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ULTRAPOWERS OF ℓ^1-MUNN ALGEBRAS AND THEIR APPLICATION TO SEMIGROUP ALGEBRAS

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Abstract

In this work, we study and investigate the ultrapowers of ℓ^1-Munn algebras. Then we show that the class of ℓ^1-Munn algebras is stable under ultrapowers. Finally, applying this result to semigroup algebras, we show that for a semigroup S, ultra-amenability of $\ell^1(S)$ and amenability of the second dual $\ell^1(S)''$ are equivalent.

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1. Introduction

Given a Banach space E and a nonprincipal ultrafilter $\mathcal{U}$ on a nonempty set I, we can form the ultrapower of E with respect to $\mathcal{U}$, which is denoted by $(E)_{\mathcal{U}}$ and defined as the quotient space

$$(E)_{\mathcal{U}} = \ell^\infty(I, E)/N_{\mathcal{U}},$$

where

$$\ell^\infty(I, E) = \Big\{(x_i)_i \subset A : \sup_i \|x_i\| < \infty\Big\}$$

and

$$N_{\mathcal{U}} = \Big\{(x_i)_i \in \ell^\infty(I, E) : \lim_{i\to\mathcal{U}} \|x_i\| = 0\Big\}.$$

It is routine to write $(x_i)_{\mathcal{U}}$ for the equivalence class it represents. There is a canonical isometry $i_E : E \to (E)_{\mathcal{U}}$ which sends $x \in E$ to the constant family $(x)_{\mathcal{U}}$. It is easily seen that if $(x_i)_{\mathcal{U}}$ represents an equivalence class in $(A)_{\mathcal{U}}$, then

$$\|(x_i)_{\mathcal{U}}\| = \lim_{i\to\mathcal{U}} \|x_i\|.$$

If A is a Banach algebra, $(A)_{\mathcal{U}}$ is also a Banach algebra; in other words, the class of Banach algebras is stable under ultrapowers. In [7], stability of some special classes of Banach algebra such as C*-algebras is investigated. In [2], the ultrapowers of Banach algebras are used to study the bidual of A. Also, in [3], Daws introduces the notion of ultra-amenability for a Banach algebra A. We say A is ultra-amenable if every ultrapower of A is amenable. In [3, 4], the ultra-amenability of C*-algebras is characterised and, for a locally compact group G, the ultra-amenability of $L^1(G)$ is investigated (see also [8]). In particular, it is shown that if G is a discrete group, $\ell^1(G)$ is ultra-amenable if and only if G is finite.

The aim of this paper is to study the concept of ultrapowers for a Banach algebra, which is the so-called ℓ^1-Munn algebras. These algebras were first introduced by Munn [9] to study some semigroup algebras. Then they were generalised by Esslamzadeh [5]. These algebras were then investigated by, for example, Esslamzadeh and Esslamzadeh [6] and Shojaee *et al.* [10], where they study contractibility, weak amenability and Connes amenability of these algebras.

We end this work by classifying the ultra-amenability of the ℓ^1-Munn algebras. Finally, as an application, we show that for a semigroup S, $\ell^1(S)$ is ultra-amenable if and only if $\ell^1(S)''$ is amenable, which is a generalisation of the case where S is a group.

2. ℓ^1-Munn algebra of ultraproduct of Banach algebras

Let A be a unital Banach algebra, let $P = (p_{ji}) \in M_{n\times m}(A)$ with $\max\{\|p_{ji}\| : i \in m, j \in n\} \leq 1$, where $m, n \in \mathbb{N}$. We regard $\mathfrak{A} = M_{m\times n}(A)$ as a Banach algebra by taking the norm to be specified by

$$\|(a_{ij})\| = \sum_{i\in\mathbb{N}_m, j\in\mathbb{N}_n} \|a_{ij}\| \quad (a = (a_{ij})_{ij} \in \mathfrak{A}),$$

and with the product

$$a \circ b = aPb,$$

for all $a, b \in \mathfrak{A}$ in the sense of matrix products. Then $\mathfrak{A}$ is called the ℓ^1-Munn algebra over A with sandwich matrix P, and it is denoted by $\mathcal{M}(A, P, m, n)$. We are interested in determining $(\mathcal{M}(A, P, m, n))_{\mathcal{U}}$, when $\mathcal{U}$ is an ultrafilter.

