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RELIABILITY ANALYSIS OF A SIMPLE REPAIRABLE SYSTEM

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Abstract

We consider a new kind of simple repairable system consisting of a repairman with multiple delayed-vacation strategy. A common technique in reliability studies is to substitute the steady-state reliability indexes for instantaneous ones because the dynamic solution of the system is difficult or even impossible to obtain. However, this substitution is not always valid. Therefore, it is important to study the existence, uniqueness and expression for the system's dynamic solution, and to discuss the system's stability. The purpose of this paper is threefold: to study the uniqueness and existence of the dynamic solution, and its expression, using C_0-semigroup theory; to discuss the exponential stability of the system by analysing the spectral distribution and quasi-compactness of the system operator; to derive some reliability indexes of the system from an eigenfunction point of view, which is different from the traditional Laplace transform technique, and present a profit analysis to determine the optimal vacation time in order to achieve the maximum system profit.

2000 *Mathematics subject classification*: primary 93D20; secondary 93E20.

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1. Introduction

Reliability is an important concept at the planning, design and operation stages of various complex systems. The need for obtaining highly reliable systems has become increasingly important with the development of modern technologies. The simple repairable system (namely a repairable system consisting of one component and a single repairman) is a basic and important topic studied in reliability and has been

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given much attention in previous research. Cao and Wu [4] and Guo and Cao [7] studied the system with multiple states under different assumptions and obtained some reliability quantities using renewal theory. Tang [18] considered the system with reliability problems such as aged reliability, remaining reliability and total reliability. Utkin and Gurov [20] studied the steady-state reliability of the system by combining probability and possibility assumptions. Barlow and Hunter [1] presented a minimal repair model. The system failure rate is not affected by performing a minimal repair. Brown and Proschan [2] studied an imperfect repair model with probability p of a perfect repair and probability $1 - p$ of a minimal repair. Gopalan and Murulidhar [6] presented a cost analysis of the system subject to online preventive maintenance or repair (or both). The explicit expression for the expected total cost incurred on the system in a specified time interval was obtained via the regeneration point technique. Lam [14] introduced a geometrical process to model the system, in which two kinds of replacement policies, T (the system is replaced whenever its working age reaches T) and N (the system is replaced at the time of the Nth failure since the last replacement), were studied. Zhang [22] considered a bivariate replacement policy (T, N) and showed that it is better than the univariate policies T and N. Wang and Zhang [21] studied the system with preventive repair and failure repair. They determined an optimal mixed policy $(R, N)^*$ (the system is preventively repaired when its reliability drops to the critical reliability R and replaced at the time of the Nth failure) such that the long-run average cost per unit time is minimized. Liu and Tang [15] presented a new model of the system, in which the concept of a repairman with multiple delayed-vacation strategy was introduced. They obtained some primary reliability indexes by using the supplementary variables approach and generalized Markov process method.

In classical repairable systems, it is assumed that the repairman or server remains idle until a component fails. In such situations, there is a waste of valuable resources because the cost incurred in hiring a skilled repairman is high. To save time and cost, it is economical to assign alternative jobs such as desk work or maintenance work to an idle repairman. In practice, the manager of a system would usually consider whether or not the repairman should be encouraged to rest or do other work in his idle time in order to reduce cost or increase the total profit of the system. This introduces the problems of how the reliability indexes will be affected in this situation and how the repairman vacation time or extra work time outside of the system should be suitably restricted to increase the total system profit but not affect his primary work. Because the study of repairable systems with server vacation (which was originally applied in queuing theory [13, 23]) is very important in both theory and practice, Jain et al. [11], Ke and Wang [12] and Liu and Tang [15] introduced the "server vacation model" and discussed some important reliability indexes.

One thing to note is that in traditional reliability research, it is well known that the dynamic solution of a system is difficult or even impossible to obtain. Therefore, it is common to substitute the steady-state indexes of a system for the dynamic ones. In particular, it is usual to replace the instantaneous availability with the steady-state availability due to the importance of the latter. However, this replacement is not always

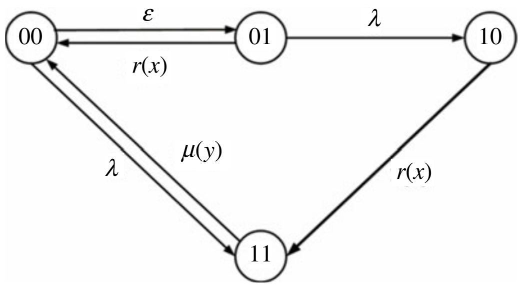

Figure 1. State transition for the system.

valid. To explain why, we now consider our model from this paper with different repair rates as an example.

Figure 1 shows the state transition of the model. Here 00 denotes the state in which the system is working and the repairman is preparing for vacation, 01 the state in which the system is working and the repairman is on vacation, 10 the state in which the system has failed and the repairman is on vacation, and 11 the state in which the repairman is repairing the unit. Moreover, ϵ is the constant delayed-vacation rate of the repairman, λ the constant failure rate of the unit, $r(x)$ the time-dependent vacation rate of the repairman, and $\mu(y)$ the time-dependent failure rate of the unit. Consider the following four scenarios.

