals in Diffeology

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Abstract. We establish a formula for the variation of integrals of differential forms on cubic chains in the context of diffeological spaces. Then we establish the diffeological version of Stokes' theorem, and we apply that to get the diffeological variant of the Cartan–Lie formula. Still in the context of Cartan–De Rham calculus in diffeology, we construct a chain-homotopy operator K, and we apply it here to get the homotopic invariance of De Rham cohomology for diffeological spaces. This is the chain-homotopy operator that is used in symplectic diffeology to construct the moment map.

Introduction

The formula for the variation of the integral of differential forms on chains and the chain-homotopy operator are two essential tools for the Cartan–De Rham calculus in diffeology, and are used to establish some important formulas like the Cartan–Lie formula, linking Lie derivatives, contractions, and exterior differential of forms. These constructions are also interesting in differential geometry of manifolds even if it does not give new results. The possibility of using differential forms and the diffeological differential techniques on the space of paths of a manifold — which is never a manifold (except in trivial cases) — radically simplifies some proofs of fundamental theorems, and may shed new light on their structural nature. But, even if diffeology can be used as a shortcut in the classical framework, the purpose of this work is not to give new proofs for old theorems. I felt it necessary to publish these constructions and theorems independently, because as the main tools for the Cartan–De Rham calculus in diffeology, I used them in parallel works. In particular, I have used them in the construction of the moment maps in diffeology, and to show (by the way) how every symplectic manifold is a coadjoint orbit of its group of symplectomorphisms [Piz07].

I would emphasize the fact that all these constructions apply to a large category of spaces, from quotients of manifolds to spaces of smooth functions. Diffeology preserves the main classical results and theorems without ever involving any of the functional topology or analysis heavy tools. And that is a huge saving in terms of technical investment.

Note Diffeology is an extension of the category of smooth real domains. It is a Cartesian closed category, complete and cocomplete, and contains fully and faithfully the category of manifolds. The axiomatics of Diffeology have been formulated by J.-M Souriau in [Sou81]. They are a variant of Chen's structure [Che77]. They have been extended then by his students in [Don84, DI85, Igl85], etc. A few years ago I

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began to write a textbook on diffeology [Piz05], where the reader can find all the details of constructions used here.

1 Differential Forms on Diffeological Spaces

I remind the reader of the main constructions related to differential calculus in diffeology; the basics of the general theory are assumed to be known. The proofs of the claims can be found in [Piz05].

1.1 Differential Forms on Diffeological Spaces

Let X be a diffeological space. A *differential k-form* of X (or *defined on* X) is a map α that associates with any plot P of X a *smooth k-form* $\alpha(P)$ defined on the domain of P such that, for every smooth parametrization F of the domain of P,

$$\alpha(P \circ F) = F^*(\alpha(P)).$$

Let us summarize formally: α is a k-form of X if and only if

(1) for all integers n, for all n-plots $P\colon U \to X$, $\alpha(P) \in \mathcal{C}^\infty(U, \Lambda^k(\mathbf{R}^n))$, where $\Lambda^k(\mathbf{R}^n)$ denotes the space of linear k-forms of $\mathbf{R}^n$;

(2) for all m domains V, for all smooth parametrizations $F\colon V \to U$, for all $v \in V$ and for all k vectors $\xi_1 \dots \xi_k \in \mathbf{R}^m$,

$$\alpha(P \circ F)(v)(\xi_1) \cdots (\xi_k) = \alpha(P)(F(v))(D(F)(v)(\xi_1)) \cdots (D(F)(v)(\xi_k)),$$

where $D(F)(v)$ denotes the tangent linear map of F at v.

The condition $\alpha(P \circ F) = F^*(\alpha(P))$ is called the *smooth compatibility condition*, and we say that $\alpha(P)$ *represents* the differential form α in the plot P. The space of differential k-forms of X will be denoted by $\Omega^k(X)$, and we shall see that it is naturally a diffeological vector space.

Note 1 Let U be an n-domain, that is, an open subset of $\mathbf{R}^n$. There is a subtle difference between a smooth k-form $a \in \mathcal{C}^\infty(U, \Lambda^k(\mathbf{R}^n))$ on U and the differential k-form $[a]$ on U (regarded as a diffeological space) defined by $[a](F) = F^*(a)$. We may identify $a = [a](\mathbf{1}_U)$ and $[a]$, by abuse of notation.

Note 2 As an example of differential form, let us give this one, adapted from [Don94], and defined on the group $G = \mathrm{Diff}(S^1)$. We consider the universal covering $\widetilde{G}$ represented by the diffeomorphisms f of $\mathbf{R}$ satisfying $f(x+2\pi) = f(x) + 2\pi$. Let $P\colon U \to \widetilde{G}$ be an n-plot, let $r \in U$, $\delta r \in \mathbf{R}^n$, and let $\widetilde{\alpha}$ be defined by

$$\widetilde{\alpha}(P)_r(\delta r) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\partial}{\partial s} \left\{ P(r)^{-1} \circ P(s)(n) \right\}_{s=r} (\delta r).$$

One can check that $\widetilde{\alpha}$ is a left invariant 1-form on $\widetilde{G}$, and invariant by $2\pi\mathbf{Z}$. It defines a unique left invariant 1-form α on $\mathrm{Diff}(S^1)$. The coadjoint orbit of this *momentum* by $\mathrm{Diff}(S^1)$ is an example of a nonregular coadjoint orbit in the sense of Kirillov.

1.2 Functional Diffeology of the Space of Forms

The set $\Omega^k(X)$ of all differential k-forms on a diffeological space X is a real vector space. For all $\alpha, \alpha' \in \Omega^k(X)$, for all real number s, and for all plots P of X,

$$(\alpha + \alpha')(P) = \alpha(P) + \alpha'(P)$$
$$(s \times \alpha)(P) = s \times \alpha(P).$$

The set of parametrizations $\phi \colon V \mapsto \Omega^k(X)$ defined by the following condition, is a diffeology of vector spaces.

♣ For every plot $P \colon U \to X$, the map $(s, r) \mapsto \phi(s)(P)(r)$ defined from $V \times U$ to $\Lambda^k(\mathbf{R}^n)$ is smooth, that is, $[(s, r) \mapsto \phi(s)(P)(r)] \in \mathcal{C}^\infty(V \times U, \Lambda^k(\mathbf{R}^n))$.

We call this diffeology, the (*standard*) *functional diffeology* of $\Omega^k(X)$.

1.3 Pullbacks of Differential Forms

Let X and X' be two diffeological spaces. Let $\alpha' \in \Omega^k(X')$, and let $f \colon X \to X'$ be a smooth map. The *pullback* $f^*(\alpha')$ of α' by f, defined for all plots P of X by

$$(f^*(\alpha'))(P) = \alpha'(f \circ P),$$

is a differential k-form on X. The pullback of differential forms is *contravariant*. Let X, X', and X'' be three diffeological spaces, and let $f \colon X \to X'$ and $g \colon X' \to X''$ be two smooth maps. Let $\alpha'' \in \Omega^k(X'')$, then

$$(g \circ f)^*(\alpha'') = f^*\big(g^*(\alpha'')\big).$$

Moreover, the pullback operation

$$f^* \colon \Omega^k(X') \to \Omega^k(X')$$

is a smooth linear map for the functional diffeology of the spaces of forms.

Note Let $\alpha \in \Omega^k(X)$ and $P \colon U \to X$ be a plot. The differential k-form $P^*(\alpha)$ defined on U, regarded as a diffeological space, is characterized by $\alpha(P) = P^*(\alpha)(\mathbf{1}_U)$. Therefore, α is just defined by the family of its pullbacks by the plots of X.

1.4 Exterior Differentials of Forms

Let X be a diffeological space. Let α be a p-form of X. The exterior differential $d\alpha$ of α is the $(p+1)$-differential form defined by

$$(d\alpha)(P) = d(\alpha(P)),$$

for all plots P of X. The linear operator so defined, $d \colon \Omega^p(X) \to \Omega^{p+1}(X)$, is smooth, where $\Omega^p(X)$ and $\Omega^{p+1}(X)$ are equipped with their functional diffeology defined in Subsection 1.2.

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Thanks to the commutativity between pullback and exterior differential of smooth forms, the pullback of differential forms on diffeological spaces commutes with the exterior differential. Let X' be another diffeological spaces and let $f\colon X \to X'$ be a smooth map. Then, for all differential forms α' of X',

$$d\big(f^*(\alpha')\big) = f^*(d\alpha').$$

1.5 Exterior Products of Differential Forms

Let X be a diffeological space, let $\alpha \in \Omega^k(X)$ and $\beta \in \Omega^\ell(X)$. There exists a $k+\ell$ differential form $\alpha \wedge \beta$, defined on X by

$$(\alpha \wedge \beta)(P) = \alpha(P) \wedge \beta(P),$$

for all plots P of X. The form $\alpha \wedge \beta$ is called the *exterior product* of α with β. The exterior product, regarded as a map:

$$\wedge\colon \Omega^k(X) \times \Omega^\ell(X) \to \Omega^{k+\ell}(X) \quad \text{with} \quad \wedge(\alpha,\beta) = \alpha \wedge \beta,$$

is a smooth bilinear map, for the functional diffeology of the spaces of forms.

The basic properties of the exterior product of smooth forms extend naturally to the exterior product of differential forms on diffeological spaces:

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha. \quad \text{and} \quad f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta),$$

where f is a smooth map.

2 Lie Derivatives and Contractions

In this section, we define the *Lie derivative* of differential forms of diffeological spaces, along *slidings*, that is, (germs of) paths of diffeomorphisms centered at the identity. We define also the notion of *contractions* of differential forms by *arcs* of plots, which extend the contractions by slidings.

2.1 The Lie Derivative of Differential Forms

Let X be a diffeological space, let $\alpha \in \Omega^k(X)$, and let $h\colon \mathbf{R} \to \mathrm{Diff}(X)$ be a 1-plot for the functional diffeology, centered at the identity, that is, $h(0) = \mathbf{1}_X$. Then there exists a k-form of X denoted by $\mathfrak{L}_h(\alpha)$, called the *Lie derivative* of α by h and defined by

$$\mathfrak{L}_h(\alpha) = \frac{\partial}{\partial t}\Big\{ h(t)^*\alpha \Big\}_{t=0}.$$

This means precisely that, for every integer n, for every n-plot $P\colon U \to X$, for every $r \in U$ and any set of k vectors $v_1, \ldots, v_k \in \mathbf{R}^n$,

$$\mathfrak{L}_h(\alpha)(P)(r)(v_1)\cdots(v_k) = \frac{\partial}{\partial t}\Big\{ \alpha(h(t)\circ P)(r)(v_1)\cdots(v_k)\Big\}_{t=0}.$$

The Lie derivative, $\mathfrak{L}_h\colon \Omega^k(X) \to \Omega^k(X)$ is a smooth linear map for the functional diffeology of forms.

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Note The Lie derivative $\mathfrak{L}_h(\alpha)$ depends only on the "1-jet" of the plot h at 0; that is, if h and h' are two 1-plots of $\mathrm{Diff}(X)$ centered at the identity, such that $h' = h \circ f$, with $f(0) = 0$ and $D(f)(0) = 1$, then $\mathfrak{L}_h(\alpha) = \mathfrak{L}_{h'}(\alpha)$. This is the case in particular when h and h' coincide on an open neighborhood of 0 (have the same germ at 0).

Proof Let us show that $\mathfrak{L}_h(\alpha)$ is a well-defined differential form of X. Let $P\colon U \to X$ be a n-plot of X, and let $h \cdot P$ be the $(p+1)$-plot defined by

$$h \cdot P(t,r) = (h(t) \circ P)(r) = h(t)(P(r)), \quad \text{where} \quad (t,r) \in \mathbf{R} \times U.$$

By definition, $\alpha(h \cdot P)$ is a differential form on $\mathbf{R} \times U$. But $\alpha(h(t) \circ P)$ is the pull-back of $\alpha(h \cdot P)$ by the injection $j_t\colon r \mapsto (t,r)$ from U into $\mathbf{R} \times U$, *i.e.*, $\alpha(h(t) \circ P)$ is the restriction of $\alpha(h \cdot P)$ on $\{t\} \times U$. Since $\alpha(h(t) \circ P)$ is a smooth form, the map $t \mapsto \alpha(h(t) \circ P) \restriction \{t\} \times U$ is a smooth parametrization in $\mathbf{\Lambda}^k(\mathbf{R}^n)$. Thus, its derivative is still a smooth parametrization of $\mathbf{\Lambda}^k(\mathbf{R}^n)$. Therefore, the definition of $\mathfrak{L}_h(\alpha)$ makes sense.

Now let us check now the fundamental property of differential forms (Subsection 1.1). Let $F\colon V \to U$ be a smooth parametrization in U, then

$$\begin{aligned}
\mathfrak{L}_h(\alpha)(P \circ F) &= \frac{\partial}{\partial t}\Big\{ \alpha\big(h(t) \circ (P \circ F)\big) \Big\}_{t=0} = \frac{\partial}{\partial t}\Big\{ \alpha\big((h(t) \circ P) \circ F\big) \Big\}_{t=0} \\
&= \frac{\partial}{\partial t}\Big\{ F^*\big(\alpha(h(t) \circ P)\big) \Big\}_{t=0} = F^*\Big(\frac{\partial}{\partial t}\big\{ \alpha(h(t) \circ P)\big\}_{t=0}\Big) \\
&= F^*\big(\mathfrak{L}_h(\alpha)(P)\big).
\end{aligned}$$

Hence, $\mathfrak{L}_h(\alpha)$ is a differential k-form of X.