Theorem 2.1. *Let A be a Banach algebra, $m, n \in \mathbb{N}$, and let $\mathcal{U}$ be an ultrafilter. If $\mathfrak{A} = \mathcal{M}(A, P, m, n)$ is an ℓ^1-Munn algebra over A with sandwich matrix $P = (p_{ji})$, then there exists $Q \in M_{n\times m}((A)_{\mathcal{U}})$ such that*

$$(\mathfrak{A})_{\mathcal{U}} \simeq \mathcal{M}((A)_{\mathcal{U}}, Q, m, n).$$

Proof. Define $Q = i_A(P) \in M_{n\times m}((A)_{\mathcal{U}})$ by $Q_{ji} = i_A(P_{ji})$ for each i, j, and consider the map

$$\Psi : ((\mathcal{M}(A, P, m, n)))_{\mathcal{U}} \to \mathcal{M}((A)_{\mathcal{U}}, Q, m, n)$$

with $\Psi((a_i)_{\mathcal{U}}) = ((a^i_{kl})_{\mathcal{U}})_{kl}$, where $a_i = (a^i_{kl})$ for each i. This definition is easily seen to be independent of the choice of the representation; indeed, Ψ is an isometry.

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For $(a_i) \in \mathfrak{A}$,

$$\begin{aligned}\|\Psi((a_i)_{\mathcal{U}})\| &= \|((a^i_{kl})_{\mathcal{U}})_{kl}\| \\ &= \sum_{k=1}^{m}\sum_{l=1}^{n} \|(a^i_{kl})_{\mathcal{U}}\| \\ &= \sum_{k=1}^{m}\sum_{l=1}^{n} \lim_{i\to\mathcal{U}} \|a^i_{kl}\| \\ &= \lim_{i\to\mathcal{U}} \sum_{k=1}^{m}\sum_{l=1}^{n} \|a^i_{kl}\| \\ &= \lim_{i\to\mathcal{U}} \|a_i\| = \|(a_i)_{\mathcal{U}}\|.\end{aligned}$$

Therefore Ψ is one-to-one. Linearity of Ψ is obvious. Now, for $(a_i)_{\mathcal{U}}, (b_i)_{\mathcal{U}} \in \mathfrak{A}$,

$$((a_i)_{\mathcal{U}} \circ (b_i)_{\mathcal{U}})_{kl} = \sum_{j=1}^{n}\sum_{t=1}^{m} a^i_{kj} P_{jt} b^i_{tl}.$$

So, for $k \in \mathbb{N}_m$ and $l \in \mathbb{N}_n$,

$$\begin{aligned}(\Psi((a_i)_{\mathcal{U}}) \circ (b_i)_{\mathcal{U}})_{kl} &= \bigg(\bigg(\sum_{j=1}^{n}\sum_{t=1}^{m} a^i_{kj} P_{jt} b^i_{tl}\bigg)_{\mathcal{U}}\bigg)_{kl} \\ &= (\Psi((a_i)_{\mathcal{U}}) \circ \Psi((b_i)_{\mathcal{U}}))_{kl}.\end{aligned}$$

This shows that Ψ is a homomorphism. One may check that Ψ is also onto and so is an isomorphism. □

Corollary 2.2. *Let A be a Banach algebra, let $\mathcal{U}$ be an ultrafilter, and $n \in \mathbb{N}$. Then*

$$M_n((A)_{\mathcal{U}}) \simeq (M_n(A))_{\mathcal{U}}.$$

Theorem 2.3. *Let A be a Banach algebra, let $\mathcal{U}$ be an ultrafilter, and let $\mathfrak{A} = \mathcal{M}(A, P, m, n)$ be the ℓ^1-Munn algebra over A with sandwich matrix $P = (p_{ji})$, where $m, n \in \mathbb{N}$. Then the following statements are equivalent.*

(a) *$\mathfrak{A}$ is ultra-amenable.*

(b) *A is ultra-amenable, $m = n$, and P is invertible.*

Proof. (a) ⇒ (b). Since $\mathfrak{A}$ is ultra-amenable, $\mathfrak{A}$ is amenable by [3]. So, $m = n$, P is invertible and $\mathfrak{A} \simeq M_n(A)$ by [1, Proposition 2.16]. Furthermore, by Corollary 2.2, $M_n((A)_{\mathcal{U}}) \simeq (M_n(A))_{\mathcal{U}}$. By our hypothesis, for each $\mathcal{U}$, $(M_n(A))_{\mathcal{U}}$ is amenable. Hence, $M_n((A)_{\mathcal{U}})$ and so $(A)_{\mathcal{U}}$ is amenable for each $\mathcal{U}$ by [1, Proposition 2.7(i)]. This shows that A is ultra-amenable.

