



A Short Proof of the Characterization of Model Sets by Almost Automorphy

Jean-Baptiste Aujogue

Abstract. The aim of this note is to provide a conceptually simple demonstration of the fact that repetitive model sets are characterized as the repetitive Meyer sets with an almost automorphic associated dynamical system.

1 Introduction

In this note we will be interested in certain point sets of a locally compact Abelian group \mathbb{G} that are nowadays called *Meyer sets*. Meyer introduced such sets—under the name harmonious sets—in the context of harmonic analysis [14], and observed that some specific examples of Meyer sets, that he called *model sets*, could be produced through an elementary geometric method. More than a decade later and independent of Meyer’s work, the geometric method to produce model sets was reintroduced by several authors in an attempt to answer a problem in solid state physics, viz to produce point sets in \mathbb{R}^2 and \mathbb{R}^3 having a diffraction picture displaying rotational symmetries that cannot hold for fully periodic point sets, as it was first realized by considering the vertex sets of the renowned Penrose tilings of the plane [13]. Currently, many interesting examples of model sets and tilings affiliated with model sets can be found in the literature [4], illustrating the elegance and fruitfulness of Meyer’s method.

It has early been observed that a certain property holds for the dynamical system associated with any model set: It is always an almost 1-to-1 cover of some Kronecker system. This property was first recognized by Robinson [17], who asserted that the dynamical system of a Penrose tiling is a minimal almost 1–1 extension of an \mathbb{R}^2 -action over the torus \mathbb{T}^4 by rotation. This may be explained by the existence of an underlying model set structure for such tilings. Similarly, by a direct use of the model set construction, such a result was obtained by Baake, Hermisson, and Pleasants [5] for the Fibonacci model set of the line, whose dynamical system was shown to be a minimal almost 1–1 extension of an \mathbb{R} -action by rotation over the torus \mathbb{T}^2 . These results were generalized for certain repetitive model sets of a Euclidean space by Forrest, Hunton, and Kellendonk [8], and independently by Schlottmann [19] for arbitrary repetitive model sets of a σ -compact, locally compact Abelian group, where in both works it is shown that the model set construction immediately describes the dynamical system of

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a repetitive model set as a minimal almost 1–1 extension of a Kronecker system. This is the so-called *torus parametrization* of a model set (this terminology going back to [5]). Dynamical systems with this property appeared long before in topological dynamics, in particular, in the work of Veech [21] who termed them *almost automorphic*.

It was thus reasonable to think that almost automorphy may characterize on a dynamical side (repetitive) model sets among the wider class of (repetitive) Meyer sets. Proof of such a characterization came with the work of by Baake, Lenz, and Moody [6] and Lee and Moody [11], who showed using the geometric construction developed in [7] that repetitive model sets satisfying the additional condition of regularity are characterized by a strong form of almost automorphy on their associated dynamical system. Although regularity is a very reasonable condition notably concerning the diffraction theory of point sets [9, 19], it is not necessary for almost automorphy to hold. Thus, since regularity played a crucial role in the arguments of [6, 11] a dynamical characterization that applies for all repetitive (not necessarily regular) model sets remained a challenging open problem. This was solved in [1], where for $\mathbb{G} = \mathbb{R}^d$, the theorem following was proved.

Main Theorem *A repetitive model set is precisely a repetitive Meyer set whose associated dynamical system is almost automorphic.*

The proof set in [1] involves Ellis semigroup and regional proximality for Meyer sets, two objects coming from abstract topological dynamics [3], for which a detailed account in the context of point sets can be found in [2]. It is also detailed in [2] how the result of [6, 11] and the above theorem are part of a natural hierarchical description of repetitive Meyer sets involving their associated dynamical systems.

In this note, we aim to revisit this result by providing a greatly simplified proof, which is shown here to hold for point sets in arbitrary second countable, σ -compact locally compact Abelian groups. Our starting point here is a result of Schlottmann upon characterizing repetitive model sets by an intrinsic condition on the return times of the set [18], that is unexpectedly very close to almost automorphy. The proof we give here consists of a combination of Schlottmann's original result (Theorem 4.1) with a general statement on almost automorphy (Theorem 5.1), and with an intermediate statement connecting the two (Proposition 4.2). In particular, we do not make use of the torus parametrization anywhere in this work.