(1) Obviously, since the derivative is local, the Lie derivative is local, and $\mathfrak{L}_h(\alpha)$ only depends on the germ of h. Now, if $h' = h \circ f$ with $f(0) = 0$, then

$$\mathfrak{L}_{h'}(\alpha)(P)(r)[v] = D\big(s \mapsto \alpha\big(h(f(s)\big) \circ P\big)(r)[v])(0)(1),$$

where $[v]$ denotes the k vectors $(v_1) \cdots (v_k)$. But

$$\big[s \mapsto \alpha(h(f(s)) \circ P)(r)[v]\big] = \big[t \mapsto \alpha(h(t) \circ P)(r)[v]\big] \circ \big[s \mapsto f(s)\big],$$

thus

$$\begin{aligned}
\mathfrak{L}_{h'}(\alpha)(P)(r)[v] &= D\big(t \mapsto \alpha(h(t) \circ P)(r)[v]\big)(0)\big(D(f)(0)(1)\big) \\
&= D(f)(0)(1) \times \mathfrak{L}_h(\alpha)(P)(r)[v].
\end{aligned}$$

Therefore, if $D(f)(0)(1) = 1$, then $\mathfrak{L}_h(\alpha) = \mathfrak{L}_{h'}(\alpha)$.

(2) It is clear that the Lie derivative is linear. Let us prove that $\mathfrak{L}_h$ is smooth. Let $\phi\colon V \to \Omega^k(X)$ and $P\colon U \to X$ be two plots. We want to check that $(s,r) \mapsto$

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$\mathfrak{L}_h(\phi(s))(P)(r)$ is smooth. But,

$$\begin{aligned}\mathfrak{L}_h(\phi(s))(P)(r) &= \frac{\partial}{\partial t}\big(h(t)^*(\phi(s))(P)(r)\big)_{t=0}\\ &= \frac{\partial}{\partial t}\big(\phi(s)(h(t)\circ P)(r)\big)_{t=0}\\ &= \frac{\partial}{\partial t}\big(j_t^*(\phi(s)(h\cdot P))(r)\big)_{t=0}.\end{aligned}$$

Now, $(s,t,r)\mapsto \phi(s)(h\cdot P)(t,r)$ is a smooth map, and $j_t^*(\phi(s)(h\cdot P))(r)$ is just its restriction to $V\times\{t\}\times U$. Thus, $\mathfrak{L}_h(\phi(s))(P)(r)$ is a partial derivative of a smooth map, hence smooth. Therefore, the Lie derivative $\mathfrak{L}_h$ is a smooth endomorphism of $\Omega^k(X)$. ■

2.2 Lie Derivative along Homomorphisms

Let X be a diffeological space, and let h be a smooth homomorphism from $\mathbf{R}$ to $\mathrm{Diff}(X)$. Then, for all t_0 in $\mathbf{R}$ we have,

$$\frac{\partial h(t)^*(\alpha)}{\partial t}\Big|_{t=t_0} = h(t_0)^*(\mathfrak{L}_h(\alpha)).$$

In particular, if $\mathfrak{L}_h(\alpha)=0$, then $h(t)^*(\alpha)=\alpha$, for all t. Note that it is not necessary for h to be defined on all $\mathbf{R}$; it is enough to be defined on a small interval centered at the origin and such that $h(t+t')=h(t)\circ h(t')$ whenever it makes sense.

Proof Let us compute the derivative of $[t\mapsto h(t)^*(\alpha)]$ at the point t_0,

$$\begin{aligned}\frac{\partial h(t)^*(\alpha)}{\partial t}\Big|_{t=t_0} &= \lim_{\epsilon\to 0}\frac{h(t_0+\epsilon)^*(\alpha)-h(t_0)^*(\alpha)}{\epsilon}\\ &= \lim_{\epsilon\to 0} h(t_0)^*\Big[\frac{h(\epsilon)^*(\alpha)-\alpha}{\epsilon}\Big].\end{aligned}$$

Let us denote $\beta_\epsilon=(h(\epsilon)^*(\alpha)-\alpha)/\epsilon$. Then, for every n-plot $P\colon U\to X$, every point $r\in U$, every k vectors $u_1,\dots,u_k\in\mathbf{R}^n$,

$$\begin{aligned}\frac{\partial h(t)^*(\alpha)}{\partial t}\Big|_{t=t_0}(P)(r)(u_1)\cdots(u_k) &= \lim_{\epsilon\to 0} h(t_0)^*(\beta_\epsilon)(P)(r)(u_1)\cdots(u_k)\\ &= \lim_{\epsilon\to 0}\beta_\epsilon(h(t_0)\circ P)(r)(u_1)\cdots(u_k)\\ &= \mathfrak{L}_h(\alpha)(h(t_0)\circ P)(r)(u_1)\cdots(u_k)\\ &= h(t_0)^*(\mathfrak{L}_h(\alpha))(P)(r)(u_1)\cdots(u_k).\end{aligned}$$

Hence,

$$\frac{\partial h(t)^*(\alpha)}{\partial t}\Big|_{t=t_0} = h(t_0)^*(\mathfrak{L}_h(\alpha)).$$

Now, if $\mathfrak{L}_h(\alpha)=0$, then $\partial[h(t)^*(\alpha)]/\partial t=0$ for all t and $h(t)^*(\alpha)$ is constant, that is, equal to $\alpha=h(0)^*(\alpha)$. ■

2.3 Contraction of Differential Forms

Let X be a diffeological space, and let $\mathcal{D}$ be its diffeology. Let $P\colon U \to X$ be an n-plot and let $F\colon\ ]-\varepsilon, +\varepsilon[\ \to \mathcal{D}$ be an *arc of* $\mathcal{D}$*, centered at* P; that is,

(a) $F(0) = P$, and for all $s \in\ ]-\varepsilon, \varepsilon[$, $F(s)$ is a plot of X defined on the same fixed domain U than P;

(b) $\bar{F}\colon (s, r) \mapsto F(s)(r)$, defined on $]-\varepsilon, \varepsilon[\ \times U$, is a plot of X.

In particular, F is a smooth path of $\mathcal{D}$, equipped with the functional diffeology [Piz05]. Now, let α be a k-form of X, $k \geq 1$. For all $r \in U$, $\alpha(\bar{F})$ is a k-form of $]-\varepsilon, +\varepsilon[\ \times U$. The restriction to $\{0\} \times U$ of the contraction of $\alpha(\bar{F})$ with the vector $(1, 0) \in \mathbf{R} \times \mathbf{R}^n$ is a smooth $(k-1)$-form of U. That is, for all r in U, and for all $(k-1)$ vectors $v_2, \ldots, v_k$ of $\mathbf{R}^n$,

$$r \mapsto \left[(v_2, \ldots, v_k) \mapsto \alpha(\bar{F})_{\binom{0}{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \cdots \begin{pmatrix} o \\ v_k \end{pmatrix} \right] \in \mathcal{C}^\infty(U, \Lambda^{k-1}(\mathbf{R}^n)).$$

Let F and F' be two arcs of $\mathcal{D}$, centered at P. We shall say that F and F' define the same *variation* δP of the n-plot P, if for all k-forms α of X, $k \geq 1$, for all r in U, for any $(k-1)$ vectors $v_2, \ldots, v_k \in \mathbf{R}^n$, we have

$$\alpha(\bar{F}')_{\binom{0}{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ v_k \end{pmatrix} = \alpha(\bar{F})_{\binom{0}{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \cdots \begin{pmatrix} o \\ v_k \end{pmatrix}. \tag{2.1}$$

Thus, the variation δP of the plot P can be regarded as the class of the arc F of $\mathcal{D}$ for the equivalence relation defined by (2.1), and F *represents* δP.

For every differential form $\alpha \in \Omega^k(X)$, $k \geq 1$. For every n-plot $P\colon U \to X$, for every variation δP of the plot P, we call *contraction* of α by δP, the smooth $(k-1)$-form defined on U by

$$\alpha(\delta P)(r)(v_2) \cdots (v_k) = \alpha(\bar{F})_{\binom{0}{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \cdots \begin{pmatrix} o \\ v_k \end{pmatrix}, \tag{2.2}$$

where F represents δP, $r \in U$, and $v_2, \ldots v_k \in \mathbf{R}^n$,

$$\alpha(\delta P) \in \mathcal{C}^\infty(U, \Lambda^{k-1}(\mathbf{R}^n)).$$

Note 1 We shall denote by $\alpha \rfloor \delta P \in \Omega^{k-1}(U)$ the differential form of U, defined by $(\alpha \rfloor \delta P)(\mathbf{1}_U) = \alpha(\delta P)$, where U is regarded as a diffeological space.

Note 2 The contraction of a k-form of X, by some arc of plots, is not a $(k-1)$-form of X, since it is not defined on the plots of X but on the domain of the *target* P of the arc of plots. However, in some particular situations, for example in Subsection 2.4, this definition gives rise to a true $(k-1)$-form of X.

Note 3 In the definition of the variation δP, the value $s = 0$, where the variation is computed, does not really matter. We can also define the variation, denoted by δP_s, to be the variation for $s = 0$ of the translated arc $F_s\colon s' \mapsto F(s' + s)$, which gives

$$\alpha(\delta P_s)(r)(v_2) \cdots (v_k) = \alpha(\bar{F})_{\binom{s}{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \cdots \begin{pmatrix} o \\ v_k \end{pmatrix}.$$

2.4 Contracting Differential Forms on Slidings

Let X be a diffeological space, and let $\mathrm{Diff}(X)$ be its group of diffeomorphisms, equipped with the functional diffeology [Piz05]. We call a *sliding* of X any 1-plot F of $\mathrm{Diff}(X)$ centered at the identity, or more conveniently,

$$F\colon\ ]-\epsilon, +\epsilon[\ \to \mathrm{Diff}(X) \quad \text{and} \quad F(0) = \mathbf{1}_X,$$

where ϵ is any positive real number. Let $P\colon U \to X$ be any n-plot, for some integer n, we denote by $F \cdot P$ the following $(n+1)$-plot of X

$$F \cdot P\colon\ ]-\epsilon, +\epsilon[\ \times U \to X \quad \text{with} \quad F \cdot P\colon (t, r) \mapsto F(t)(P(r)).$$

Next, let $\alpha \in \Omega^p(X)$ be any differential p-form with $p \geq 1$. The contraction of α by the arc of plots $t \mapsto [r \mapsto F \cdot P(t, r)]$ (Subsection 2.3), will be denoted by $i_F(\alpha)(P)$. It is defined, for every $r \in U$, and for any $(p-1)$ vectors $v_2, \ldots, v_p \in \mathbf{R}^n$, by

$$i_F(\alpha)(P)_r(v_2)\cdots(v_p) = \alpha(F \cdot P)_{\binom{0}{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ v_p \end{pmatrix}. \tag{2.3}$$

(1) The sliding F being given, the map $i_F(\alpha)$ defined by (2.3) is a differential $(p-1)$-form of X. It will be called the *contraction of* α *by the sliding* F.

(2) The map $(F, \alpha) \mapsto i_F(\alpha)$ is smooth, where F belongs to the space of slidings of X and α belongs to $\Omega^p(X)$, equiped with their respective functional diffeologies. In particular, the contraction operation $i_F\colon \Omega^p(X) \to \Omega^{p-1}(X)$, with $p \geq 1$, is a smooth linear map.

Note To use the notation of Subsection 2.3, $i_F(\alpha)(P) = \alpha(\delta P)$, where δP is the variation of the arc of plots $[t \mapsto [r \mapsto F(t)(P(r))]]$.