(I) Let $\varepsilon = 1$, $r(x) \equiv 1$, $\lambda = 0.5$ and

$$\mu(y) = \begin{cases} 0.4 & \text{for } y \in [0, 2), \\ \mu_0 & \text{for } y \in [2, \infty). \end{cases}$$

The system's instantaneous availability $A(t)$ is shown in Figure 2(a) for $\mu_0 = 0.4, 0.3, 1$ by the curves labelled (1), (2), (3), respectively.

(II) Let $\varepsilon = 1$, $r(x) \equiv 1$, $\lambda = 0.5$ and

$$\mu(y) = \begin{cases} 0.2 & \text{for } y \in [0, 2.3), \\ 1.5 & \text{for } y \in [2.3, 3.3), \\ \mu_0 & \text{for } y \in [3.3, \infty). \end{cases}$$

The system's instantaneous availability $A(t)$ is shown in Figure 2(b) for $\mu_0 = 0.4, 0.2, 0.6$ by curves (1), (2), (3), respectively.

(III) Let $\varepsilon = 1$, $r(x) \equiv 1$, $\lambda = 0.5$ and

$$\mu(y) = \begin{cases} 0.5 & \text{for } y \in [0, 3), \\ 2.0 & \text{for } y \in [3, 4), \\ 0.2 & \text{for } y \in [4, 5.5), \\ 2.0 & \text{for } y \in [5.5, 7), \\ \mu_0 & \text{for } y \in [7, \infty). \end{cases}$$

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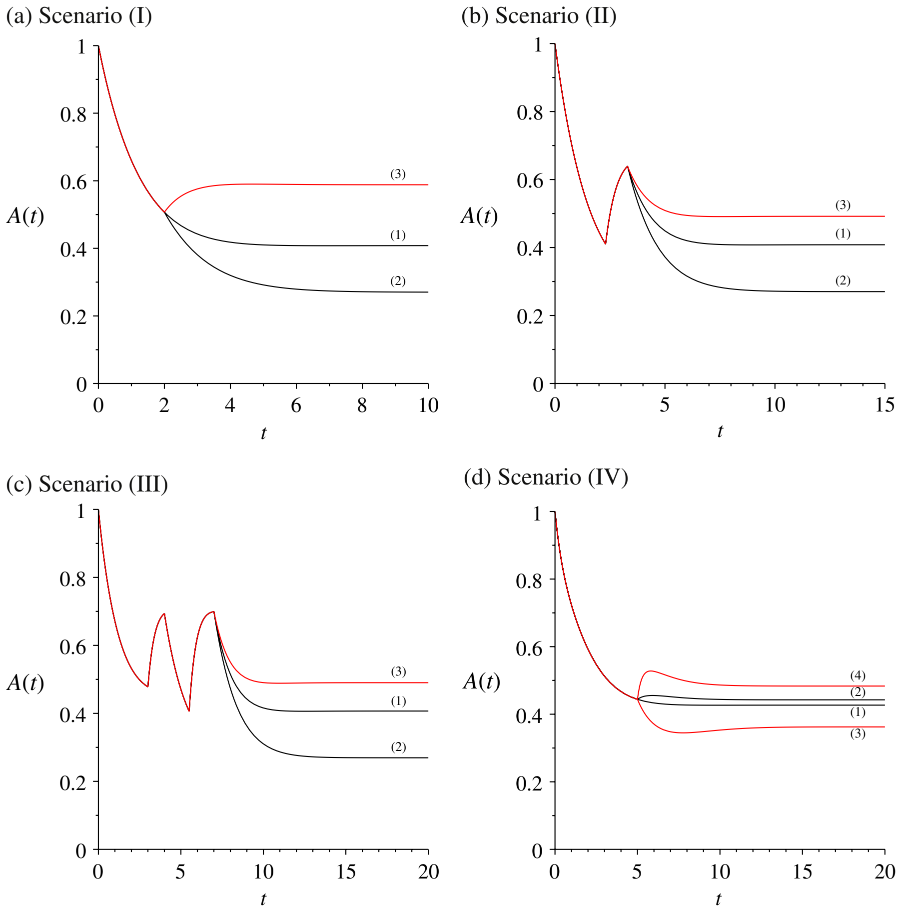

FIGURE 2. Instantaneous availability of the system with different failure rates.

The system's instantaneous availability $A(t)$ is shown in Figure 2(c) for $\mu_0 = 0.4, 0.2, 0.6$ by curves (1), (2), (3), respectively.

(IV) Let $\varepsilon = 1$, $r(x) \equiv 0.3$, $\lambda = 0.5$ and

$$\mu(y) = \begin{cases} 1.2 & \text{for } y \in [0, 5), \\ \mu_0 & \text{for } y \in [5, \infty). \end{cases}$$

The system's instantaneous availability $A(t)$ is shown in Figure 2(d) for $\mu_0 = 1.2, 1.5, 0.6, 3.5$ by curves (1), (2), (3), (4), respectively.

One can deduce from these examples that when the repair time follows an exponential distribution, the availability is monotonically decreasing, and thereby the steady-state availability can substitute for the instantaneous one. Indeed, it is accepted

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in reliability research but not proved in theory that an exponentially distributed repair time yields a monotonically decreasing availability. However, if the repair time follows an arbitrary distribution then the substitution is not always valid, unless a safety factor (the difference between the steady-state availability $\lim_{t\to\infty} A(t)$ and the secure availability $\inf_{t\geq 0} A(t)$) or a reliable interval of the repair rate (an interval where $\lim_{t\to\infty} A(t) = \inf_{t\geq 0} A(t)$) is considered, which was generally ignored in previous literature.