2 Meyer Sets

In this note we consider a locally compact Abelian (LCA) group \mathbb{G} , which is assumed to be σ -compact and second countable. These assumptions guarantee metrizable of all the involved point set dynamical systems; see Section 3. Second countability is a standard assumption, but is actually not essential here. However, if one gets rid of it, then all statements involving metrics have to be reformulated using uniformities, so we shall keep it for convenience.

A subset Λ of \mathbb{G} is a *Meyer set* whenever it fulfills the following three conditions: First it should be uniformly discrete; that is, there is a neighborhood U of 0 in \mathbb{G} such that no two different points $t, s \in \Lambda$ can yield $t - s \in U$. Second it should be relatively

dense, meaning that there is K compact in \mathbb{G} such that $K + \Lambda = \mathbb{G}$. And third, there should exist a finite subset $F \subset \mathbb{G}$ such that $\Lambda - \Lambda \subseteq \Lambda + F$. This is one of the various ways to define a Meyer set, and the reader will find different yet equivalent formulations in [15] in the case $\mathbb{G} = \mathbb{R}^d$, and [20] in our slightly more general setting.

A trivial example of a Meyer set is a uniformly discrete and relatively dense subgroup Γ , a lattice, of \mathbb{G} , and any sum $\Gamma + F$ with F finite. Much less trivial examples of Meyer sets are the *model sets*, also called *Cut and Project sets*: To construct a model set one begins by considering a product $H \times \mathbb{G}$ with a LCA group H together with a lattice $\Gamma \subset H \times \mathbb{G}$ in it. The triple (H, Γ, \mathbb{G}) is called a *Cut and Project scheme*. Then, choosing a compact subset $W \subseteq H$ that is supposed equal to the closure of its interior, one ends up with the point set

$$\mathfrak{P}(W) := \{t \in \mathbb{G} : \exists(t', t) \in \Gamma, t' \in W\}.$$

One usually supposes that Γ projects in a 1–1 way into \mathbb{G} , but this can be achieved by modding out the discrete subgroup $\Gamma_0 := \Gamma \cap H \times \{0\}$ in H and considering the Cut and Project scheme $(H/\Gamma_0, \Gamma/\Gamma_0, \mathbb{G})$. It is, moreover, often required that the projection of Γ in H has dense image, which can be obtained by taking the closure H' of this range in replacement of the whole H . None of these operations alters the model sets as given above.

Remark 2.1 There are natural generalizations of the definition of model sets. For instance, instead of requiring W to have dense interior we could ask this interior to be only non-empty, or even empty at all [16]. One can also formulate this in terms of the indicator function of the subset W and then replace it by different types of function (continuous [12, 20], Riemann measurable, Baire class 1). Ultimately, one can also drop the commutativity requirement on all involved groups.

In our case, we will be particularly interested in a certain type of model set. A model set $\mathfrak{P}(W)$ is called *non-singular* when there is no point of Γ whose projection in H belongs to the boundary of W . In other terms, a model set is non-singular when it satisfies

$$\mathfrak{P}(\overset{\circ}{W}) = \mathfrak{P}(W)$$

3 The Dynamical System of a Meyer Set

We wish to characterize the Meyer sets that are non-singular model sets in terms of a certain dynamical system canonically associated with each of them. Given a Meyer set Λ , the construction of its dynamical system (X_Λ, \mathbb{G}) goes as follows: We consider the *hull* X_Λ of Λ , consisting of all Meyer sets in which any finite subset appears somewhere in Λ ,

$$X_\Lambda := \{\Lambda' \subset \mathbb{G} : \forall K \subset \mathbb{G} \text{ compact } \exists t \in \mathbb{G}, \Lambda' \cap K = (\Lambda - t) \cap K\}.$$

This set is endowed with a topology that comes from a uniformity (see [10, Chapter 6] for background), for which a basis is given, for an arbitrary compact set K of \mathbb{G} and U any neighborhood of 0 in \mathbb{G} , by the sets

$$\mathcal{U}_{K,U} := \{(\Lambda', \Lambda'') \in X_\Lambda \times X_\Lambda : \exists t, t' \in U, (\Lambda' - t) \cap K = (\Lambda'' - t') \cap K\}.$$