Proof Let us prove that $i_F(\alpha)$ is a differential form on X. Then let $P\colon U \to X$ be an n-plot of X and let $\psi\colon V \to U$ be a smooth parametrization. Let us check the compatibility condition $i_F(\alpha)(P \circ \psi) = \psi^*(i_F(\alpha)(P))$. Let $s \in V$ and $u_2, \ldots, u_p \in \mathbf{R}^m$; we have

$$i_F(\alpha)(P \cdot \psi)_s(u_2, \ldots, u_p) = \alpha(F \cdot (P \circ \psi))_{\binom{0}{s}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ u_2 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ u_p \end{pmatrix}.$$

But

$$\begin{aligned} F \cdot (P \circ \psi)(t, s) &= F(t)\big(P \circ \psi(s)\big) = F(t)\big(P(\psi(s))\big) \\ &= F \cdot P\big(t, \psi(s)\big) = (F \cdot P) \circ (\mathbf{1} \times \psi)(t, s), \end{aligned}$$

where $\mathbf{1} \times \psi(t, v) = (t, \psi(v))$. Let us use the more compact notation

$$\begin{pmatrix} 0 \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ u_2 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ u_p \end{pmatrix}.$$

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We have, from the above,

$$\begin{aligned}\alpha(F\cdot(P\circ\psi))_{\binom{0}{s}}\binom{1}{0}\binom{0}{u} &= \alpha((F\cdot P)\circ(\mathbf{1}\times\psi))_{\binom{0}{s}}\binom{1}{0}\binom{0}{u}\\ &= (\mathbf{1}\times\psi)^*(\alpha(F\cdot P))_{\binom{0}{s}}\binom{1}{0}\binom{0}{u}\\ &= \alpha(F\cdot P)_{\binom{0}{\psi(s)}}\binom{1}{0}\binom{0}{v},\end{aligned}$$

where

$$\binom{0}{v} = \binom{0}{v_2}\cdots\binom{0}{v_p} \quad \text{with} \quad v_i = D(\psi)_s(u_i).$$

Thus, denoting $D(\psi)_s(u)$ for $(D(\psi)_s(u_2))\cdots(D(\psi)_s(u_p))$, we get finally

$$\begin{aligned}\alpha(F\cdot(P\circ\psi))_{\binom{0}{s}}\binom{1}{0}\binom{0}{u} &= i_F(\alpha)(P)_{\psi(s)}(D(\psi)_s(u))\\ &= \psi^*(i_F(\alpha)(P))_s(u).\end{aligned}$$

Hence, $i_F(\alpha)(P\circ\psi) = \psi^*(i_F(\alpha)(P))$, and $i_F(\alpha)$ is a $(p-1)$-form of X.

Let us prove now that $(F,\alpha)\mapsto i_F(\alpha)$ is smooth. Let $Q\colon s\mapsto(F_s,\alpha_s)$ be a plot of $\mathcal{C}^\infty_{\mathrm{loc}}(\mathbf{R},\mathrm{Diff}(X))\times\Omega^p(X)$, where $\mathcal{C}^\infty_{\mathrm{loc}}(\mathbf{R},\mathrm{Diff}(X))$ is the space of 1-plots of $\mathrm{Diff}(X)$ and F_s is centered at the identity for all s, $F_s(0)=\mathbf{1}_X$. According to the definition of the functional diffeology of a diffeology [Piz05], for any $s_0\in\mathrm{dom}(Q)$ there exists a small neighborhood W of s_0 such that $\mathrm{dom}(F_s)=\mathrm{dom}(F_{s_0})$ for all $s\in W$. We can restrict the domain of Q to W and choose $\mathrm{dom}(F_s)=\,]-\epsilon,+\epsilon[$ for some $\epsilon>0$. For a parametrization $s\mapsto F_s$ to be a plot means that $(s,t,x)\mapsto F_s(t)(x)$ is smooth. Now let us check that $s\mapsto i_{F_s}(\alpha_s)$ is a plot of $\Omega^{p-1}(X)$. Let $P\colon U\to X$ be a plot; we have

$$i_{F_s}(\alpha_s)(P)_r(v) = \alpha_s(F_s\cdot P)_{\binom{0}{r}}\binom{1}{0}\binom{0}{v},$$

with the same notations as above for v. But $(s,t,r)\mapsto(s,t,P(r))\mapsto F_s(t)(P(r)) = (F_s\cdot P)(t,r)$ is a plot of X. Thus, by the definition of the functional diffeology of $\Omega^p(X)$, $(s,t,r)\mapsto\alpha_s(F_s\cdot P)_{(t,r)}$ is smooth. Hence,

$$(r,s)\mapsto\alpha_s(F\cdot P)_{\binom{0}{r}}\binom{1}{0}$$

is smooth. Therefore, $(r,s)\mapsto i_F(\alpha_s)(P)_r$ is locally a plot of $\Omega^{p-1}(X)$, thus a plot, and $(F,\alpha)\mapsto i_F(\alpha)$ is a smooth map that is clearly linear in α. ∎

3 Integration of Differential Forms

In this section we shall study some properties of the integration of differential forms on chains, in diffeology. For the sake of simplicity, we integrate differential forms on *cubic chains*, which are related to *cubic homology*, because they only depend on the computation of multiple integrals in real spaces, which is a simple procedure. For more details on cubic chains and associated cubic homology; see [Piz05].

3.1 Integrating Forms on Chains

Let us consider the real vector space $\mathbf{R}^p$, *oriented* by its canonical basis $\mathcal{B} = (\mathbf{e}_1, \dots, \mathbf{e}_p)$; that is, oriented by the canonical volume vol_p associated with $\mathcal{B}$, $\mathrm{vol}_p = \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p$ or $d\mathbf{x}^1 \wedge \cdots \wedge d\mathbf{x}^p$ with $\mathbf{e}_i = dx^i$. As we know, any smooth p-form ω, on a real domain $U \subset \mathbf{R}^p$, is proportional to vol_p. That is, for all $\omega \in \mathcal{C}^\infty(U, \Lambda^p(\mathbf{R}^p))$, there exists a unique $f \in \mathcal{C}^\infty(\mathbf{R}^p, \mathbf{R})$ such that

$$\omega = f \times \mathrm{vol}_p, \quad \text{and} \quad f(x) = \omega_x(\mathbf{e}_1, \dots, \mathbf{e}_p).$$

Now, let α be a p-form on a diffeological space X. And let $\sigma \in \mathrm{Cub}_p(X) = \mathcal{C}^\infty(\mathbf{R}^p, X)$ be a *smooth p-cube*, that is, a smooth map from $\mathbf{R}^p$ to X. The *integral* of the p-form α on the p-cube σ is defined by

$$\int_\sigma \alpha = \int_{\mathrm{I}^p} \alpha(\sigma), \tag{3.1}$$

where $I = [0, 1]$. Since $\alpha(\sigma)$ is a smooth p-form of $\mathbf{R}^p$, there exists a smooth function f_σ such that $\alpha(\sigma) = f_\sigma \times \mathrm{vol}_p$. Hence, the integral of α over σ can also be written

$$\int_\sigma \alpha = \int_{\mathrm{I}^p} f_\sigma \times d\mathbf{x}^1 \wedge \cdots \wedge d\mathbf{x}^p \quad \text{with} \quad \alpha(\sigma) = f_\sigma \times \mathrm{vol}\,.$$

Then, the integral of α on σ becomes

$$\int_\sigma \alpha = \int_0^1 dx_1 \cdots \int_0^1 f_\sigma(x)\, dx_p, \quad \text{with} \quad x = (x_1, \dots, x_p),$$

where $f_\sigma(x) = \alpha(\sigma)_x(\mathbf{e}_1, \dots, \mathbf{e}_p)$. Note that, the space $\mathbf{R}^p$ having been oriented once and for all, there is no ambiguity on the value or the sign of the function f_σ.

The integral of p-forms on *cubic p-chains* is defined by linear extension of the integral of p-forms on p-cubes,

$$\int_c \alpha = \sum_\sigma n_\sigma \int_\sigma \alpha \quad \text{for all} \quad c = \sum_\sigma n_\sigma \sigma \in C_p(X).$$

The space $C_p(X)$ of cubic p-chains in X is the set of linear combinations of p-cubes in X, with coefficients in $\mathbf{Z}$, and finitely supported. The *support* of the p-chain $c = \sum_\sigma n_\sigma \sigma$ is the set of p-cubes σ such that $n_\sigma \neq 0$.

3.2 Pairing Chains and Forms

The *pairing operation* $(c, \alpha) \mapsto \int_c \alpha$, with $(c, \alpha) \in C_p(X) \times \Omega^p(X)$, satisfies the following properties:

(1) The pairing is a *bilinear* operation:

$$\int_{nc+n'c'} (s\alpha + s'\alpha') = ns \int_c \alpha + ns' \int_c \alpha' + n's \int_{c'} \alpha + n's' \int_{c'} \alpha'.$$

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(2) On cubes, the pairing is smooth:

$$\left[(\sigma, \alpha) \mapsto \int_\sigma \alpha\right] \in \mathcal{C}^\infty(\mathrm{Cub}_p(X) \times \Omega^p(X), \mathbf{R}),$$

where $\mathrm{Cub}_p(X)$ is equipped with its functional diffeology of space of smooth maps from $\mathbf{R}^p$ to X and $\Omega^p(X)$ is equipped with its functional diffeology of space of forms.

(3) A p-form vanishes identically if and only if its integral on any smooth p-cube vanishes,

$$\alpha = 0 \text{ if and only if } \int_\sigma \alpha = 0, \text{ for all } \sigma \in \mathrm{Cub}_p(X),$$

which is equivalent to saying that two p-forms coincide if and only if their integrals on any p-cube coincide.

Proof The bilinearity of the pairing is a direct consequence of the definitions of sum of chains and sums of forms.

For the second point, let $r \mapsto (\sigma_r, \alpha_r)$ be a plot of $\mathcal{C}^\infty(\mathrm{Cub}_p(X) \times \Omega^p(X))$, and then

$$\int_{\sigma_r} \alpha_r = \int_{\mathrm{I}^p} \alpha_r(\sigma_r).$$

The parametrization $r \mapsto \alpha_r$ being a plot of $\Omega^p(X)$ means that, for any plot $P\colon U \to X$, the map $(r, s) \mapsto \alpha_r(P)(s)$ is smooth (Subsection 1.1). On the other hand, $s \mapsto \sigma_s$ is a plot of $\mathrm{Cub}_p(X)$, which means that $S\colon (s, t) \mapsto \sigma_s(t)$ is a plot of X. Then $(r, s, t) \mapsto \alpha_r(S)(s, t)$ is smooth. But $[(r, t) \mapsto \alpha_r(\sigma_r)(t)]$ is the restriction of $(r, s, t) \mapsto \alpha_r(S)(s, t)$ for $s = r$. Thus, $[(r, t) \mapsto \alpha_r(\sigma_r)(t)]$ is smooth, which means that there exists a smooth function $(r, t) \mapsto f_r(t)$ such that $\alpha_r(\sigma_r)(t) = f_r(t) \times \mathrm{vol}_p$. Therefore,

$$\int_{\sigma_r} \alpha_r = \int_{\mathrm{I}^p} f_r \times \mathrm{vol}_p\,.$$

And $r \mapsto \int_{\sigma_r} \alpha_r$ is smooth, because the integration over the cube I^p is a smooth operation.

For the third point, let us assume that $\int_\sigma \alpha = 0$ for all p-cube σ and that $\alpha \neq 0$. Then there exists a p-plot $P\colon U \to X$ such that $\alpha(P) \neq 0$. That means that there exists $r \in U$, such that $\alpha(P)(r) \neq 0$. But α is a p-form on a domain of dimension p. Hence, there exists $f \in \mathcal{C}^\infty(U, \mathbf{R})$ such that $\alpha(P) = f \times \mathrm{vol}$, and $\alpha(P)(r) \neq 0$ means that $f(r) \neq 0$. Let us assume that $f(r) > 0$ (it would be equivalent to assume that $f(r) < 0$). By continuity, there exists a small cube C, centered at the point r, such that $f(r') > 0$ for all $r' \in C$. Since $f \restriction C$ is positive, the integral of f over the cube C is positive,

$$\int_C f \times \mathrm{vol}_p > 0.$$

But there exists a positive diffeomorphism φ from $\mathbf{R}^p$ onto an open neighborhood of r, mapping the standard cube I^p to C. Hence, $\sigma = P \circ \varphi$ is a smooth p-cube of X and

$$\int_\sigma \alpha = \int_{\mathrm{I}^p} \alpha(P \circ \varphi) = \int_{\mathrm{I}^p} \varphi^*(\alpha(P)) = \int_{\mathrm{I}^p} \varphi^*(f \times \mathrm{vol}_p).$$

But since φ is a positive diffeomorphism, by a change of coordinates, we get

$$\int_{\mathrm{I}^p} \varphi^*(f \times \mathrm{vol}_p) = \int_{\mathrm{I}^p} f \circ \varphi \times \det(D(\varphi)) \times \mathrm{vol}_p = \int_C f \times \mathrm{vol}_p\,.$$

Then $\int_\sigma \alpha > 0$, and we have a smooth p-cube σ on which the integral of α is non zero, which contradicts with the hypothesis. Therefore, for each plot P of X, $\alpha(P) = 0$, that is, $\alpha = 0$. ■

3.3 Pulling Back and Forth Forms and Chains

Let X and X' be two diffeological spaces and $f \in \mathcal{C}^\infty(X, X')$. For all p-cube $\sigma \in \mathrm{Cub}_p(X)$, the *pushforward* of σ by f is denoted by $f_*(\sigma)$ and is defined by

$$f_*(\sigma) = f \circ \sigma \quad \text{and} \quad f_*\colon \mathrm{Cub}_p(X) \to \mathrm{Cub}_p(X').$$

The pushforward of p-chains by f is defined by linear extension of the pushforward of p-cubes. Let $c = \sum_\sigma n_\sigma \sigma$ be a p-chain, then

$$f_*(c) = \sum_{\sigma'} n_{\sigma'}\, \sigma', \quad \text{where} \quad n_{\sigma'} = \sum_{\substack{\sigma \in \mathrm{Supp}(c) \\ \sigma' = f_*(\sigma)}} n_\sigma.$$

Now, for all $\alpha' \in \Omega^p(X')$, we have

$$\int_{f_*(c)} \alpha' = \int_c f^*(\alpha').$$

Note Since for any p-cube $\sigma = \mathbf{1}_{p*}(\sigma)$, where $\mathbf{1}_p\colon \mathbf{R}^p \to \mathbf{R}^p$ is the identity, we have an equivalent formulation of the integral of a p-form on a p-cube,

$$\int_\sigma \alpha = \int_{\mathbf{1}_p} \sigma^*(\alpha).$$

This expression will be used in the paragraph about the variations of the integrals of forms on chains.