Moreover, in much of the previous literature on reliability studies, all results were obtained by the method of Laplace transforms and Laplace–Stieltjes transforms of semi-Markov processes, based on two hypotheses. The first is that the system concerned has a unique nonnegative time-dependent solution, and the second is that the system's solution is asymptotically stable. Both of these hypotheses hold if the system is a Markov model (the working time and repair time of the system, as well as other parameters, all follow exponential distributions). However, whether they hold for a non-Markov model remains an open question.

Therefore, it is necessary to study the existence and uniqueness as well as the expression for the dynamic solution of a system to determine the safety factor or reliable interval of the repair rate. It is also necessary to discuss the stability (especially the exponential stability) of the system. These can not only provide a strict theoretical basis for reliability research, but also extend its applications in management, technology, and so on.

This paper is concerned with the simple repairable system proposed by Liu and Tang [15], but under a slightly different assumption. We demonstrate that the system of interest has a unique nonnegative dynamic solution and that it is exponentially stable by using C_0-semigroup theory and by analysing the spectral distribution and quasi-compactness of the system operator. We also discuss some primary reliability indexes of the system and present a profit analysis to determine the optimal vacation time of the repairman in order to obtain the maximum system profit.

The rest of this paper is organised as follows. The system model is presented and translated into an abstract Cauchy problem in Section 2. Existence and uniqueness of the solution is discussed in Section 3, and its exponential stability in Section 4. Some reliability indexes of the system and a profit analysis are discussed in Section 5.

2. System formulation

The system model of interest was proposed by Liu and Tang [15]. The difference between their model and the one in this paper is that here the working time of the unit is assumed to follow an exponential distribution (as opposed to an arbitrary distribution). Our system model can be described specifically as follows.

At the initial time $t = 0$, the unit is new, the system begins to work and the repairman prepares for his vacation. If the unit fails in the delayed vacation period (the period in which the repairman prepares for his vacation), the repairman will deal with it immediately and cancel his vacation preparation. Otherwise, he will leave for a

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vacation after the delayed period. When the vacation ends, the repairman either deals with the failed unit or prepares for another vacation. That is, the vacation strategy is one of multiple delayed vacations. The system state transition plot is presented in Figure 1 and all detailed parameters therein are independent of each other. The system after repair is as good as new.

As shown by Liu and Tang [15], by the supplementary variables technique, the model of the system can be formulated as follows:

$$\Big(\frac{d}{dt} + \varepsilon + \lambda\Big)P_{00}(t) = \int_0^\infty r(x)P_{01}(t, x)\,dx + \int_0^\infty \mu(y)P_{11}(t, y)\,dy, \tag{2.1}$$

$$\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda + r(x)\Big)P_{01}(t, x) = 0, \tag{2.2}$$

$$\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + r(x)\Big)P_{10}(t, x) = \lambda P_{01}(t, x), \tag{2.3}$$

$$\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu(y)\Big)P_{11}(t, y) = 0, \tag{2.4}$$

with boundary conditions

$$P_{01}(t, 0) = \varepsilon P_{00}(t), \quad P_{10}(t, 0) = 0, \quad P_{11}(t, 0) = \lambda P_{00}(t) + \int_0^\infty r(x)P_{10}(t, x)\,dx \tag{2.5}$$

and initial conditions

$$P_{00}(0) = 1, \quad P_{01}(0, x) = P_{10}(0, x) = P_{11}(0, x) = 0. \tag{2.6}$$

Here $P_{00}(t)$ is the probability that the system is in state 00 at time t, $P_{01}(t, x)\,dx$ the probability that the system is in state 01 with elapsed vacation time lying in $[x, x + dx)$ at time t, $P_{10}(t, x)\,dx$ the probability that the system is in state 10 with elapsed vacation time lying in $[x, x + dx)$ at time t, and $P_{11}(t, y)\,dy$ the probability that the system is in state 11 with elapsed repair time lying in $[y, y + dy)$ at time t.

Based on the practical background, we can assume that

$$0 \le \sup_{x\in[0,\infty)} r(x) < \infty, \quad 0 \le \sup_{y\in[0,\infty)} \mu(y) < \infty, \quad \int_0^\infty r(x)\,dx = \int_0^\infty \mu(y)\,dy = \infty$$

and

$$\int_0^T r(x)\,dx < \infty, \quad \int_0^T \mu(y)\,dy < \infty \quad \text{for all } 0 < T < \infty.$$

Furthermore, we know that, in practice, many repairs or services are periodical. For example, a system works in the daytime and is maintained at night. So we can assume (see, for example, Hu et al. [9, 10]) that

$$0 < \lim_{x\to\infty} \frac{1}{x} \int_0^x r(s)\,ds = \hat{r} < \infty, \quad 0 < \lim_{y\to\infty} \frac{1}{y} \int_0^y \mu(s)\,ds = \hat{\mu} < \infty. \tag{2.7}$$

We translate the system equations (2.1)–(2.6) into an abstract Cauchy problem in the Banach space

$$X = \Big\{ P = (P_1, P_2, P_3, P_4)^{\mathrm{T}} \in \mathbb{R} \times (L^1[0,\infty))^3 \;\Big|\; \|P\| = |P_1| + \sum_{i=2}^{4} \|P_i\|_{L^1[0,\infty)} < \infty \Big\}.$$