The set X_Λ endowed with the topology coming from this uniformity is a compact Hausdorff space [19], and since \mathbb{G} is assumed to be σ -compact and second countable, this uniformity has a countable basis and is thus metrizable [10, Chapter 6, Theorem 13]. Finally, X_Λ is naturally endowed with a \mathbb{G} -action by $t.\Lambda := \Lambda - t$ and this action is by homeomorphism. By construction, the Meyer set Λ now viewed as an element of X_Λ is a transitive element; that is, its \mathbb{G} -orbit is dense in the dynamical system (X_Λ, \mathbb{G}) .

In this note we will be interested exclusively in Meyer sets that are repetitive: A Meyer set Λ is repetitive whenever for any compact subset K of \mathbb{G} the set

$$\Lambda_K := \{t \in \mathbb{G} : (\Lambda - t) \cap K = \Lambda \cap K\}$$

is relatively dense in \mathbb{G} . It admits an elegant interpretation in terms of the associated dynamical system (X_Λ, \mathbb{G}) as follows. Observe first that for a compatible metric d on X_Λ , for any compact set K and open neighborhood U of 0 in \mathbb{G} there must exist an $\varepsilon > 0$ such that whenever $d(\Lambda, \Lambda') < \varepsilon$ then $(\Lambda, \Lambda') \in \mathcal{U}_{K,U}$, and, conversely, given $\varepsilon > 0$ there exists a pair of a compact set K and an open neighborhood U of 0 in \mathbb{G} such that the reverse implication holds. Now let us introduce the collection of sets given for $\varepsilon > 0$ by

$$P_\varepsilon := \{t \in \mathbb{G} : d(t.\Lambda, \Lambda) < \varepsilon\}.$$

What we can observe is that for any pair of a compact set K and an open neighborhood U of 0 in \mathbb{G} , there exists a $\varepsilon > 0$ such that $P_\varepsilon \subseteq \Lambda_K + U$, and conversely, for any $\varepsilon > 0$, there are K and U as before such that $\Lambda_K + U \subseteq P_\varepsilon$. From this observation we conclude that the Meyer set Λ is repetitive precisely when the P_ε 's are relatively dense in \mathbb{G} , that is, when Λ is an almost periodic point in the dynamical system (X_Λ, \mathbb{G}) . It is a general fact of topological dynamics that this happens if and only if the dynamical system (X_Λ, \mathbb{G}) is minimal.

Thus, when a Meyer set Λ is repetitive, its dynamical system (X_Λ, \mathbb{G}) is equally generated by any other Meyer set in it, in which case we may hide the subscript Λ and simply write (X, \mathbb{G}) , and refer to it as a *minimal system of Meyer sets*. When a minimal system of Meyer sets (X, \mathbb{G}) contains a non-singular model set, or equivalently, is generated by a non-singular model set (these are always repetitive), then we call it a *minimal system of model sets* and call its elements *repetitive model sets*. For more details, see, for instance, [1].

4 Schlottmann's Characterization of Model Sets

Let us recall the result of Schlottmann.

Theorem 4.1 ([18, Theorem 2]) *For a Meyer set Λ , the following statements are equivalent:*

- (i) Λ is a non-singular model set;
- (ii) for any compact K in \mathbb{G} , the set Λ_K is relatively dense, and there is K' compact with $\Lambda_{K'} - \Lambda_{K'} \subseteq \Lambda_K$.

The complete proof of this can be found in Schlottmann's original paper [18]. We warn the reader that non-singularity is termed "regularity" in [18], which is unrelated

with today's notion of regularity used for model sets. We shall see now that, given a Meyer set Λ with associated dynamical system (X_Λ, \mathbb{G}) , Schlottmann's characterization can be expressed using the collection $(P_\varepsilon)_\varepsilon$ introduced in Section 3 for some compatible metric d on X_Λ .