Proof By definition, for any smooth p-cube σ, we have

$$\int_{f_*(\sigma)} \alpha' = \int_{\mathbf{1}_p} [f_*(\sigma)]^*(\alpha') = \int_{\mathbf{1}_p} (f \circ \sigma)^*(\alpha') = \int_{\mathbf{1}_p} \sigma^*(f^*(\alpha')) = \int_\sigma f^*(\alpha').$$

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Now, let $c = \sum_\sigma n_\sigma\,\sigma$, and let $c' = f_*(c)$. Thus, on the one hand we have

$$\int_{c'} \alpha' = \sum_{\sigma'} n_{\sigma'} \int_{\sigma'} \alpha' = \sum_{\sigma'} \Big[\sum_{\substack{\sigma\in \mathrm{Supp}(c)\\ f_*(\sigma)=\sigma'}} n_\sigma \Big] \int_{f_*(\sigma)} \alpha',$$

and on the other hand,

$$\int_c f^*(\alpha') = \sum_\sigma n_\sigma \int_\sigma f^*(\alpha') = \sum_\sigma n_\sigma \int_{f_*(\sigma)} \alpha'$$
$$= \sum_{\sigma'} \Big[\sum_{\substack{\sigma\in \mathrm{Supp}(c)\\ \sigma'=f_*(\sigma)}} n_\sigma \Big] \int_{f_*(\sigma)} \alpha'.$$

Therefore, $\int_{f_*(c)} \alpha' = \int_c f^*(\alpha')$. ∎

3.4 Changing the Coordinates of a Cube

Let X be a diffeological space. Let $\sigma \in \mathrm{Cub}_p(X)$ and $\alpha \in \Omega^p(X)$. Let φ be a *positive diffeomorphism* of I^p, that is, $\varphi \in \mathrm{Diff}(\mathbf{R}^p)$, $\varphi(\mathrm{I}^p) = \mathrm{I}^p$, and for all $x \in \mathrm{I}^p$, $\det[D(\varphi)(x)] > 0$. Then

$$\int_{\sigma_*(\varphi)} \alpha = \int_\sigma \alpha.$$

Note that, in the notation $\sigma_*(\varphi)$, φ is regarded as a smooth cube and σ a smooth map. Note that φ does not need to be defined on the whole $\mathbf{R}^p$, but only on any open neighborhood of the cube $\mathrm{I}^p \subset \mathbf{R}^p$.

Proof This proposition is a result of the change of variables of a multiple integral. Let f such that $\alpha(\sigma) = f \times \mathrm{vol}_p$. Thus,

$$\int_{\sigma_*(\varphi)} \alpha = \int_{\mathrm{I}^p} \alpha(\sigma\circ\varphi) = \int_{\mathrm{I}^p} \varphi^*(\alpha(\sigma)) = \int_{\mathrm{I}^p} \varphi^*(f \times \mathrm{vol}_p)$$

But $\varphi^*(f \times \mathrm{vol}_p) = (f\circ\varphi) \times \varphi^*(\mathrm{vol}_p)$. Hence,

$$\int_{\mathrm{I}^p} \varphi^*(f \times \mathrm{vol}_p) = \int_{\mathrm{I}^p} (f\circ\varphi) \times \varphi^*(\mathrm{vol}_p) = \int_{\mathrm{I}^p} (f\circ\varphi) \times \det(D(\varphi)) \times \mathrm{vol}_p\,.$$

But, since $\det(D(\varphi)) > 0$, we have

$$\int_{\mathrm{I}^p} (f\circ\varphi) \times \det(D(\varphi)) \times \mathrm{vol}_p = \int_{\mathrm{I}^p} (f\circ\varphi) \times |\det(D(\varphi))| \times \mathrm{vol}_p\,.$$

But, by application of the change of variables $x \mapsto \varphi(x)$, in a multiple integral, we get

$$\int_{\mathrm{I}^p} (f\circ\varphi) \times |\det(D(\varphi))| \times \mathrm{vol}_p = \int_{\varphi(\mathrm{I}^p)} f \times \mathrm{vol}_p = \int_{\mathrm{I}^p} f \times \mathrm{vol}_p = \int_\sigma \alpha.$$

Therefore, $\int_{\sigma_*(\varphi)} \alpha = \int_\sigma \alpha$. ∎

4 Variations of the Integrals of Forms on Chains

In this section we establish some theorems relative to the variations of integrals of forms on chains. We begin with the diffeological version of the Stokes theorem. Then we give a formula for any variation of the integral of a p-form on a p-chain. This formula mixes the form, its exterior differential, and the contractions with the variation of the chain. We deduce then the diffeological version of the Cartan–Lie formula and the homotopic invariance of the De Rham cohomology of diffeological spaces.

4.1 The Stokes Theorem

Let X be a diffeological space and let $\alpha \in \Omega^{p-1}(X)$ and $c \in C_p(X)$, $p \geq 1$, then

$$\int_c d\alpha = \int_{\partial c} \alpha.$$

Proof By linearity we need just to prove the statement for a p-cube $\sigma \in \mathrm{Cub}_p(X)$. We assume that Stokes' theorem is known for a p-form in $\mathbf{R}^{p-1}$, then

$$\int_\sigma d\alpha = \int_{\mathrm{I}^p} d\alpha(\sigma) = \int_{\mathrm{I}^p} d[\alpha(\sigma)] = \int_{\partial \mathrm{I}^p} \alpha(\sigma) = \int_{\partial\sigma} \alpha.$$

The covariant nature of diffeology makes the Stokes theorem reduce to the simplest case of smooth $(p-1)$-forms on standard p-cubes. ■

4.2 Variation of the Integral of a Form on a Cube

Let X be a diffeological space. And, let $r \mapsto (\sigma_r, \alpha_r) \in \mathcal{C}^\infty(U, \mathrm{Cub}_p(X) \times \Omega^p(X))$ be a plot of the product, defined on some real domain U of $\mathbf{R}^m$, $m \in \mathbf{N}$. Let us write simply α for α_0 and σ for σ_0. Then, since the pairing of chains and forms is a smooth map (Subsection 3.2), we have

$$\left[r \mapsto \int_{\sigma_r} \alpha_r \right] \in \mathcal{C}^\infty(U, \mathbf{R}).$$

The *variation of the integral* of α on σ, at some point $r \in U$, applied to a vector $\delta r \in \mathbf{R}^m$, is the number denoted and defined by

$$\delta \int_\sigma \alpha = \frac{\partial}{\partial r} \left\{ \int_{\sigma_r} \alpha_r \right\}_r (\delta r),$$

where the partial derivative $\partial/\partial r$ denotes the tangent linear map. Let us give an equivalent formulation of the variation of the integral. Let us consider the following map, defined on a small real interval $]-\epsilon, \epsilon[$,

$$s \mapsto \int_{\sigma_s} \alpha_s \quad \text{where} \quad \sigma_s = \sigma_{r+s\delta r} \text{ and } \alpha_s = \alpha_{r+s\delta r},$$

then

$$\delta \int_\sigma \alpha = \frac{\partial}{\partial s}\Big\{ \int_{\sigma_s} \alpha_s \Big\}_{s=0}.$$

Thus, the variations of the integral involves only 1-plot of cubes and forms. For this reason we shall continue only with 1-*plot variations* of $\int_\sigma \alpha$.

Now, for any arc of p-cube $s \mapsto \sigma_s$ of X centered at σ, for any arc of p-form $s \mapsto \alpha_s$ of X centered at α, we have the following identity:

$$\delta \int_\sigma \alpha = \int_{\mathbf{1}_p} d\alpha \rfloor \delta\sigma + \int_{\mathbf{1}_p} d[\alpha \rfloor \delta\sigma] + \int_{\mathbf{1}_p} \sigma^*(\delta\alpha), \tag{4.1}$$

where

(i) $d\alpha$ is the exterior differential of α defined in (Subsection 1.4);
(ii) $\delta\alpha$ is the p-form of X,

$$\delta\alpha = P \mapsto \frac{\partial}{\partial s}\big\{ \alpha_s(P) \big\}_{s=0},$$

defined, for every n-plot P of X, $r \in U$ and $v_1, \ldots, v_p \in \mathbf{R}^n$, by

$$\frac{\partial}{\partial s}\big\{ \alpha_s(P) \big\}_{s=0}(r)(v_1)\cdots(v_p) = \frac{\partial}{\partial s}\big\{ \alpha_s(P)(r)(v_1)\cdots(v_p) \big\}_{s=0};$$

(iii) $\alpha \rfloor \delta\sigma$ and $d\alpha \rfloor \delta\sigma$ are the contractions of the forms α and $d\alpha$ with the arc of plots $s \mapsto \sigma_s$ defined in Subsection 2.3.

Note 1 Thanks to the Stokes theorem (Subsection 4.1), the variation of the integral α on the cube σ can also be written

$$\delta \int_\sigma \alpha = \int_{\mathbf{1}_p} d\alpha \rfloor \delta\sigma + \int_{\partial\mathbf{1}_p} \alpha \rfloor \delta\sigma + \int_{\mathbf{1}_p} \sigma^*(\delta\alpha). \tag{4.2}$$

Note 2 The variation formula (4.1) still applies, *mutatis mutandi*, to the variation $\delta\sigma_s$ (Subsection 2.3, Note 3), for any $s \in \,]-\varepsilon, +\varepsilon[$.

Proof Let us consider the decomposition of the pairing

$$s \mapsto \begin{pmatrix} s \\ s \end{pmatrix} \mapsto \int_{\sigma_s} \alpha_s, \quad \text{with} \quad \begin{pmatrix} s \\ t \end{pmatrix} \mapsto \int_{\sigma_s} \alpha_t,$$

such that,

$$\frac{\partial}{\partial s}\Big\{ \int_{\sigma_s} \alpha_s \Big\}_{s=0} = \frac{\partial}{\partial s}\Big\{ \int_{\sigma_s} \alpha \Big\}_{s=0} + \frac{\partial}{\partial t}\Big\{ \int_{\sigma} \alpha_t \Big\}_{t=0}.$$

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Let us use the variable $r \in \mathbf{R}^p = \mathrm{dom}(\sigma)$, let $r = \sum_{k=1}^{p} r^k \mathbf{e}_k$ and $dr^k = \mathbf{e}^k$, $k = 1, \dots p$. The second term of the right-hand sum of this identity gives

$$\begin{aligned}
\frac{\partial}{\partial t}\left\{\int_\sigma \alpha_t\right\}_{t=0} &= \frac{\partial}{\partial t}\left\{\int_{\mathrm{I}^p} \alpha_t(\sigma)\right\}_{t=0} \\
&= \frac{\partial}{\partial t}\left\{\int_{\mathrm{I}^p} \alpha_t(\sigma)(r)(\mathbf{e}_1)\cdots(\mathbf{e}_p)\, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p\right\}_{t=0} \\
&= \int_{\mathrm{I}^p} \frac{\partial}{\partial t}\left\{\alpha_t(\sigma)(r)(\mathbf{e}_1)\cdots(\mathbf{e}_p)\right\}_{t=0} \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p \\
&= \int_{\mathrm{I}^p} (\delta\alpha)(\sigma)(r)(\mathbf{e}_1)\cdots(\mathbf{e}_p)\Big\}_{t=0} \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p \\
&= \int_\sigma \delta\alpha.
\end{aligned}$$

The variation of the pairing decomposes into two parts,

$$\delta \int_\sigma \alpha = \delta \int_\sigma \alpha\Big|_{\delta\alpha=0} + \int_\sigma \delta\alpha, \tag{4.3}$$

the first term corresponding to a zero variation of α. Let us then introduce

$$\sigma(s,r) = \sigma_s(r) \quad \text{and} \quad j_s \colon r \mapsto (s,r),$$

with σ a $(p+1)$-plot defined on $]-\epsilon,\epsilon[\,\times \mathrm{I}^p$ and j_s the injection from I^p to $]-\epsilon,\epsilon[\,\times \mathrm{I}^p$ at the height s. Now, using $\sigma_s = \sigma \circ j_s$, we have

$$\begin{aligned}
\int_{\sigma_s} \alpha &= \int_{\mathrm{I}^p} \alpha(\sigma_s)(r)(\mathbf{e}_1)\cdots(\mathbf{e}_p)\, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p \\
&= \int_{\mathrm{I}^p} \alpha(\sigma \circ j_s)(r)(\mathbf{e}_1)\cdots(\mathbf{e}_p)\, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p \\
&= \int_{\mathrm{I}^p} j_s^*(\alpha(\sigma))(r)(\mathbf{e}_1)\cdots(\mathbf{e}_p)\, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p \\
&= \int_{\mathrm{I}^p} \alpha(\sigma)\begin{pmatrix} s \\ r \end{pmatrix}\begin{pmatrix} 0 \\ \mathbf{e}_1 \end{pmatrix}\cdots\begin{pmatrix} 0 \\ \mathbf{e}_p \end{pmatrix} \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p.
\end{aligned}$$