Define the following operators in X:

$$A = \operatorname{diag}\Big(-\varepsilon - \lambda, -\frac{d}{dx} - \lambda - r(x), -\frac{d}{dx} - r(x), -\frac{d}{dy} - \mu(y) \Big),$$

$$\begin{aligned} D(A) = \Big\{ P \in X \;\Big|\; & \frac{dP_2(x)}{dx}, \frac{dP_3(x)}{dx}, \frac{dP_4(y)}{dy} \in L^1[0,\infty); \\ & P_2(x), P_3(x), P_4(y) \text{ are absolutely continuous functions satisfying} \\ & P_2(0) = \varepsilon P_1, P_3(0) = 0, P_4(0) = \lambda P_1 + \int_0^\infty r(x) P_3(x)\, dx \Big\}; \end{aligned}$$

$$BP = (0, 0, \lambda P_2(x), 0)^{\mathrm{T}}, \quad D(B) = X;$$

$$EP = \Big(\int_0^\infty r(x) P_2(x)\, dx + \int_0^\infty \mu(y) P_4(y)\, dy, 0, 0, 0 \Big)^{\mathrm{T}}, \quad D(E) = X.$$

Then (2.1)–(2.6) can be written as the following abstract Cauchy problem:

$$\begin{cases} \dfrac{dP(t,\cdot,\cdot)}{dt} = (A + B + E) P(t,\cdot), \quad t \geq 0, \\ P(t,\cdot,\cdot) \triangleq (P_{00}(t), P_{01}(t,x), P_{10}(t,x), P_{11}(t,y))^{\mathrm{T}}, \\ P(0,\cdot,\cdot) \triangleq P_0 = (1,0,0,0)^{\mathrm{T}}. \end{cases} \tag{2.8}$$

3. Existence and uniqueness of the solution

In this section, we study the existence and uniqueness of the system's solution by using C_0-semigroup theory. We not only prove the first hypothesis used in traditional reliability research mentioned in Section 1 (that there exists a unique nonnegative time-dependent solution), but also present the expression of the dynamic system solution. We first discuss some properties of the system operator $A + B + E$.

LEMMA 3.1. *$A + B + E$ is a dispersive operator with dense domain.*

PROOF. The fact that the domain $D(A)$ of $A + B + E$ is dense in X can be proved using the method of Gupur et al. [8]. It remains to prove that $A + B + E$ is a dispersive operator. For any $P \in D(A)$, choose

$$\varphi = \Big(\frac{[P_{00}]^+}{P_{00}}, \frac{[P_{01}(x)]^+}{P_{01}(x)}, \frac{[P_{10}(x)]^+}{P_{10}(x)}, \frac{[P_{11}(y)]^+}{P_{11}(y)} \Big)^{\mathrm{T}},$$

where $[f]^+$ denotes the nonnegative part of f. By the boundary conditions, it is not difficult to check that $\langle (A + B + E)P, \varphi \rangle \leq 0$. The result follows. □

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Lemma 3.2. *The set* $\{\gamma \in \mathbb{C} \mid \mathrm{Re}(\gamma) > 0$ *or* $\gamma = ia, a \in \mathbb{R}\backslash\{0\}\}$ *belongs to the resolvent set of* $A + B + E$.

Proof. If $\mathrm{Re}(\gamma) > 0$ or $\gamma = ia$ for some $a \in \mathbb{R}\backslash\{0\}$ then by considering the equation $[\gamma I - (A + B + E)]P = G$ for any $G \in X$ we easily derive that $\gamma I - (A + B + E)$ is surjective. Since $\gamma I - (A + B + E)$ is a closed operator and $D(A)$ is dense, the Banach inverse operator theorem implies that $[\gamma I - (A + B + E)]^{-1}$ exists and is bounded. The result follows. □

Lemma 3.3. *Zero is an eigenvalue of* $A + B + E$, *with algebraic multiplicity one.*

Proof. Consider the equation $(A + B + E)P = 0$. Its solution

$$P = \alpha P^* \triangleq \alpha(P^*_{00}, P^*_{01}(x), P^*_{10}(x), P^*_{11}(y))^{\mathrm{T}}, \quad \alpha \in \mathbb{C},$$

belongs to $D(A)$. Here

$$\begin{aligned} P^*_{01}(x) &= \varepsilon P^*_{00} e^{-\int_0^x (\lambda + r(s))\,ds}, \\ P^*_{10}(x) &= \varepsilon P^*_{00} (1 - e^{-\lambda x}) e^{-\int_0^x r(s)\,ds}, \\ P^*_{11}(y) &= [\lambda + \varepsilon(1 - f)] P^*_{00} e^{-\int_0^y \mu(s)\,ds}, \end{aligned}$$

where $f = \int_0^\infty r(x) e^{-\int_0^x (\lambda + r(s))\,ds}\,dx$ and P^*_{00} is an arbitrary real number. Without loss of generality, we take $P^*_{00} > 0, \alpha > 0$. Then $P > 0$. Hence P is not only the nonnegative eigenfunction of $A + B + E$ corresponding to the zero eigenvalue, but also the nonnegative steady-state solution of the system. To prove that the algebraic multiplicity of the zero eigenvalue in X is one, we only need to prove that its algebraic index is one, because its geometric multiplicity is one. This can be readily obtained by a contradiction argument. □

We now apply the Lumer–Phillips theorem [17, p. 14] and Lemmas 3.1 and 3.2 to obtain the main result of this section.