Proposition 4.2 *For Λ a Meyer set and (X_Λ, \mathbb{G}) its associated dynamical system, the following two statements are equivalent:*

- (i) *for any compact K in \mathbb{G} , the set Λ_K is relatively dense and there is K' compact with $\Lambda_{K'} - \Lambda_{K'} \subseteq \Lambda_K$;*
- (ii) *for any $\varepsilon > 0$, the set P_ε is relatively dense and there exists $\delta > 0$ such that $P_\delta - P_\delta \subseteq P_\varepsilon$.*

Proof We already discussed the equivalence between relative denseness of the Λ_K 's and of the P_ε ' in Section 3. Now one easily shows that (i) implies (ii): Since d is compatible, from the discussion of Section 3 we have for any given $\varepsilon > 0$ a compact K and a neighborhood U of 0 in \mathbb{G} with $\Lambda_K + U \subseteq P_\varepsilon$, and taking K' and U' such that $\Lambda_{K'} - \Lambda_{K'} \subseteq \Lambda_K$ and $U' - U' \subseteq U$, and then choosing $\delta > 0$ with $P_\delta \subseteq \Lambda_{K'} + U'$ gives $P_\delta - P_\delta \subseteq P_\varepsilon$, as desired.

Let us show that, conversely, (ii) implies (i): Again from the discussion of Section 3, we deduce that for K chosen and any open neighborhood U of 0 in \mathbb{G} there exists, using our assumption, a K' compact as well as another open neighborhood U' of 0 in \mathbb{G} , that can be chosen arbitrarily small, such that $(\Lambda_{K'} + U') - (\Lambda_{K'} + U') \subseteq \Lambda_K + U$. This yields, in particular, $\Lambda_{K'} - \Lambda_{K'} \subseteq \Lambda_K + U$. Now let us observe that for a compact K that intersect Λ the point set Λ_K belongs to the difference set $\Lambda - \Lambda$. Indeed, having $t \in \Lambda_K$ means that $\Lambda \cap K$ and $(\Lambda - t) \cap K$ are equal, and selecting some point p in it ensures that both p and $p + t$ belong to Λ , providing $t = (p + t) - p \in \Lambda - \Lambda$. Thus, applying this to K' yields that $\Lambda_{K'} - \Lambda_{K'}$ belongs to $\Lambda - \Lambda + \Lambda - \Lambda$ independently on K' , provided it is large enough to intersect Λ . Now as we can freely reduce U (although this would change K'), we can select it small enough so that $(\Lambda - \Lambda + \Lambda - \Lambda) \cap (\Lambda_K + U)$ is actually contained in Λ_K : Indeed one can check that any open set U whose intersection with $\Lambda - \Lambda + \Lambda - \Lambda + \Lambda - \Lambda$ is $\{0\}$ works, and such an open set always exists from the Meyer property of Λ . Thus, for such a choice of open U , we end up with a compact K' that satisfies the desired inclusion $\Lambda_{K'} - \Lambda_{K'} \subseteq \Lambda_K$. ■

5 A General Result on Almost Automorphy

In this section we consider an abstract dynamical system (X, \mathbb{G}) , where X is a compact metric space and \mathbb{G} a LCA group that acts jointly continuously on X . We moreover assume the existence of a transitive point $x \in X$, that is, a point whose \mathbb{G} -orbit is dense in X .

Recall that a factor map $\pi: X \rightarrow Y$ from a dynamical system (X, \mathbb{G}) to another dynamical system (Y, \mathbb{G}) is a continuous onto mapping that is a \mathbb{G} -map, that is, such that $\pi(t.x) = t.\pi(x)$ occurs for any $x \in X$ and any $t \in \mathbb{G}$. A dynamical system (X, \mathbb{G}) always admit a greatest factor whose \mathbb{G} -action is equicontinuous, its so-called *maximal equicontinuous factor* $(X_{\text{eq}}, \mathbb{G})$, which is uniquely defined up to conjugacy; see

[2] for details. The factor map from a dynamical system onto its maximal equicontinuous factor is, in particular, unique up to left-composing the factor map with a \mathbb{G} -commuting homeomorphism of X_{eq} .

