Notes

99.27 A relationship between Pell numbers and triangular square numbers

It was shown in [1] that the nth triangular square number is

$$u_n = \frac{1}{32}\left[\left(17 + 12\sqrt{2}\right)^n + \left(17 - 12\sqrt{2}\right)^n - 2\right] \qquad (1)$$

for $n \geqslant 0$. The first six triangular square numbers are 0, 1, 36, 1225, 41616, 1413721. The Pell numbers are defined recursively by $P_0 = 0$, $P_1 = 1$ and $P_{n+2} = 2P_{n+1} + P_n$ for all $n \geqslant 0$. The first ten Pell numbers are 0, 1, 2, 5, 12, 29, 70, 169, 408, 985. The purpose of this note is to show a relationship between the Pell numbers and the triangular square numbers.

The Binet formula for P_n is

$$P_n = \frac{1}{2\sqrt{2}}\left[\left(1 + \sqrt{2}\right)^n - \left(1 - \sqrt{2}\right)^n\right]$$

for $n \geqslant 0$. Since $\left(1 + \sqrt{2}\right)^4 = 17 + 12\sqrt{2}$ and $\left(1 - \sqrt{2}\right)^4 = 17 - 12\sqrt{2}$, equation (1) can be written as

$$\begin{aligned} u_n &= \frac{1}{32}\left[\left(1 + \sqrt{2}\right)^{4n} + \left(1 - \sqrt{2}\right)^{4n} - 2\right] \\ &= \frac{1}{4}\left(\frac{1}{8}\right)\left[\left(1 + \sqrt{2}\right)^{2n} - \left(1 - \sqrt{2}\right)^{2n}\right]^2 \\ &= \frac{1}{4}\left\{\frac{1}{2\sqrt{2}}\left[\left(1 + \sqrt{2}\right)^{2n} - \left(1 - \sqrt{2}\right)^{2n}\right]\right\}^2 \\ &= \frac{1}{4}P_{2n}^2. \end{aligned}$$

Reference

1. Problem E 954, *Amer. Math. Monthly*, **58** (1951), p. 568.

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99.28 A generalisation of an intriguing ratio

It is interesting to note how a subtle interplay between numbers can lead to visually appealing results. In [1], the following pattern is studied, in various bases:

$$\frac{9876543\mathbf{12}}{123456789} = 8, \text{ in base 10;}$$

$$\frac{543\mathbf{12}}{12345} = 4, \text{ in base 6.}$$