Conversely, if $m = n$ and P is invertible, then $\mathfrak{A} \simeq M_n(A)$. Now, by Corollary 2.2, $(\mathfrak{A})_{\mathcal{U}} \simeq M_n((A)_{\mathcal{U}})$. Since $(A)_{\mathcal{U}}$ is amenable, $M_n((A)_{\mathcal{U}})$ is amenable by [1, Proposition 2.7(i)]. Hence $(\mathfrak{A})_{\mathcal{U}}$ is amenable for each $\mathcal{U}$. □

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Before proceeding further, we need to fix some notation. For a semigroup S, $\ell^1_0(S)$ denotes the set of all $f \in \ell^1(S)$ such that $\sum_{s\in S} f(s) = 0$.

Let G be a group, and $G^o = G \cup \{o\}$. Let

$$S = \{(g)_{ij} : g \in G, i \in \mathbb{N}_m, j \in \mathbb{N}_n\} \cup \{o\},$$

where $m, n \in \mathbb{N}$ and $(g)_{ij}$ denotes the element of $M_{m\times n}(G^o)$ with g in the (i, j)th position and o elsewhere, and o is a matrix with 0 everywhere. Let $P = (p_{ji})$ be an $n \times m$ matrix over G^o. Then the set S with the composition

$$(a)_{ij} \circ o = o \circ (a)_{ij} = o \quad \text{and} \quad (a)_{ij} \circ (b)_{lk} = (ap_{jl}b)_{ik}, \quad ((a)_{ij}, (b)_{lk} \in S)$$

is a semigroup which is called a Rees matrix semigroup with a zero over G and will be denoted by $S = \mathcal{M}^o(G, P, m, n)$.

Recall that for $g \in G$, $(g)_{ij}$ is identified with the element of $M_{m\times n}(\ell^1(G))$ which has δ_g in the (i, j)th position and 0 elsewhere, and o is identified with δ_0. Furthermore, we identify $P \in M_{n\times m}(G^o)$ with a matrix $P \in M_{n\times m}(\ell^1(G))$ as follows: if the first matrix P has $g \in G$ in the (i, j)th position, then the new matrix P has the point mass δ_g in the (i, j)th position; if the first matrix P has o in the (i, j)th position, then the new matrix P has 0 in the (i, j)th position. Thus we can write

$$\ell^1(S)/\mathbb{C}\delta_o = \mathcal{M}(\ell^1(G), P, m, n).$$

The product in $\ell^1(S)$ also satisfies $f \star \delta_o = \delta_o \star f = \sum_{s\in S} f(s)\delta_o$, and

$$(f)_{ij} \star (g)_{kl} = \begin{cases} (f \star \delta_{p_{jk}} \star g)_{il} & \text{if } p_{jk} \neq o, \\ \displaystyle\sum_{s,t\in S} f(s)g(t)\delta_o & \text{if } p_{jk} = o \end{cases}$$

for all $f, g \in \ell^1(S)$; $j, l \in \mathbb{N}_n$ and $i, k \in \mathbb{N}_m$. For more details, see [1].

Corollary 2.4. *Let G be a group and $S = \mathcal{M}^0(G, P, m, n)$ be a Rees matrix semigroup with a zero over G, and let $\mathcal{U}$ be an ultrafilter. Then the following are equivalent.*

(a) *$\ell^1(S)$ is ultra-amenable.*

(b) *$\ell^1(G)$ is ultra-amenable, P is invertible and $m = n$.*

(c) *S is finite, and $\ell^1(S)$ is amenable.*

Proof. (a) ⇒ (b). We can identify $(\ell^1(S))_{\mathcal{U}}/\mathbb{C}\delta_o$ with $(\mathcal{M}(\ell^1(G), P, m, n))_{\mathcal{U}}$ (see [1, Corollary 2.3.2]). Furthermore, by Theorem 2.1,

$$(\mathcal{M}(\ell^1(G), P, m, n))_{\mathcal{U}} \simeq \mathcal{M}((\ell^1(G))_{\mathcal{U}}, P, m, n).$$

Now, amenability of $\mathcal{M}((\ell^1(G))_{\mathcal{U}}, P, m, n)$ for each ultrafilter $\mathcal{U}$ implies that $(\ell^1(G))_{\mathcal{U}}$ is amenable, P is invertible and $m = n$ by [5]. So, $\ell^1(G)$ is ultra-amenable, and (b) follows.

(b) ⇒ (c). By [3], ultra-amenability of $\ell^1(G)$ implies that G, and so S is finite.