Theorem 3.4. $A + B + E$ *generates a positive* C_0*-semigroup of contraction* $T(t)$.

Theorem 3.5. *The system* (2.8) *has a unique nonnegative time-dependent solution* $P(t, \cdot, \cdot)$ *which satisfies* $\|P(t, \cdot, \cdot)\| = 1$ *for all* $t \in [0, \infty)$.

Proof. By Theorem 3.4 and Theorem 1.4 of Pazy [17, p. 104], the system (2.8) has a unique nonnegative solution $P(t, \cdot, \cdot)$ given by

$$P(t, \cdot, \cdot) = T(t)P_0 \quad \text{for all } t \in [0, \infty). \tag{3.1}$$

Since $P(t, \cdot, \cdot)$ satisfies equations (2.1)–(2.6), it follows that $d\|P(t, \cdot, \cdot)\|/dt = 0$. Therefore, $\|P(t, \cdot, \cdot)\| = \|T(t)P_0\| = \|P_0\| = 1$, for all $t \in [0, \infty)$. □

Remark. Since $P_0 \notin D(A)$, the solution obtained in (3.1) is the generalized solution. It is also the classical solution when $t > 0$.

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4. Exponential stability of the system

In this section, we study the exponential stability of the system by discussing its quasi-compactness, proving the second hypothesis used in traditional reliability research mentioned in Section 1 (that of asymptotic stability). The operator A also generates a C_0-semigroup $S(t)$. Hence it is sufficient to prove the quasi-compactness of $S(t)$ because of the compactness of E [16, p. 215]. We define two new operators:

$$\overline{A} = \operatorname{diag}\Big(-\varepsilon - \lambda, -\frac{d}{dx} - \lambda - r(x), -\frac{d}{dx} - r(x), -\frac{d}{dy} - \mu(y)\Big),$$
$$D(\overline{A}) = D = \Big\{P \in X \;\Big|\; \frac{dP_{01}(x)}{dx}, \frac{dP_{10}(x)}{dx}, \frac{dP_{11}(y)}{dy} \in L^1[0, \infty);$$
$$P_{01}(x), P_{10}(x), P_{11}(y) \text{ are absolutely continuous functions}\Big\};$$
$$A_0 = \overline{A}\big|_{D(A_0)}, \quad D(A_0) = \{P \in D \mid P_{01}(0) = P_{10}(0) = P_{11}(0) = 0\}.$$

Both $\overline{A}$ and A_0 are closed operators with dense domains in X.

Lemma 4.1. *$A_0 + B$ generates a quasi-compact semigroup $T_0(t)$.*

Proof. Consider the following abstract Cauchy problem:

$$\begin{cases} \dfrac{dP(t,\cdot,\cdot)}{dt} = (A_0 + B)P(t,\cdot,\cdot), \\ P(t,\cdot,\cdot) = (P_{00}(t), P_{01}(t,x), P_{10}(t,x), P_{11}(t,y))^{\mathrm{T}}, \\ P(0,\cdot,\cdot) = \Phi, \end{cases}$$

where $\Phi = (\phi_{00}, \phi_{01}(x), \phi_{10}(x), \phi_{11}(y))^{\mathrm{T}} \in X$. It is not hard to prove that $A_0 + B$ generates a C_0-semigroup $T_0(t)$ satisfying

$$(T_0(t)\Phi)(x,y) = \begin{cases} \begin{pmatrix} \phi_{00} e^{-(\varepsilon+\lambda)t} \\ 0 \\ 0 \\ 0 \end{pmatrix} & x, y < t, \\ \begin{pmatrix} \phi_{00} e^{-(\varepsilon+\lambda)t} \\ \phi_{01}(x-t) e^{-\int_{x-t}^{x} (\lambda + r(s))\,ds} \\ [\phi_{10}(x-t) + \phi_{01}(x-t)(1 - e^{-\lambda t})] e^{-\int_{x-t}^{x} r(s)\,ds} \\ \phi_{11}(y-t) e^{-\int_{y-t}^{y} \mu(s)\,ds} \end{pmatrix} & x, y \geq t. \end{cases}$$

From the assumption (2.7) we derive $\|T_0(t)\| \leq e^{-\min\{\varepsilon+\lambda, \hat{r}-\varepsilon, \hat{\mu}-\varepsilon\}t}$. Therefore, we have $W_{\mathrm{ess}}(A_0 + B) \leq W(A_0 + B) < 0$, where $W(G)$ and $W_{\mathrm{ess}}(G)$ denote respectively the growth bound and essential growth bound of an operator G. □

For $\gamma > 0$, $P \in X$, set

$$(\Phi_\gamma(P))(x,y) = \operatorname{diag}\Big(\int_0^\infty \Gamma(s)P(s)\,ds\Big) E_\gamma(x,y), \quad x, y \geq 0,$$

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where $E_\gamma(x, y) = (0, e^{-\int_0^x(\gamma+\lambda+r(s))\,ds}, e^{-\int_0^x(\gamma+r(s))\,ds}(2 - e^{-\lambda x}), e^{-\int_0^y(\gamma+\mu(s))\,ds})^{\mathrm{T}}$ and

$$\Gamma(x) = \begin{pmatrix} e^{-x} & 0 & 0 & 0 \\ \varepsilon e^{-x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda e^{-x} & 0 & r(x) & 0 \end{pmatrix}.$$

Note that $E_\gamma(x, y) \in \mathrm{Ker}[\gamma I - (\overline{A} + B)]$ and that Φ_γ is a compact operator.