Note that $\alpha(\sigma)$ is a p-form on the $(p+1)$-domain $]-\epsilon,\epsilon[\,\times \mathbf{R}^p$. Then let us introduce the $(p+1)$ coordinates $\mathbf{a}_i = [(s,r) \mapsto a_i]$, with $i = 0, 1, \dots, p$, and let $\mathbf{e}^0 = ds$. We have

$$\begin{aligned}
\alpha(\sigma)(s,r) = a_0\, &\mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3 \wedge \cdots \wedge \mathbf{e}^p \\
&+ a_1\, \mathbf{e}^0 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3 \wedge \cdots \wedge \mathbf{e}^p \\
&+ a_2\, \mathbf{e}^0 \wedge \mathbf{e}^1 \wedge \mathbf{e}^3 \wedge \cdots \wedge \mathbf{e}^p \\
&\quad \vdots \\
&+ a_p\, \mathbf{e}^0 \wedge \mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \cdots \wedge \mathbf{e}^{p-1},
\end{aligned}$$

or, equivalently

$$\alpha(\sigma)(s,r) = \sum_{k=0}^{p} a_k\, \mathbf{e}^0 \wedge \cdots [\mathbf{e}^k] \cdots \wedge \mathbf{e}^p,$$

where the bracket $[\mathbf{e}^k]$ means that $\mathbf{e}^k$ is omitted. Thus,

$$\int_{\sigma_s} \alpha = \int_{\mathrm{I}^p} a_0\, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p \quad \text{with} \quad a_0 = \alpha(\sigma)\begin{pmatrix} s \\ r \end{pmatrix}\begin{pmatrix} 0 \\ \mathbf{e}_1 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ \mathbf{e}_p \end{pmatrix}.$$

Now, since everything is smooth, integration and derivation commute, and the derivative with respect to the variable s becomes

$$\frac{\partial}{\partial s}\left\{ \int_{\sigma_s} \alpha \right\}_{s=0} = \int_{\mathrm{I}^p} \frac{\partial a_0}{\partial s}\Big|_{s=0} \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p.$$

Next, let us introduce the exterior differential of $\alpha(\sigma)$,

$$d[\alpha(\sigma)]_{\binom{s}{r}} = d\alpha(\sigma)_{\binom{s}{r}} = \left\{ \frac{\partial a_0}{\partial s} - \frac{\partial a_1}{\partial r^1} + \cdots + (-1)^p \frac{\partial a_p}{\partial r^p} \right\} \mathbf{e}^0 \wedge \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p.$$

After contracting the two terms of this identity by the vector of coordinates $(1,0) \in \mathbf{R} \times \mathbf{R}^p$, we get

$$\frac{\partial a_0}{\partial s}\, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p = d\alpha(\sigma)_{\binom{s}{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \left\{ \sum_{k=1}^{p} (-1)^k \frac{\partial a_k}{\partial r^k} \right\} \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p.$$

The first term of the right-hand side is the inner product of the value of the p-form $d\alpha(\sigma)$ at the point (s,r), with the vector of coordinates $(1,0)$. Evaluated at the point $(0,r)$, and restricted to $\mathbf{R}^p$, it is exactly the contraction $d\alpha$ with the arc of p-cubes $s \mapsto \sigma_s$ from (2.2). Hence,

$$\frac{\partial a_0}{\partial s}\Big|_{s=0} \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p = d\alpha(\delta\sigma)_r - \left\{ \sum_{k=1}^{p} (-1)^k \frac{\partial \mathbf{a}_k(0,r)}{\partial r^k} \right\} \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p.$$

The second term of the right-hand side of this identity is just the exterior differential of the contraction $\alpha(\delta\sigma)$. Indeed, let $v_2, \ldots, v_p$ be $p-1$ vectors of $\mathbf{R}^p$, using the above expression of $\alpha(\sigma)$, we get

$$\begin{aligned} &\alpha(\delta\sigma)_r(v_2)\cdots(v_p) \\ &\quad = \alpha(\sigma)_{\binom{0}{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ v_2 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ v_p \end{pmatrix} \\ &\quad = \left[\sum_{k=0}^{p} \mathbf{a}_k(0,r)\, dr^0 \wedge \cdots [dr^k] \cdots \wedge dr^p \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ v_2 \end{pmatrix} \cdots \begin{pmatrix} 0 \\ v_p \end{pmatrix} \\ &\quad = \left[\sum_{k=1}^{p} \mathbf{a}_k(0,r)\, dr^1 \wedge \cdots [dr^k] \cdots \wedge dr^p \right] (v_2)\cdots(v_p). \end{aligned}$$

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Thus,

$$\alpha(\delta\sigma)_r = \sum_{k=1}^{p} \mathbf{a}_k(0,r)\, dr^1 \wedge \cdots [dr^k] \cdots \wedge dr^p.$$

Then

$$\begin{aligned} d\big[\alpha(\delta\sigma)\big]_r &= \sum_{k=1}^{p} \frac{\partial \mathbf{a}_k(0,r)}{\partial r^k}\, \mathbf{e}^k \wedge \mathbf{e}^1 \wedge \cdots [\mathbf{e}^k] \cdots \wedge \mathbf{e}^p \\ &= \sum_{k=1}^{p} (-1)^{p-1} \frac{\partial \mathbf{a}_k(0,r)}{\partial r^k}\, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^k \wedge \cdots \wedge \mathbf{e}^p \\ &= -\Big\{ \sum_{k=1}^{p} (-1)^k \frac{\partial \mathbf{a}_k(0,r)}{\partial r^k} \Big\}\, \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p. \end{aligned}$$

Hence,

$$\frac{\partial a_0}{\partial s}\Big|_{s=0} \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p = d\alpha(\delta\sigma)_r + d[\alpha(\delta\sigma)]_r.$$

Therefore,

$$(4.4) \qquad \begin{aligned} \frac{\partial}{\partial s}\Big\{ \int_{\sigma_s} \alpha \Big\}_{s=0} &= \frac{\partial a_0}{\partial s}\Big|_{s=0} \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^p = \int_{\mathrm{I}^p} d\alpha(\delta\sigma) + d\big[\alpha(\delta\sigma)\big] \\ &= \int_{\mathbf{1}_p} d\alpha \rfloor \delta\sigma + \int_{\mathbf{1}_p} d[\alpha \rfloor \delta\sigma] \end{aligned}$$

Finally, combining (4.3) and (4.4) we obtain the formula (4.1) of the variation of the integral of a p-form on a p-cube. ■

4.3 Variation of the Integral of Forms on Chains

Let X be a diffeological space. Let $c = \sum_\sigma n_\sigma\, \sigma$ be a cubic p-chain. Let α be a differential p-form. The pullback of α by c is defined by linearity,

$$c^*(\alpha) = \sum_\sigma n_\sigma\, \sigma^*(\alpha) \in \Omega^p(\mathbf{R}^p).$$

And since the support of c is finite, the sums involved are finite. We can also write $\alpha(c) = \sum_\sigma n_\sigma\, \alpha(\sigma) \in \mathcal{C}^\infty(\mathbf{R}^p, \Lambda^p(\mathbf{R}^p))$. Now, the parametrizations $r \mapsto c_r$, from U to $C_p(X)$, satisfying the following property define a diffeology.

♠ For all $r_0 \in U$, there exists a open neighborhood V of r_0, a finite family of indices $\mathcal{I}$, a family $\{n_i\}_{i\in\mathcal{I}}$, with $n_i \in \mathbf{Z}$, a family $\{\sigma_i\}_{i\in\mathcal{I}}$, with $\sigma_i \in \mathcal{C}^\infty(V, \mathrm{Cub}_p(X))$, such that $c_r = \sum_{i\in\mathcal{I}} n_i\, \sigma_{i,r}$ for all $r \in V$.

The proof that ♠ defines a diffeology is left to the reader. Now, let $s \mapsto c_s$ be an arc of cubic p-chains centered at c, that is, a 1-plot of $C_p(X)$, defined on an interval $]-\varepsilon, +\varepsilon[$

such that $c_0 = c$. Let $c_s = \sum_\sigma n_\sigma \, \sigma_s$, where the σ_s are a finite family of p-cubes of X. We define the variation δc as the sum of the variations $\delta\sigma$, $\delta c = \sum_\sigma n_\sigma \, \delta\sigma$. That is for all p-forms $\alpha \in \Omega^p(X)$,

$$\alpha(\delta c) = \alpha\Big(\sum_\sigma n_\sigma \delta\sigma\Big) = \sum_\sigma n_\sigma \, \alpha(\delta\sigma).$$

Or directly, according to the definition (Subsection 2.3),

$$\begin{aligned}\alpha(\delta c)(r)(v_2)\cdots(v_p) &= \alpha(\mathbf{c})_{\binom{0}{r}}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\v_2\end{pmatrix}\cdots\begin{pmatrix}0\\v_p\end{pmatrix}\\ &= \sum_\sigma n_\sigma \, \alpha(\sigma)_{\binom{0}{r}}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\v_2\end{pmatrix}\cdots\begin{pmatrix}0\\v_p\end{pmatrix},\end{aligned}$$

where $\mathbf{c}$ is the plot $(s,r) \mapsto c_s(r)$, with $(s,r) \in \,]-\varepsilon, +\varepsilon[\, \times \mathbf{R}^p$, and $\mathbf{c} = \sum_\sigma n_\sigma \, \sigma$, with $\sigma(s,r) = \sigma_s(r)$. Now, thanks to the linearity of all these constructions, we have the formula of the variation of the integral of p-forms on p-chains,

$$\delta \int_c \alpha = \int_{\mathbf{1}_p} d\alpha \rfloor \delta c + \int_{\mathbf{1}_p} d[\alpha \rfloor \delta c] + \int_{\mathbf{1}_p} c^*(\delta\alpha).$$

Note As well as for the variation of the integral of α on a cube, applying Stokes' theorem, we have the alternative formula

$$\delta \int_c \alpha = \int_{\mathbf{1}_p} d\alpha \rfloor \delta c + \int_{\partial \mathbf{1}_p} \alpha \rfloor \delta c + \int_{\mathbf{1}_p} c^*(\delta\alpha).$$

4.4 The Cartan-Lie Formula

Let X be a diffeological space, and let its group of diffeomorphisms $\mathrm{Diff}(X)$ be equipped with the functional diffeology. Let $F\colon \mathbf{R} \to \mathrm{Diff}(X)$ be a sliding, that is, a 1-plot centered at the identity (Subsection 2.4), and let α be any differential k-form of X, with $k \geq 1$. The Lie derivative $\mathfrak{L}_F(\alpha)$ of α by F (Subsection 2.1) satisfies the identity

$$\mathfrak{L}_F(\alpha) = i_F(d\alpha) + d(i_F(\alpha)), \tag{4.5}$$

where i_F is the contraction by F defined in (Subsection 2.4). This is the extension to diffeological spaces of the classical Cartan–Lie formula for manifolds.

Proof Let $F\colon \mathbf{R} \to \mathrm{Diff}(X)$ be a 1-plot, and let $\sigma \in \mathrm{Cub}_k(X)$ be a k-cube on X, and let

$$\begin{aligned}\alpha_t &= F(t)_*(\alpha) = (F(t)^{-1})^*(\alpha)\\ \sigma_t &= (F(t))_*(\sigma) = F(t) \circ \sigma.\end{aligned}$$

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Then, thanks to Subsection 3.3, for all $t \in \mathbf{R}$:

$$\int_{\sigma_t} \alpha_t = \int_{F(t)_*(\sigma)} F(t)_*(\alpha) = \int_\sigma F(t)^* \circ F(t)_*(\alpha) = \int_\sigma \alpha.$$

Now, by differentiation with respect to the parameter t, we get on the one hand

$$\delta \int_{\sigma_t} \alpha_t = \delta \int_\sigma \alpha = 0, \quad \text{with} \quad \delta = \frac{\partial}{\partial t}\Big|_{t=0},$$

and on the other hand, by the formula of the variation of integral of differential forms (4.1),

$$\delta \int_{\sigma_t} \alpha_t = \int_{\mathbf{1}_p} d\alpha \rfloor \delta\sigma + \int_{\mathbf{1}_p} d[\alpha \rfloor \delta\sigma] + \int_{\mathbf{1}_p} \delta\sigma^*(\alpha). \tag{4.6}$$

But:

(a) $\delta\alpha = \dfrac{\partial \alpha_t}{\partial t}\Big|_{t=0} = \dfrac{\partial}{\partial t} F(t)_*(\alpha)\Big|_{t=0} = \dfrac{\partial}{\partial t}(F(t)^{-1})^*(\alpha)\Big|_{t=0} = -\mathfrak{L}_F(\alpha);$

(b) $\sigma_t = F(t) \circ \sigma$ and $\delta = \dfrac{\partial}{\partial t}\Big|_{t=0}$ implies $\alpha \rfloor \delta\sigma = \sigma^*(i_F(\alpha))$, for all α.

