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NORMS OF POWERS IN THE VOLTERRA ALGEBRA

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We prove a conjecture of Willis saying that for each f in the Volterra algebra the sequence $(\|f^n\|)_n$ is a weight sequence which is regulated at 1.

Let $(w_n)_{n\in\mathbb{N}_0}$ be a sequence of reals with the following property:

(i) $w_n > 0 \quad (n \in \mathbb{N}_0)$,
(ii) $w_0 = 1$,
(iii) $w_{n+m} \leqslant w_n w_m$,
(iv) $w_n^{1/n} \to 0$ as $n \to \infty$.

Such a sequence is called a *weight* (sequence). For a weight $w = (w_n)_{n\in\mathbb{N}}$ we denote by $l^1(w)$ the Banach space

$$\Big\{x \in C(\mathbb{N}_0) \Big| \, \|x\| = \sum |x(n)|\, w_n < \infty\Big\}.$$

Then $l^1(w)$ is a radical Banach algebra with adjoint unit under the usual convolution.

Let V be the Volterra algebra $(L_1([0,1]), *)$. For $f \in V$ we set $\alpha(f) = \inf \operatorname{supp}(f) > 0$. Let V^0 be given by $\{f \in V \mid \alpha(f) = 0\}$ and let V_+^0 be given by $\{h \in V^0 \mid h \geqslant 0 \text{ almost everywhere}\}$.

For $f \in V$, we construct a sequence $w_f = (w_n)_{n\in\mathbb{N}_0}$ by $w_0 = 1$; $w_n = \|f^n\|$ $(n \in \mathbb{N})$. Clearly w_f satisfies (ii)–(iv) for all $f \in V$, and by Titchmarsh's convolution theorem w_f satisfies (i) if and only if $\alpha(f) = 0$.

DEFINITION: A weight sequence $w = (w_n)_{n\in\mathbb{N}}$ is called *star shaped* if $\|w_n\|^{1/n} \searrow 0$ monotonically. The sequence $w = (w_n)_{n\in\mathbb{N}}$ is said to be *regulated* at $p \in \mathbb{N}$ if $w_{n+p}/w_n \to 0$ as $n \to \infty$.

The star shaped weights are essentially the class of weights such that all closed ideals in $l^1(w)$ are standard [4].

In this note we investigate the rate of decrease for the sequence w_f for $f \in V^0$. In [1] Allan showed that w_f is *star shaped* for each $f \in V_+^0$. It is easy to see that if w

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is star shaped then w is regulated at 1; and if w is regulated at some $p \in \mathbb{N}$ then w is regulated at every $q \geqslant p$. Thus w_f is regulated at 1 for each $f \in V_+^0$. In [2] Willis proved that for every $f \in V^0$ either

(i) w_f regulated at 1 or
(ii) w_f is not regulated at any $p \geqslant 1$.

It was conjectured in [2] that w_f is regulated at 1 for every $f \in V^0$; that is, condition (ii) above never occurs. We prove that this is indeed true.

Proposition. *The weight* $w_f = (w_n)_{n \in \mathbb{N}}$ *where*

$$w_n = \begin{cases} 1 & \text{if } n = 0 \\ \|f^n\| & \text{if } n > 0 \end{cases}$$

is regulated at 1 *for all* $f \in V^0$.

Proof: For every $f \in V$ the multiplication operator $T_f : V \to V$ given by $T_f(g) = f * g$ is a compact operator. Consider the bounded sequence $x_n = f^n / \|f^n\|$. Since T_f is compact we only have to prove that every convergent subsequence of

$$y_n = f^{n+1} / \|f^n\| = T_f(x_n)$$

tends to zero.

Let (y_{n_k}) be a subsequence of (y_n) converging to $g \in V$, say. Consider the map $D : V \to V$ given by $D(h)(t) = th(t)$. It is easily seen that D is a bounded derivation, and hence for every $n \in \mathbb{N}$ we have $D(f^n) = nf^{n-1} * D(f)$. So for each $k \in \mathbb{N}$

$$\begin{aligned} f * D(y_{n_k}) &= f * D\big(f^{n_k+1} / \|f^{n_k}\|\big) = (n_k + 1) \frac{f^{n_k+1}}{\|f^{n_k}\|} * D(f) \\ &= (n_k + 1) y_{n_k} * D(f). \end{aligned}$$

Since D is bounded

$$(n_k + 1) y_{n_k} * D(f) \to f * D(g)$$

and we also have

$$y_{n_k} * D(f) \to g * D(f).$$

It follows that $y_{n_k} * D(f) \to 0$ as $k \to \infty$, and hence $g * D(f) = 0$. Since $\alpha(D(f)) = \alpha(f) = 0$ we conclude by Titchmarsh's convolution theorem that $g = 0$. This completes the proof. ∎

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