LEMMA 4.2. *$S(t) - T_0(t)$ is a compact operator for any $t \geq 0$.*

PROOF. From the equation $[\gamma I - (A + B)](I + \Phi_\gamma) = \gamma I - (A_0 + B)$, it is readily shown that $S(t) \geq T_0(t)$ for all $t \geq 0$. For $P \in D(A_0)$, set

$$\Psi(s)P = S(t - s)(I + \Phi_\gamma)T_0(s)P, \quad \gamma > 0, 0 \leq s \leq t.$$

Then $\Psi'(s)P = S(t - s)\Phi_\gamma[-\gamma I + (A_0 + B)]T_0(s)P$, and $[\Psi(t) - \Psi(0)]P = \int_0^t \Psi'(s)P\,ds$. We obtain

$$S(t)P - T_0(t)P = -\int_0^t S(t - s)\Phi_\gamma[-\gamma I + (A_0 + B)]T_0(s)P\,ds + \Phi_\gamma T_0(t)P - S(t)\Phi_\gamma P.$$

Note that, by the compactness of Φ_γ, the right-hand side of the above equation is the sum of three compact operators. Hence $S(t) - T_0(t)$ is compact. □

We now present the main results of this section.

THEOREM 4.3. *The C_0-semigroup $T(t)$ generated by $A + B + E$ is quasi-compact.*

PROOF. By Lemmas 4.1 and 4.2, and Proposition 9.20 of Clément et al. [5, p. 231], $W_{\mathrm{ess}}(A + B) \leq W(A_0 + B) < 0$, which shows that $S(t)$ is quasi-compact. It follows that $T(t)$ is compact, by the perturbation theory of compact operators [16, p. 215]. □

THEOREM 4.4. *The time-dependent solution of the system (2.1)–(2.6) strongly converges to its steady-state solution. That is, $\lim_{t\to\infty} P(t, \cdot) = \hat{P}$. Moreover,*

$$\|P(t, \cdot) - \hat{P}\| \leq Me^{-\delta t}$$

for some constants $\delta > 0, M \geq 1$, where $\hat{P} = P^*/\|P^*\|$ with P^* an eigenfunction of $A + B + E$ corresponding to the zero eigenvalue (see Lemma 3.3).

PROOF. Combining Theorem 2.1 of Nagel [16, p. 343] with the results of Section 3, we decompose the C_0-semigroup $T(t)$ generated by $A + B + E$ into $T(t) = \overline{P}_0 + R(t)$, where $\overline{P}_0$ is the residue corresponding to the zero eigenvalue of $A + B + E$, and $\|R(t)\| \leq Me^{-\delta t}$ for suitable constants $\delta > 0, M \geq 1$. Moreover, by Theorem 3.5, the nonnegative solution of the system (2.1)–(2.6) can be expressed as $P(t, \cdot, \cdot) = T(t)P_0$, $t \in [0, \infty)$. From Theorem 12.3 of Taylor and Lay [19, p. 247], we obtain

$$P(t, \cdot, \cdot) = T(t)P_0 = (\overline{P}_0 + R(t))P_0 = \langle P_0, Q^*\rangle\hat{P} + R(t)P_0 = \hat{P} + R(t)P_0,$$

where $Q^* = (1, 1, 1, 1)^{\mathrm{T}}$ is an eigenfunction of the adjoint operator $(A + B + E)^*$ of $A + B + E$ corresponding to the zero eigenvalue. The result follows. □

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5. Reliability indexes and profit analysis

In this section, we first discuss some reliability indexes of the system, and then present a profit analysis to show how the repairman vacation affects the total system profit and how to determine the optimal vacation time to achieve the maximum profit.

THEOREM 5.1. *The steady-state availability of the system is*

$$A_v = \frac{1 + \varepsilon n}{1 + \varepsilon m + \lambda k(1 + \varepsilon n)}, \tag{5.1}$$

where $m = \int_0^\infty e^{-\int_0^x r(s)\,ds}\,dx$, $n = \int_0^\infty e^{-\int_0^x (\lambda + r(s))\,ds}\,dx$, $k = \int_0^\infty e^{-\int_0^y \mu(s)\,ds}\,dy$.

PROOF. The instantaneous availability at time t is $A_v(t) = P_{00}(t) + \int_0^\infty P_{01}(t, x)\,dx$. Taking $t \to \infty$, we obtain the steady-state availability

$$A_v = \frac{P^*_{00} + P^*_{01}}{M} = \frac{1 + \varepsilon n}{1 + \varepsilon m + \lambda k(1 + \varepsilon n)},$$

where

$$\begin{aligned} M = P^*_{00} + P^*_{01} + P^*_{10} + P^*_{11} &\triangleq P^*_{00} + \int_0^\infty P^*_{01}(x)\,dx + \int_0^\infty P^*_{10}(x)\,dx + \int_0^\infty P^*_{11}(y)\,dy \\ &= [1 + \varepsilon m + \lambda k(1 + \varepsilon n)]P^*_{00} \end{aligned}$$

and P^*_{00}, $P^*_{01}(x)$, $P^*_{10}(x)$ and $P^*_{11}(y)$ are as in Lemma 3.3. □