Hence, the above identity (4.6) becomes

$$0 = \int_\sigma i_F[d\,\alpha] + \int_\sigma d\left[i_F(\alpha)\right] - \int_\sigma \mathfrak{L}_F(\alpha) = \int_\sigma i_F[d\,\alpha] + d\left[i_F(\alpha)\right] - \mathfrak{L}_F(\alpha).$$

This is satisfied for any k-cube σ. But, a k-form whose integral vanishes on any k-cube vanishes identically (Subsection 3.2), thus $i_F[d\ \alpha)] + d\ [i_F(\alpha)] - \mathfrak{L}_F(\alpha) = 0$, and $\mathfrak{L}_F(\alpha) = i_F[d\ \alpha)] + d\ [i_F(\alpha)]$. ■

4.5 A Few Additional Examples of Application

Here are a few more examples of application of the variation of the integral of a p-form on a p-chain and of the Cartan–Lie formula; see [Piz05]. Let X be a connected diffeological space:

(1) Let $h \in \mathrm{Hom}^\infty(\mathbf{R}, \mathrm{Diff}(X))$ such that $h(t)^*(\omega) = \lambda(t) \times \omega$, where λ is a real smooth function and ω a differential p-form on X. We call such homomorphisms *Liouville's rays*. Thus, ω is exact and $i_h(\omega)$ is a primitive.

(2) Let α be a closed 1-form on X, the integral $\int_\ell \alpha$, where $\ell \in \mathrm{Loops}(X)$, does not depends on the free homotopy class of ℓ. This also applies for the integrals of a p-forms on homotopic p-cubes.

(3) If there exists a 1-form whose the integral on a loop is not zero, then the space X is not simply connected. This is used to easily show that $\mathrm{Diff}(S^1)$ is not simply connected by considering the pullback of the fundamental 1-form θ on S^1 by the orbit map $\widehat{1}\colon f \mapsto f(1)$, and the loop mapping S^1 into $\mathrm{Diff}(S^1)$ as the group of rotations.

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(4) Any closed 1-form α on a simply connected diffeological space X is exact. Moreover, if the group of period $P_\alpha = \{\int_\ell \alpha \mid \ell \in \mathrm{Loops}(X)\}$ is diffeologically discrete, which is equivalent to being a strict subgroup of $\mathbf{R}$, then there exists a smooth function $f\colon X \to T_\alpha$, such that $f^*(\theta) = \alpha$, where $T_\alpha = \mathbf{R}/P_\alpha$ and θ is the fundamental 1-form on T_α. In any case there exists a unique smallest Galoisian covering of X, with structure group P_α, on which the pullback of α is exact. This solves the universal problem of integrating a closed 1-form.

(5) If ω is a closed 2-form on X and G is a diffeological group preserving ω, then there exists a *Paths Moment Map* $\Psi\colon X \to \mathcal{G}^*$, where $\mathcal{G}^*$ denotes the space of left momenta of G, from which is derived the generalization of the moment map in diffeology [Piz07]. Related to the example (Subsection 1.1, Note 2), the moment map μ of $d\alpha$ on $\mathrm{Diff}(S^1)$ is then given by

$$\mu(\varphi)(P)_r(\delta r) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\partial}{\partial s} \big\{ \varphi^{-1} \circ P(r)^{-1} \circ P(s) \circ \varphi(n) \big\}_{s=r}(\delta r).$$

Because a momentum is characterized by its values on arcs centered at the identity, and because every arc γ centered at the identity in $\widetilde{G}$ is tangential to some ray $h \in \mathrm{Hom}^\infty(\mathbf{R}, \widetilde{G})$, the computation of the stabilizer of α, for the coadjoint action, is reduced to Donato's computation in [Don94], and it coincides with the orbits of $2\pi\mathbf{Z}$. Thus, $d\alpha$ passes to the coadjoint orbit $\mathcal{O}_\alpha$, which is actually diffeomorphic to $\mathrm{Diff}(S^1)$ itself, and symplectic according to the meaning we define in *op. cit.*.

(6) If ω is a closed 2-form on X and if the group of periods P_ω of ω is discrete, then there exists a principal fiber bundle $\pi\colon Y \to X$, with structure group P_ω and a connection form λ on Y with curvature ω. This solves a universal problem about integrating closed 2-forms in the general framework of diffeology,[1] and applies in particular to manifolds.

5 The Chain-Homotopy Operator

The *chain-homotopy* operator on a diffeological space X is a linear map

$$K\colon \Omega^p(X) \to \Omega^{p-1}(\mathrm{Paths}(X))$$

that satisfies $K \circ d + d \circ K = \widehat{1}^* - \widehat{0}^*$ (Subsection 5.6), with $\mathrm{Paths}(X) = \mathcal{C}^\infty(\mathbf{R}, X)$, $\widehat{0}$ and $\widehat{1}$ map a path γ to its source $\widehat{0}(\gamma) = \gamma(0)$ and its target $\widehat{1}(\gamma) = \gamma(1)$. Since the space of paths of X is naturally a diffeological space, it is legitimate to consider differential forms on $\mathrm{Paths}(X)$ and its subspaces. The advantage of the diffeological approach is the ability to stay in the same category and avoid parallel constructions and tedious constructions. The chain-homotopy operator has multiple applications; it implies, for example, the homotopic invariance of the De Rham cohomology (Subsection 6.1), and it is crucial for the construction of the moment map of a closed 2-form; see [Piz07].

[1] This is a work in progress.

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5.1 Integration Operator of Forms Along Paths

Let X be a diffeological space, and let $\mathrm{Paths}(X) = \mathcal{C}^\infty(\mathbf{R}, X)$ be the space of smooth paths of X, equipped with the functional diffeology. Let us consider, for each $t \in \mathbf{R}$ the *evaluation map* of paths, at the point t:

$$\hat{t} \colon \mathrm{Paths}(X) \to X \quad \text{with} \quad \hat{t}(\gamma) = \gamma(t).$$

The map $\hat{t}$ is a smooth map; this is an immediate consequence of the definition of the functional diffeology.

(1) For all $p > 0$, there exists a map $\Phi \colon \Omega^p(X) \to \Omega^p(\mathrm{Paths}(X))$, which we call *Integration Operator*, and which we denote symbolically by

$$\Phi(\alpha) = \int_0^1 \hat{t}^*(\alpha)\,\mathrm{dt}, \text{ for all } \alpha \in \Omega^p(X).$$

This map is precisely defined, for any n-plot $P \colon U \to \mathrm{Paths}(X)$, $n \in \mathbf{N}$, for all $r \in U$ and $v = (v_1 \cdots v_p)$, p vectors of $\mathbf{R}^n$, by

$$\Phi(\alpha)(P)(r)(v) = \int_0^1 \alpha(\hat{t} \circ P)(r)(v)\,\mathrm{dt}, \quad \hat{t} \circ P = \big[r \mapsto P(r)(t)\big].$$

Then Φ maps any differential p-form α of X to the p-form of $\mathrm{Paths}(X)$, given by integrating α along the paths.

(2) The Integration Operator Φ is linear. Let α and α' be any two p-forms of X, and let $s \in \mathbf{R}$:

$$\Phi(\alpha + \alpha') = \Phi(\alpha) + \Phi(\alpha') \quad \text{and} \quad \Phi(s\alpha) = s \times \Phi(\alpha).$$

(3) The Integration Operator Φ is a smooth map,

$$\Phi \in \mathcal{C}^\infty\big(\Omega^p(X), \Omega^p(\mathrm{Paths}(X))\big).$$

Proof (1) Let us check first that $\Phi(\alpha)$ is a well-defined p-form on $\mathrm{Paths}(X)$. Let $P \colon U \to X$ be a plot and let $F \in \mathcal{C}^\infty(V, U)$, where V is any real domain. Thus,

$$\begin{aligned}\Phi(\alpha)(P \circ F) &= \int_0^1 \alpha(\hat{t} \circ P \circ F)\,\mathrm{dt} = \int_0^1 F^*(\alpha(\hat{t} \circ P))\,\mathrm{dt} \\ &= F^*\left(\int_0^1 \alpha(\hat{t} \circ P)\,\mathrm{dt}\right) = F^*(\Phi(\alpha)(P)).\end{aligned}$$

(2) Now, let us check that the Integration Operator is linear. Let α and α' be any two p-forms of X. Let $P_t = \hat{t} \circ P$, we have

$$\begin{aligned}\Phi(\alpha + \alpha') &= \left[P \mapsto \int_0^1 (\alpha + \alpha')(P_t)\,\mathrm{dt}\right] \\ &= \left[P \mapsto \int_0^1 \alpha(P_t)\,\mathrm{dt} + \int_0^1 \alpha'(P_t)\,\mathrm{dt}\right] \\ &= \Phi(\alpha) + \Phi(\alpha').\end{aligned}$$

And, for any $s \in \mathbf{R}$,

$$\Phi(s\alpha) = \left[P \mapsto \int_0^1 s\alpha(P_t)\,\mathrm{dt}\right] = \left[P \mapsto s\int_0^1 \alpha(P_t)\,\mathrm{dt}\right] = s \times \Phi(\alpha).$$

(3) Since the map $\hat{t}\colon \mathrm{Paths}(X) \to X$ is smooth and since integration preserves smoothness, the integration operator Φ is a smooth linear map from $\Omega^p(X)$ to $\Omega^p(\mathrm{Paths}(X))$. ■

5.2 The Operator Φ is a Morphism of De Rham Complexes

The Integration Operator Φ of a diffeological space X (Subsection 5.1) is a morphism of the De Rham complex of X, to the De Rham complex of Paths(X),

$$d \circ \Phi = \Phi \circ d.$$

This is summarized by the following commutative diagram, for all $p > 0$:

$$\begin{array}{ccc} \Omega^p(X) & \xrightarrow{\Phi} & \Omega^p(\mathrm{Paths}(X)) \\ \downarrow d & & \downarrow d \\ \Omega^{p+1}(X) & \xrightarrow[\Phi]{} & \Omega^{p+1}(\mathrm{Paths}(X)) \end{array}$$

Proof Let α be a p-form on X, $p > 0$. Then with the above notation, $P_t = \hat{t} \circ P$, where P is a plot of Paths(X), we have:

$$\begin{aligned}\Phi(d\alpha)(P) &= \int_0^1 (d\alpha)(P_t)\,\mathrm{dt} = \int_0^1 d\big[\alpha(P_t)\big]\,\mathrm{dt} = d\Big(\int_0^1 \alpha(P_t)\,\mathrm{dt}\Big) \\ &= d\big(\Phi(\alpha)(P)\big).\end{aligned}$$

Thus, $\Phi(d\alpha) = d(\Phi(\alpha))$. ■

5.3 Variance of the Integration Operator Φ

Let X and X' be two diffeological spaces, let $f\colon X \to X'$ be a smooth map. The map f leads to a smooth correspondence

$$f_{\mathcal{P}}\colon \mathrm{Paths}(X) \to \mathrm{Paths}(X') \quad \text{with} \quad f_{\mathcal{P}}(\gamma) = f \circ \gamma.$$

The map f also induces the two pullbacks:

$$f^*\colon \Omega^\star(X') \to \Omega^\star(X) \quad \text{and} \quad f^*_{\mathcal{P}}\colon \Omega^\star(\mathrm{Paths}(X')) \to \Omega^\star(\mathrm{Paths}(X)).$$

Let Φ_X et $\Phi_{X'}$ be the two associated Integration Operators (Subsection 5.1), then

$$\Phi_X \circ f^* = f^*_{\mathcal{P}} \circ \Phi_{X'}.$$

This is summarized by the following commutative diagram:

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$$\begin{array}{ccc} \Omega^p(X') & \xrightarrow{\Phi_{X'}} & \Omega^p(\mathrm{Paths}(X')) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle f_{\mathcal{P}}^*} \\ \Omega^p(X) & \xrightarrow[\Phi_X]{} & \Omega^p(\mathrm{Paths}(X)) \end{array}$$

Proof Let α be a differential p-form of X' and $P\colon U \to \mathrm{Paths}(X)$ be a plot. Let us use the above notation $P_t = \widehat{t} \circ P$. On the one hand we have

$$\big[(\Phi_X \circ f^*)(\alpha)\big](P) = \big[\Phi_X(f^*(\alpha))\big](P) = \int_0^1 f^*(\alpha)(P_t)\,\mathrm{dt} = \int_0^1 \alpha(f \circ P_t)\,\mathrm{dt},$$

and, on the other hand,

$$\big[(f_{\mathcal{P}}^* \circ \Phi_{X'})(\alpha)\big](P) = \big[f_{\mathcal{P}}^*(\Phi_{X'}(\alpha))\big](P) = \big[\Phi_{X'}(\alpha)\big](f_{\mathcal{P}} \circ P) = \int_0^1 \alpha\big[(f_{\mathcal{P}} \circ P)_t\big]\,\mathrm{dt}\,.$$

But, for any $r \in U$,

$$\begin{aligned} (f \circ P_t)(r) &= f(P_t(r)) = f(P(r)(t)), \\ (f_{\mathcal{P}} \circ P)_t(r) &= (f_{\mathcal{P}} \circ P)_t(r) = f(P(r)(t)). \end{aligned}$$

Thus, $f \circ P_t = (f_{\mathcal{P}} \circ P)_t$, and finally $\Phi_X \circ f^* = f_{\mathcal{P}}^* \circ \Phi_{X'}$. ∎

5.4 Derivation along Time Reparametrization

Let X be a diffeological space, and let $\mathrm{Paths}(X)$ be the space of its smooth paths, equipped with the functional diffeology. The group of translations $(\mathbf{R}, +)$ acts smoothly by reparametrization on $\mathrm{Paths}(X)$ as a 1-parameter group of diffeomorphisms. Let us denote by τ this action. For all $\gamma \in \mathrm{Paths}(X)$, and for all $e \in \mathbf{R}$,

$$\tau(e)\colon \gamma \mapsto \gamma \circ T_e \quad \text{with} \quad T_e\colon t \mapsto t + e.$$

Let α be a p-form of X, the Lie derivative (Subsection 2.1) of the p-form $\Phi(\alpha)$ by the 1-parameter group τ satisfies:

$$\mathfrak{L}_\tau(\Phi(\alpha)) = \widehat{1}^*(\alpha) - \widehat{0}^*(\alpha).$$

Proof Let us first check that $\tau\colon e \mapsto [\gamma \mapsto \gamma \circ T_e]$ is a smooth homomorphism from $(\mathbf{R}, +)$ into $\mathrm{Diff}(\mathrm{Paths}(X))$. Let us check that τ takes its values in $\mathrm{Diff}(\mathrm{Paths}(X))$:

(a) for all $e \in \mathbf{R}$, $\tau(e)(\gamma) = \tau(e)(\gamma')$ implies $\gamma = \gamma'$, so $\tau(e)$ is injective, and the inverse is given by $\tau(e)^{-1} = \tau(-e)$, thus $\tau(e)$ is bijective;

(b) for all $e \in \mathbf{R}$, $\tau(e)$ is smooth.