(c) ⇒ (a). Since S is finite, $\ell^1(S)$ is finite-dimensional, and $\ell^1(S) \simeq (\ell^1(S))_{\mathcal{U}}$ for each ultrafilter $\mathcal{U}$. Now, amenability of $\ell^1(S)$ shows that it is ultra-amenable. □

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Before we present our result on semigroup algebras, we need an easy but useful lemma.

Lemma 2.5. *Let S be a semigroup and let I be an ideal of S. Then $\ell^1_0(I)$ is an ideal of $\ell^1(S)$ and*

$$\ell^1(S/I) \cong \ell^1(S)/\ell^1_0(I).$$

Proof. We can identify $\ell^1(I)$ with the set of all functions in $\ell^1(S)$ such that their support is contained in I. In this way, $\ell^1_0(I)$ is an ideal of $\ell^1(S)$. Furthermore, the natural map $\pi : S \longrightarrow S/I$, extends to an algebra homomorphism

$$\pi : \ell^1(S) \longrightarrow \ell^1(S/I)$$

such that

$$\pi\Big(\sum_{s\in S} \alpha_s\delta_s\Big) = \sum_{s\in S} \alpha_s\delta_{\pi(s)}.$$

We claim that $\ker\pi = \ell^1_0(I)$. Clearly, $\ell^1_0(I) \subseteq \ker\pi$. Now, suppose that $f \in \ker\pi$ and $\operatorname{supp} f \nsubseteq I$. So there exists $s_0 \in S\setminus I$ such that $f(s_0) \neq 0$. Since $f \in \ker\pi$, $\pi(f)(\pi(x)) = 0$ for all $x \in S$. On the other hand, $\pi(f)(\pi(s_0)) = f(s_0) \neq 0$, a contradiction. This shows that $\operatorname{supp} f \subseteq I$; equivalently we may suppose that $f \in \ell^1(I)$. The fact that $f \in \ker\pi$ implies that $f \in \ell^1_0(I)$, and therefore, $\ker\pi \subseteq \ell^1_0(I)$. Hence, $\ker\pi = \ell^1_0(I)$. □

Corollary 2.6. *Let S be a semigroup. Then the following are equivalent.*

(a) *$\ell^1(S)$ is ultra-amenable.*
(b) *S is finite and $\ell^1(S)$ is amenable.*
(c) *$\ell^1(S)''$ is amenable.*

Proof. (a) ⇒ (b). As $\ell^1(S)$ is ultra-amenable, $\ell^1(S)$ is amenable by [3, Corollary 5.5]. By [1, Theorem 10.12], S has the principal series

$$K(S) = S_1 \trianglelefteq S_2 \cdots \trianglelefteq S_n = S,$$

where $K(S)$ is an amenable group, and each quotient S_{i+1}/S_i has the form $\mathcal{M}^o(G, P, n)$ such that G is an amenable group, and P is invertible in $\ell^1(G)$. Also, $\ell^1(K(S))$ and $\ell^1(S_2)$ are closed ideals in $\ell^1(S)$ having bounded approximate identities (in fact identities), and by [3, Proposition 5.2], they are ultra-amenable. Since $K(S)$ is a group, it is finite by [3, Theorem 5.11].

Now consider the semigroup algebra $\ell^1(S_2/S_1)$. By Lemma 2.5, $\ell^1(S_2/S_1) \simeq \ell^1(S_2)/\ell^1_0(S_1)$, and so it is ultra-amenable by [3, Corollary 5.5]. On the other hand, S_2/S_1 is a completely 0-simple semigroup. Hence, ultra-amenability of $\ell^1(S_2/S_1)$ implies that S_2/S_1 is finite by Corollary 2.4. Finiteness of S_2/S_1 and S_1 together shows that S_2 is finite. Iterate this argument to deduce that S is finite.

(b) ⇒ (c). Since S is finite, $\ell^1(S)$ is finite-dimensional and therefore isomorphic with its second dual. So $\ell^1(S)''$ is amenable, as required.

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(c) ⇒ (a). Identifying $\ell^1(S)''$ with $M(\beta S)$, and using [1, Theorem 11.8], it follows that S is finite. So $\ell^1(S)$ is finite-dimensional, and therefore $\ell^1(S) \simeq (\ell^1(S))_{\mathcal{U}}$ for each ultrafilter $\mathcal{U}$. Furthermore, since $\ell^1(S)''$ and $\ell^1(S)$ are isomorphic by finite-dimensionality, amenability of $\ell^1(S)''$ implies that $\ell^1(S)$ is amenable. So $\ell^1(S)$ is ultra-amenable. □

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