THEOREM 5.2. *The steady-state failure frequency of the system is*

$$W_f = \lambda A_v. \tag{5.2}$$

PROOF. Let

$$P_{01}(t) = \int_0^\infty P_{01}(t, x)\,dx, \quad P_{10}(t) = \int_0^\infty P_{10}(t, x)\,dx, \quad P_{11}(t) = \int_0^\infty P_{11}(t, y)\,dy$$

and

$$r_{01}(t) = \frac{\int_0^\infty r(x)P_{01}(t, x)\,dx}{P_{01}(t)}, \quad r_{10}(t) = \frac{\int_0^\infty r(x)P_{10}(t, x)\,dx}{P_{10}(t)},$$

$$\mu(t) = \frac{\int_0^\infty \mu(y)P_{11}(t, y)\,dy}{P_{11}(t)}.$$

Then the instantaneous failure frequency at time t can be expressed as $W_f(t) = \lambda(P_{00}(t) + P_{01}(t))$ [3, Theorem 6.1.7, p. 195]. Taking $t \to \infty$, we obtain the steady-state failure frequency $W_f = \lambda(P^*_{00} + P^*_{01})/M = \lambda A_v$. □

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THEOREM 5.3. *The steady-state probability of the repairman vacation is*

$$P_v = \frac{\varepsilon m}{1 + \varepsilon m + \lambda k(1 + \varepsilon n)}. \tag{5.3}$$

PROOF. The instantaneous probability of the repairman vacation at time t is $P_v(t) = P_{01}(t) + P_{10}(t)$, where $P_{01}(t)$, $P_{10}(t)$ are as in Theorem 5.2. Taking $t \to \infty$ yields the steady-state probability $P_v = (P_{01}^* + P_{10}^*)/M = \varepsilon m/(1 + \varepsilon m + \lambda k(1 + \varepsilon n))$. □

From the above discussion we see that our steady-state indexes are obtained by using the eigenfunction corresponding to the zero eigenvalue of the system operator. This method is simpler than the traditional Laplace transform method.

From (5.1), we see that the repairman vacation reduces the system availability. One may wonder whether or not it will increase the total system profit. If so, a question arises of how to increase the system profit by restricting the vacation time. Let us discuss the problem in the steady state. That is, we take $\mu(y) \equiv \mu$, $r(x) \equiv r$. Set

$$I(r) = c_1 A_v - c_2 W_f + c_3 P_v, \quad D(r) = c_1(A_v - A_v^0) - c_2(W_f - W_f^0) + c_3(P_v - P_v^0),$$

where

$$A_v^0 = \frac{1}{1 + \lambda k}, \quad W_f^0 = \lambda A_v^0, \quad P_v^0 = 0$$

are the corresponding indexes of the classical simple repairable system without repairman vacation (that is, let $\varepsilon \to 0$ in (5.1)–(5.3)) and c_1, c_2, c_3 represent the income for a working unit per unit time, the loss for a failed unit per unit time, and the income for the repairman vacation per unit time, respectively.

The repairman can take a vacation to increase the total profit $I(r)$ of the system when $D(r) > 0$. We now present a profit analysis to determine the optimal value M_r of r and the long-run expected maximum profit M_I of the system under the condition $D(r) > 0$. Given $\varepsilon = 1$, $c_1 = 50$, $c_2 = 30$, $c_3 = 100$, we discuss the above problem in two cases: fixing the failure rate as $\lambda = 0.05$, and fixing the repair rate as $\mu = 1$.

CASE 1. Let $\lambda = 0.05$. Table 1 shows a profit analysis of the system with ten different repair rates and Figure 3(a) plots three of these, with $\mu = 0.1, 0.9, 6$ (shown in bold in Table 1). Here $M_I = \max_{r \geq 0} I(r)$, $M_r = \{r \geq 0 \mid I(r) = M_I\}$, $L_D = \{r \geq 0 \mid D(r) > 0\}$. From the data we arrive at the following conclusions.

(i) The expected long-run maximum profit M_I of the system increases as the repair rate μ increases.

(ii) The repairman can leave for a vacation to increase the system profit $I(r)$ with a fixed probability. However, to achieve the expected long-run maximum profit M_I, the repairman cannot take a vacation when $\mu \leq 0.1$, and as μ increases, the optimal expected vacation time $1/M_r$ is shortened.

CASE 2. Let $\mu = 1$. Table 2 shows a profit analysis with ten different failure rates and Figure 3(b) plots four of these, with $\lambda = 0.01, 0.09, 0.1, 0.65$ (shown in bold in Table 2). From the data we arrive at the following results.

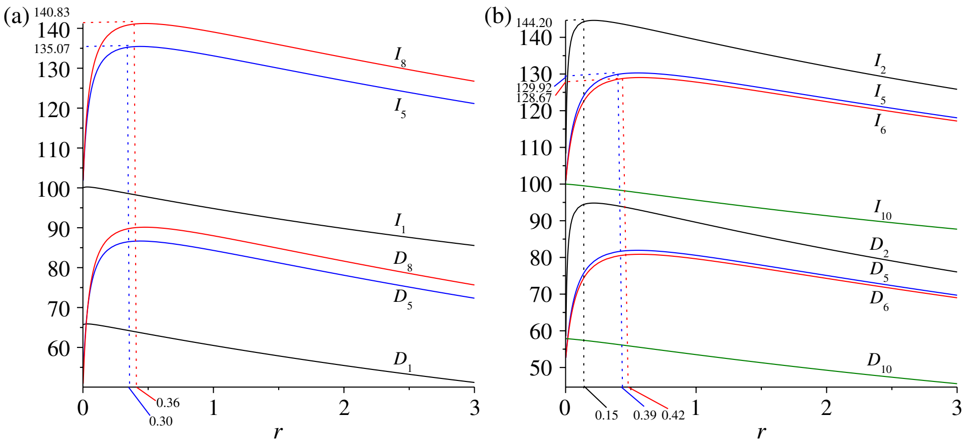

Figure 3. Profit analysis with (a) $\lambda = 0.05$ and $\mu = 0.1, 0.9, 6$; (b) $\mu = 1$ and $\lambda = 0.01, 0.09, 0.1, 0.65$.