Indeed, let P be a plot of Paths(X), that is, $(r,t) \mapsto P(r)(t)$ is a plot of X. Then $(\tau(e) \circ P)(r)(t) = (\tau(e)(P(r))(t) = P(r)(t+e)$, and the map $(r,t) \mapsto (r, t+e) \mapsto P(r)(t+e)$, being a composite of smooth maps, is smooth. Thus $\tau(e)$ is smooth, and since $\tau(e)^{-1} = \tau(-e)$, $\tau(e)$ is a diffeomorphism of Paths(X). Next, τ is clearly an homomorphism, $\tau(e+e') = \tau(e) \circ \tau(e')$. Then, let us check that τ is smooth, that is, τ is a plot of Diff(Paths(X)). The parametrization τ is a plot of Diff(Paths(X) if and only if $(e,\gamma) \mapsto \tau(e)(\gamma)$ and $(e,\gamma) \mapsto \tau(e)^{-1}(\gamma) = \tau(-e)(\gamma)$ are smooth. That is, for all plots P of Paths(X), if and only if $(e,r) \mapsto \tau(e)(P(r)) = P(r) \circ T_e$ is smooth, that is, $(e,r,t) \mapsto P(r)(t+e)$ is smooth, which indeed is the case. Thus, $\tau \in \mathrm{Hom}^\infty(\mathbf{R}, \mathrm{Diff}(\mathrm{Paths}(X)))$.

Now, let us denote $\alpha = \Phi(\alpha)$. We have then, for any plot P de Paths(X),

$$[\mathfrak{L}_\tau \alpha](P) = \frac{\partial}{\partial t}\Big\{ [\tau(t)^*\alpha](P)\Big\}_{t=0} = \frac{\partial}{\partial t}\Big\{ \alpha(\tau(t) \circ P)\Big\}_{t=0}.$$

But $\tau(t) \circ P \colon r \mapsto P(r) \circ T_t$, thus

$$\alpha\big[\tau(t) \circ P\big] = \int_0^1 \alpha\big[(\tau(t) \circ P)_s\big]\, \mathrm{d}s = \int_0^1 \alpha\big[r \mapsto P(r)(t+s)\big]\, \mathrm{d}s\,.$$

Let $u = t + s$, we get

$$\alpha\big[\tau(t) \circ P\big] = \int_t^{1+t} \alpha[r \mapsto P(r)(u)]\, \mathrm{d}u\,.$$

After derivation with respect to t, for $t = 0$, we get

$$[\mathfrak{L}_\tau \alpha](P) = \alpha\big[r \mapsto P(r)(1)\big] - \alpha\big[r \mapsto P(r)(0)\big] = [\widehat{1}^*\alpha - \widehat{0}^*\alpha](P),$$

for all plots P of X; that is, $\mathfrak{L}_\tau(\Phi(\alpha)) = \widehat{1}^*\alpha - \widehat{0}^*\alpha$. ∎

5.5 Variance of the Time Reparametrization

Let X and X' be two diffeological spaces, let $f\colon X \to X'$ be a smooth map. Let $f_{\mathcal{P}}\colon \mathrm{Paths}(X) \to \mathrm{Paths}(X')$ be the action of f on paths defined in (Subsection 5.3) and $f_{\mathcal{P}}^*\colon \Omega^p(\mathrm{Paths}(X')) \to \Omega^p(\mathrm{Paths}(X))$ be the induced action of $f_{\mathcal{P}}$ at the level of p-forms. Let τ and τ' denote the action $(\mathbf{R}, +)$ on Paths(X) and Paths(X'), as defined in Subsection 5.4. Let i_τ and $i_{\tau'}$ be the contraction associated with these 1-parameter groups of diffeomorphisms (Subsection 2.4). Then $i_\tau \circ f_{\mathcal{P}}^* = f_{\mathcal{P}}^* \circ i_{\tau'}$, summarized by the following commutative diagram:

$$\begin{array}{ccc}
\Omega^p(\mathrm{Paths}(X')) & \xrightarrow{\;f_{\mathcal{P}}^*\;} & \Omega^p(\mathrm{Paths}(X)) \\
\Big\downarrow{\scriptstyle i_{\tau'}} & & \Big\downarrow{\scriptstyle i_\tau} \\
\Omega^{p-1}(\mathrm{Paths}(X')) & \xrightarrow[\;f_{\mathcal{P}}^*\;]{} & \Omega^{p-1}(\mathrm{Paths}(X))
\end{array}$$

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Proof Let β be a p-form on Paths(X'), $P\colon U \to \text{Paths}(X)$ be a n-plot, $r \in U$, and let v represent $(p-1)$ vectors of $\mathbf{R}^n$. By definition of the contraction of a p-form by a 1-parameter group of diffeomorphisms (Subsection 2.4), we have:

(1) On the one hand,

$$\left[(i_\tau \circ f_{\mathcal{P}}^*)(\beta)\right](P)_r(v) = \left[i_\tau(f_{\mathcal{P}}^*(\beta))\right](P)_r(v) = f_{\mathcal{P}}^*(\beta)(\tau \cdot P)_{\binom{0}{r}} \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\v\end{pmatrix}$$
$$= \beta(f_{\mathcal{P}} \circ (\tau \cdot P))_{\binom{0}{r}} \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\v\end{pmatrix},$$

where $\tau \cdot P(t, r) = \tau(t)(P(r))$. But,

$$\begin{aligned}(5.1) \qquad \left[f_{\mathcal{P}} \circ (\tau \cdot P)\right](t, r) &= f_{\mathcal{P}}\big((\tau \cdot P)(t, r)\big) = f_{\mathcal{P}}\big(\tau(t)(P(r))\big)\\ &= f_{\mathcal{P}}\left[s \mapsto \tau(t)(P(r))(s)\right] = f_{\mathcal{P}}\left[s \mapsto P(r)(s+t)\right]\\ &= \left[s \mapsto f(P(r)(s+t))\right].\end{aligned}$$

(2) On the other hand,

$$\left[(f_{\mathcal{P}}^* \circ i_{\tau'})(\beta)\right](P)_r(v) = \left[f_{\mathcal{P}}^*(i_{\tau'}(\beta))\right](P)_r(v) = \left[(i_{\tau'}(\beta)(f_{\mathcal{P}} \circ P)\right]_r(v)$$
$$= \beta\big(\tau' \cdot (f_{\mathcal{P}} \circ P)\big)_{\binom{0}{r}} \begin{pmatrix}1\\0\end{pmatrix} \begin{pmatrix}0\\v\end{pmatrix}.$$

But,

$$\begin{aligned}(5.2) \qquad \left[\tau' \cdot (f_{\mathcal{P}} \circ P)\right](t, r) &= \tau'(t)(f_{\mathcal{P}} \circ P(r)) = \left[s \mapsto (f_{\mathcal{P}} \circ P(r))(s+t)\right]\\ &= \left[s \mapsto f(P(r)(s+t))\right].\end{aligned}$$

Now, comparing (5.1) and (5.2), we get

$$\left[(i_\tau \circ f_{\mathcal{P}}^*)(\beta)\right](P)_r(v) = \left[(f_{\mathcal{P}}^* \circ i_{\tau'})(\beta)\right](P)_r(v),$$

that is, finally, $i_\tau \circ f_{\mathcal{P}}^* = f_{\mathcal{P}}^* \circ i_{\tau'}$. ∎

5.6 The Chain-Homotopy Operator K

Let X be a diffeological space. Let Φ be the integration operator defined in Subsection 5.1, let τ be the time-reparametrization defined in (Subsection 5.4), and let i_τ be the contraction by the sliding τ (Subsection 2.4). The operator $K\colon \Omega^p(X) \to \Omega^{p-1}(\text{Paths}(X))$ defined, for all integers $p > 0$, by

$$(5.3) \qquad K = i_\tau \circ \Phi,$$

satisfies

$$(5.4) \qquad K \circ d + d \circ K = \widehat{1}^* - \widehat{0}^*.$$

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The operator K will be called the *chain-homotopy operator*. Moreover, K is smooth and linear, which is denoted by

$$K \in \mathrm{L}^\infty\big(\Omega^p(X), \Omega^{p-1}(\mathrm{Paths}(X))\big).$$

Let $\alpha \in \Omega^p(X)$, with $p > 1$, for $p = 1$ see (Subsection 5.7). Let $P\colon U \to \mathrm{Paths}(X)$ be an n-plot; the value of $K\alpha$ on P is explicitly given by

$$(K\alpha)(P)_r(v_2)\cdots(v_p) = \int_0^1 \alpha\Big[\begin{pmatrix} s \\ r \end{pmatrix} \mapsto P(r)(s+t)\Big]_{\binom{0}{r}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ v_2 \end{pmatrix}\cdots\begin{pmatrix} 0 \\ v_p \end{pmatrix} \mathrm{dt},$$

where $r \in U$, and $v_2, \ldots, v_p$ are $(p-1)$ vectors of $\mathbf{R}^n$.

Proof Let α be a p-form of X, $p > 0$. On the one hand (Subsection 5.4) we have

$$\mathfrak{L}_\tau(\Phi(\alpha)) = \widehat{1}^*(\alpha) - \widehat{0}^*(\alpha),$$

and on the other hand, applying the Cartan formula (Subsection 4.4) and the commutation $d \circ \Phi = \Phi \circ d$ (Subsection 5.2), we get

$$\begin{aligned}\mathfrak{L}_\tau\big(\Phi(\alpha)\big) &= d\big[i_\tau(\Phi(\alpha))\big] + i_\tau\big(d[\Phi(\alpha)]\big) = d\big[i_\tau\Phi(\alpha)\big] + i_\tau\Phi[d\alpha]\\ &= d\big[K(\alpha)\big] + K(d\alpha).\end{aligned}$$

Hence, $d[K(\alpha)] + K(d\alpha) = \widehat{1}^*(\alpha) - \widehat{0}^*(\alpha)$, that is, $K \circ d + d \circ K = \widehat{1}^* - \widehat{0}^*$. Now, because the contraction operation and the map Φ are smooth (Subsections 2.3 and 5.1), the chain-homotopy operator K is a smooth linear map. The explicit evaluation on the plot P is a direct application of the definitions. ■

5.7 The Chain-Homotopy Operator for $p = 1$

Let X be a diffeological space. Let $\mathrm{Paths}(X) = \mathcal{C}^\infty(\mathbf{R}, X)$ be the space of paths of X equipped with the functional diffeology. For every 1-form $\alpha \in \Omega^1(X)$, $F_\alpha = K(\alpha)$ belongs to $\Omega^0(\mathrm{Paths}(X)) = \mathcal{C}^\infty(\mathrm{Paths}(X), \mathbf{R})$; precisely,

$$K\colon \Omega^1(X) \to \mathcal{C}^\infty(\mathrm{Paths}(X), \mathbf{R}) \quad \text{and} \quad K(\alpha) = F_\alpha = \Big[\gamma \mapsto \int_\gamma \alpha\Big].$$

The function $F_\alpha = K(\alpha)$ can be extended, by linearity, over the whole space $C_1(X)$ of 1-chains of X, for all $\sum_\gamma n_\gamma \gamma \in C_1(X)$,

$$F_\alpha\Big(\sum_\gamma n_\gamma \gamma\Big) = \sum_\gamma n_\gamma F_\alpha(\gamma) = \sum_\gamma n_\gamma \int_\gamma \alpha.$$

And thus, for $p = 1$, the chain-homotopy operator is just the pairing of 1-forms over 1-chains. Moreover, if γ and γ' are two paths such that the juxtaposition $\gamma \vee \gamma'$ is defined, then

$$F_\alpha(\gamma \vee \gamma') = F_\alpha(\gamma + \gamma') = F_\alpha(\gamma) + F_\alpha(\gamma'). \tag{5.5}$$

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Note that if the juxtaposition of γ and γ' is not a path, then the smashed juxtaposition $\gamma \star \gamma' = \gamma^\star \vee \gamma'$ is a path [Piz05] and also satisfies

$$F_\alpha(\gamma \star \gamma') = F_\alpha(\gamma + \gamma') = F_\alpha(\gamma) + F_\alpha(\gamma'). \tag{5.6}$$

In other words, $F_\alpha = K(\alpha)$ is a morphism from $(\mathrm{Paths}(X), \star)$ to $(\mathbf{R}, +)$.