Table 1. Profit analysis with $\lambda = 0.05$.

	μ	M_I	M_r	L_D
1	**0.1**	**100.00**	**0**	**[0,** ∞**)**
2	0.2	116.05	0.21	$[0, \infty)$
3	0.5	130.12	0.30	$[0, \infty)$
4	0.7	133.25	0.30	$[0, \infty)$
5	**0.9**	**135.07**	**0.30**	**[0,** ∞**)**
6	2	138.72	0.33	$[0, \infty)$
7	4	140.42	0.36	$[0, \infty)$
8	**6**	**140.83**	**0.36**	**[0,** ∞**)**
9	8	141.25	0.39	$[0, \infty)$
10	10	141.25	0.39	$[0, \infty)$

Table 2. Profit analysis with $\mu = 1$.

	λ	M_I	M_r	L_D
1	0.001	148.04	0	$[0, \infty)$
2	**0.01**	**144.20**	**0.15**	**[0,** ∞**)**
3	0.05	135.71	0.30	$[0, \infty)$
4	0.07	132.64	0.39	$[0, \infty)$
5	**0.09**	**129.92**	**0.39**	**[0,** ∞**)**
6	**0.10**	**128.67**	**0.42**	**[0,** ∞**)**
7	0.20	118.64	0.51	$[0, \infty)$
8	0.40	106.14	0.36	$[0, \infty)$
9	0.60	100.00	0.06	$[0, \infty)$
10	**0.65**	**99.69**	**0**	**[0,** ∞**)**

(i) The expected long-run maximum profit of the system M_I decreases as the failure rate λ increases.

(ii) The repairman can leave for a vacation to increase the system profit $I(r)$ with a fixed probability. However, to achieve the expected long-run maximum profit M_I, the repairman cannot take a vacation when $\lambda \geq 0.65$ or $\lambda \leq 0.001$, and when λ lies in an interval such that the repairman can take a vacation (for example $\lambda \in [0.01, 0.6]$), the optimal expected vacation time $1/M_r$ is shortened when $\lambda \leq 0.2$ and extended when $\lambda > 0.2$.

These are consistent with practical interpretations. By comparing Tables 1 and 2, we see that the influences of the failure rate on both M_I and M_r are larger than those of the repair rate. However, when the system profit reaches the maximum value, the availability does not reach its maximum value. This is shown in Figure 4.

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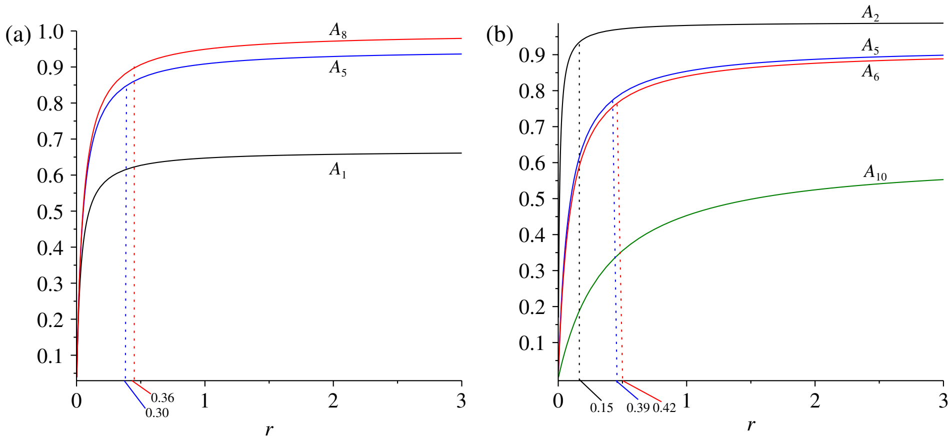

FIGURE 4. System availability with (a) $\lambda = 0.05$ and $\mu = 0.1, 0.9, 6$; (b) $\mu = 1$ and $\lambda = 0.01, 0.09, 0.1, 0.65$.

6. Conclusion

We considered a simple repairable system consisting of a repairman with multiple delayed-vacation strategy as presented by Liu and Tang [15] but under a slightly different assumption. The merit of the present paper lies mainly in the following aspects. Firstly, since the replacement of the instantaneous indexes with the steady-state ones is not valid in general, we proved that a unique solution exists and obtained its expression in order to determine a safety factor or a reliable interval of the repair rate. Secondly, we proved that the solution is exponentially stable. These two results not only provide a strict theoretical basis for reliability research but also extend its applications in management, technology, and so on. Thirdly, based on the two results, we presented some primary reliability indexes of the system from the point of view of eigenfunctions, which is simpler than the traditional Laplace transform technique. Finally, we presented a profit analysis in order to determine the optimal vacation time to achieve the maximum system profit.

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