Proof If $\gamma \vee \gamma'$ is a path of X, the identity (5.5) is just the additivity of the integral

$$\begin{aligned} F_\alpha(\gamma \vee \gamma') &= \int_0^1 \alpha(\gamma \vee \gamma')(t)\,\mathrm{dt} \\ &= \int_0^{1/2} \alpha(s \mapsto \gamma(2s))(t)\,\mathrm{dt} + \int_{1/2}^1 \alpha\big(s \mapsto \gamma'(2s-1)\big)(t)\,\mathrm{dt}\,. \end{aligned}$$

And, after a suitable change of variable, we get

$$F_\alpha(\gamma \vee \gamma') = \int_0^1 \alpha(\gamma)(t)\,\mathrm{dt} + \int_0^1 \alpha(\gamma')(t)\,\mathrm{dt} = F_\alpha(\gamma) + F_\alpha(\gamma').$$

Now, if we need to smash the paths γ and γ', the proof is the same. It is just the formula of change of variable of the integrand

$$\int_a^b f(\varphi(t))\varphi'(t)\,\mathrm{dt} = \int_{\varphi(a)}^{\varphi(b)} f(t)\,\mathrm{dt},$$

applied to the paths $\gamma^\star = \gamma \circ \lambda$ and $\gamma'^\star = \gamma' \circ \lambda$, where λ is the smashing function described in [Piz05]. ■

5.8 The Chain-Homotopy Operator for Manifolds

Let M be a manifold of finite dimension. The chain-homotopy operator K (Subsection 5.6) of M can be expressed using tangent spaces. Let $P\colon U \to \mathrm{Paths}(M)$ be a n-plot. Let us denote, for any $r \in U$ and for any vector $\delta_i r \in \mathbf{R}^n$,

$$\gamma_r = P(r)\colon t \mapsto \gamma_r(t) \quad \text{and} \quad \delta\gamma_r\colon t \mapsto D[r \to \gamma_r(t)](r)(\delta r).$$

Let $\alpha \in \Omega^p(M)$, the real $K\alpha(P)(r)(\delta_2 r)\cdots(\delta_p r)$ can be interpreted as $K\alpha$, computed at the point γ_r and applied to the $p-1$ *variations* $\delta_i\gamma_r$ associated with the $p-1$ vectors $\delta_i r$. It can be written explicitly as

$$K\alpha_{\gamma_r}(\delta_2\gamma_r)\cdots(\delta_p\gamma_r) = \int_0^1 \alpha_{\gamma_r(t)}\Big(\frac{d\gamma_r(t)}{dt}\Big)\big(\delta_2\gamma_r(t)\big)\,\cdots\big(\delta_p\gamma_r(t)\big)\,\mathrm{dt}\,.$$

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5.9 Variance of the Chain-Homotopy Operator

Let X and X' be two diffeological spaces. Let $f\colon X \to X'$ be a smooth map. Let us use the notations of the proposition (Subsection 5.3) and let K_X and $K_{X'}$ be the two chain-homotopy operators of X and X'. Then the variance of the chain-homotopy operators is given by

$$K_X \circ f^* = f_{\mathcal{P}}^* \circ K_{X'},$$

where $f_{\mathcal{P}}$ has been defined in subsection 5.3, as the action of the function f on paths. This is summarized by the following commutative diagram:

$$\begin{array}{ccc} \Omega^p(X') & \xrightarrow{\;K_{X'}\;} & \Omega^{p-1}(\mathrm{Paths}(X')) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle f_{\mathcal{P}}^*} \\ \Omega^p(X) & \xrightarrow[\;K_X\;]{} & \Omega^{p-1}(\mathrm{Paths}(X)) \end{array}$$

Proof Let us denote by τ and τ' the action of $(\mathbf{R}, +)$ on $\mathrm{Paths}(X)$ and $\mathrm{Paths}(X')$ defined in (Subsection 5.4). Let $\alpha \in \Omega^p(X')$. By definition

$$K_X\big(f^*(\alpha)\big) = (i_\tau \circ \Phi_X)(f^*(\alpha)) = i_\tau\big(\Phi_X(f^*(\alpha))\big),$$

but $\Phi_X \circ f^* = f_{\mathcal{P}}^* \circ \Phi_{X'}$ (Subsection 5.3), thus

$$K_X(f^*(\alpha)) = i_\tau(\Phi_X \circ f^*(\alpha)) = i_\tau(f_{\mathcal{P}}^* \circ \Phi_{X'}(\alpha)) = (i_\tau \circ f_{\mathcal{P}}^*)(\Phi_{X'}(\alpha)).$$

Now, thanks to Subsection 5.5, we have $i_\tau \circ f_{\mathcal{P}}^* = f_{\mathcal{P}}^* \circ i_{\tau'}$. Hence,

$$K_X(f^*(\alpha)) = (f_{\mathcal{P}}^* \circ i_{\tau'})(\Phi_{X'}(\alpha)) = f_{\mathcal{P}}^*(i_{\tau'} \circ \Phi_{X'}(\alpha)) = f_{\mathcal{P}}^*(K_{X'}(\alpha)).$$

Therefore, $K_X \circ f^* = f_{\mathcal{P}}^* \circ K_{X'}$. ∎

5.10 Chain-Homotopy Preserves Invariance

Let X be a diffeological space and let α be a differential p-form on X, $p \geq 1$. Let us denote by $\mathrm{Diff}(X, \alpha)$ the group of diffeomorphisms of X preserving α,

$$\mathrm{Diff}(X, \alpha) = \{f \in \mathrm{Diff}(X) \mid f^*(\alpha) = \alpha\}.$$

As an application of the proposition above (Subsection 5.9), we get

$$f \in \mathrm{Diff}(X, \alpha) \quad \Rightarrow \quad f_{\mathcal{P}} \in \mathrm{Diff}(\mathrm{Paths}(X), K\alpha).$$

In other words, if a diffeomorphism f of X preserves α, the action $f_{\mathcal{P}}$ of f on $\mathrm{Paths}(X)$ preserves $K\alpha$. This property is used to construct the moment maps in diffeology [Piz07].

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6 Homotopy and Differential Forms, Poincaré's Lemma

One crucial property of the De Rham cohomology in the general framework of diffeology, is its homotopic invariance, proved in this section. That is, the mapping induced on the De Rham cohomology by the pullback by a smooth map depends only of the homotopy class of the map. The equivalent theorem in classical differential geometry, for the De Rham cohomology of finite dimensional manifolds, can be regarded as a specialization of this general diffeological result. The transition through the space of paths, which is impossible in the restricted category of manifolds, offers a great simplification of this theorem even for the case of manifolds only. The main tool used in this section is the chain-homotopy operator defined in the previous section (Subsection 5.6). This proves that this invariance dwells deep in the general diffeological structure.

6.1 Homotopic Invariance of the De Rham Cohomology

Let X and X' be two diffeological spaces. Let $s \mapsto f_s$ be a smooth path in $\mathcal{C}^\infty(X, X')$. For all integers $p \geq 1$, for all $\alpha' \in \Omega^p(X')$ such that $d\alpha' = 0$, $f_0^*(\alpha')$ and $f_1^*(\alpha')$ are cohomologous. The case where $p = 0$ is trivial:

$$\alpha' \in \Omega^p(X') \text{ and } d\alpha' = 0 \;\Rightarrow\; f_1^*(\alpha') = f_0^*(\alpha') + d\beta, \text{ with } \beta \in \Omega^{p-1}(X)).$$

Denoting by $f_{\mathrm{dR}}^* \colon \mathrm{H}_{\mathrm{dR}}^\star(X') \to \mathrm{H}_{\mathrm{dR}}^\star(X)$, the action of $f \in \mathcal{C}^\infty(X, X')$ on the De Rham cohomology, the above statement means that f_{dR}^* depends only on the homotopy class of f in $\mathcal{C}^\infty(X, X')$.

Proof Let us consider the map $\varphi \colon X \to \mathcal{C}^\infty(X, \mathrm{Paths}(X'))$:

$$\varphi \colon x \mapsto \left[s \mapsto f_s(x)\right], \quad \text{and} \quad \varphi \in \mathcal{C}^\infty\big(X, \mathrm{Paths}(X')\big)$$

for the functional diffeology. The pullback by φ of the identity satisfied by the chain-homotopy operator $K \circ d + d \circ K = \hat{1}^* - \hat{0}^*$ (Subsection 5.6) gives

$$\varphi^*(Kd\alpha + dK\alpha) = \varphi^*\left(\hat{1}^*(\alpha) - \hat{0}^*(\alpha)\right).$$

Using the hypothesis $d\alpha = 0$ and the commutativity between exterior differential and pullback (Subsection 1.4), we have on the one hand $\varphi^*(Kd\alpha + dK\alpha) = \varphi^*(dK\alpha) = d(\varphi^*(K\alpha))$, and on the other hand

$$\begin{aligned}\varphi^* \circ (\hat{1}^* - \hat{0}^*)(\alpha) &= \varphi^* \circ \hat{1}^*(\alpha) - \varphi^* \circ \hat{0}^*(\alpha) = (\hat{1} \circ \varphi)^*(\alpha) - (\hat{0} \circ \varphi)^*(\alpha) \\ &= f_1^*(\alpha) - f_0^*(\alpha).\end{aligned}$$

Therefore, $f_1^*(\alpha) = f_0^*(\alpha) + d\,\beta$, with $\beta = \varphi^*(K\alpha)$. ∎

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6.2 Closed Forms on Contractible Spaces are Exact

As a corollary of the above proposition (Subsection 6.1), we deduce that every closed form on a contractible diffeological space X is exact, where *contractible* means (smoothly) homotopy equivalent to a point. This is the diffeological variant of a classic theorem due to Poincaré.

Proof Let ρ be a *deformation retraction* from a diffeological space X to a point x_0. That is,

$$\rho \in \mathrm{Paths}(\mathcal{C}^\infty(X,X)) \quad \text{such that} \quad \rho(0) = [x \mapsto x_0] \quad \text{and} \quad \rho(1) = \mathbf{1}_X.$$

With the above notations (Subsection 6.1), for any closed p-form α:

$$\rho(1)^*(\alpha) = \rho(0)^*(\alpha) + d\beta,$$

but $\rho(1)^*(\alpha) = \alpha$ and $\rho(0)^*(\alpha) = [x \mapsto x_0]^*(\alpha) = 0$, and thus $\alpha = d\,\beta$. ■

6.3 Closed Forms on Centered Paths Spaces

Let X be a diffeological space and $x_0 \in X$ be some point. Let $\mathrm{Paths}(X, x_0, \star)$ be the subspace of paths in X, *centered at* x_0, that is, the subspace of paths γ in X such that $\gamma(0) = x_0$. This space is contractible, the map $\rho = s \mapsto [\gamma \mapsto [\gamma_s\colon t \mapsto \gamma(st)]]$ is a deformation retraction from $\mathrm{Paths}(X, x_0, \star)$ to the constant path $[t \mapsto x_0]$. Therefore, any closed form on $\mathrm{Paths}(X, x_0, \star)$ is exact (Subsection 6.2). That is, $\mathrm{H}^0_{\mathrm{dR}}(\mathrm{Paths}(X, x_0, \star)) = \mathbf{R}$ and $\mathrm{H}^p_{\mathrm{dR}}(\mathrm{Paths}(X, x_0, \star)) = 0$, for all $p > 0$. This fact is used to construct geometrical realizations (as diffeological fiber bundles with connections) of closed 1-forms and 2-forms; see [Piz05].

6.4 The Poincaré Lemma

Let X be a diffeological space. As a corollary of (Subsection 6.2), if X is locally contractible [Piz05], then any closed p-form α, $p > 0$, is locally exact. That is, for each point $x \in X$ there exists a D-open neighborhood U of x, and a $(p-1)$-form β, defined on U, such that $\alpha \restriction U = d\beta$. This proposition extends Poincaré's lemma about integration of closed forms on star-shaped domains. It applies to a lot of diffeological spaces, in particular it applies to some diffeological manifolds [Piz06-b].

Note As a corollary, since manifolds are locally diffeomorphic to $\mathbf{R}^n$, they are locally contractible, and closed forms are locally exact, as we know. But this is no longer the case in diffeology in general, for example, the irrational tori [Piz05] where every form is closed but not exact or locally exact.

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