For a dynamical system (X, \mathbb{G}) with a transitive point x the maximal equicontinuous factor is a minimal *Kronecker system*: The space X_{eq} admits a compact (Hausdorff) Abelian group structure, and there exists a continuous group morphism $r: \mathbb{G} \rightarrow X_{\text{eq}}$ with dense range such that the \mathbb{G} -action on X_{eq} is given by addition $t.z := r(t) + z$ for any $z \in X_{\text{eq}}$ and $t \in \mathbb{G}$. The group structure on X_{eq} is not canonical here. In fact, any element $z \in X_{\text{eq}}$ can play the role of unit, and this is with respect to a unique compact Abelian group structure on X_{eq} . Once an element $z \in X_{\text{eq}}$ is chosen as unit, the morphism r is recovered through the \mathbb{G} -action on X_{eq} by $r(t) := t.z$ for any $t \in \mathbb{G}$.

A dynamical system (X, \mathbb{G}) is *almost automorphic* when its factor map $\pi: X \rightarrow X_{\text{eq}}$ is 1-1 over some element of X_{eq} . A point $x \in X$ where π is 1-1 is called an *almost automorphic point* (“non-singular point” would have also been a good terminology). In this section, we will characterize the property of almost automorphy of a dynamical system (X, \mathbb{G}) and of a point $x \in X$ by means of the sets, for any $\varepsilon > 0$, of ε -return times of x :

$$P_\varepsilon := \{ t \in \mathbb{G} : d(t.x, x) < \varepsilon \}.$$

Each of these sets contains the origin $0 \in \mathbb{G}$. The following result shows that almost automorphy of the system (X, \mathbb{G}) corresponds to a simple algebraic structure on the set of return times of a single transitive point.

Theorem 5.1 *Let (X, \mathbb{G}) be a dynamical system and let $x \in X$ be a transitive point. Then the following statements are equivalent:*

- (i) *For any $\varepsilon > 0$, the set P_ε is relatively dense and there exists $\delta > 0$ such that*

$$P_\delta - P_\delta \subseteq P_\varepsilon;$$

- (ii) *(X, \mathbb{G}) is almost automorphic, with $x \in X$ an almost automorphic point.*

Proof Assume that (i) holds: The collection of subsets $(P_\varepsilon)_{\varepsilon>0}$ of \mathbb{G} defines the basis $(B_\varepsilon)_{\varepsilon>0}$ of a translation-invariant and inversion-invariant uniformity on \mathbb{G} , where $B_\varepsilon \subseteq \mathbb{G} \times \mathbb{G}$ is the set of pairs (t, t') with $t - t' \in P_\varepsilon$. The topology coming from this uniformity, which we call here the $(P_\varepsilon)_\varepsilon$ -topology, turns \mathbb{G} into an Abelian (possibly non-Hausdorff) topological group, for which $(P_\varepsilon)_\varepsilon$ is a neighborhood basis of $0 \in \mathbb{G}$. This topological group admits a completion by an Abelian (Hausdorff) complete topological group \mathbb{Z} with completion morphism $r: \mathbb{G} \rightarrow \mathbb{Z}$, and

- with dense range in \mathbb{Z} ,
- having kernel $\text{Ker}(r) = \bigcap_{\varepsilon>0} P_\varepsilon$,
- which is continuous and open for the $(P_\varepsilon)_\varepsilon$ -topology on \mathbb{G} .

For details, one can see [18, Appendix A]. Then the mapping defined along the \mathbb{G} -orbit of $x \in X$ by $t.x \mapsto r(t) \in \mathbb{Z}$ is well defined, because each time we get $t.x = t'.x$, $t - t'$ belongs to any P_ε and thus to the kernel of the morphism r , giving $r(t) = r(t')$. It is, moreover, uniformly continuous and therefore extends in a continuous map $\pi: X \rightarrow \mathbb{Z}$. We need to show that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $d(t.x, t'.x) < \delta$, $t - t' \in P_\varepsilon$. To see this, observe that for $\varepsilon > 0$ there exists

by assumption a $\varepsilon' > 0$ such that $P_{\varepsilon'} - P_{\varepsilon'} \subseteq P_{\varepsilon}$. Now because the $(P_{\varepsilon})_{\varepsilon}$ are relatively dense, the transitive point x is almost periodic; *i.e.*, (X, \mathbb{G}) is minimal, so the set of translates $s.B(x, \varepsilon')$, $s \in \mathbb{G}$, covers X . One is thus able to extract a finite subcover given by $s \in F$ with F finite, which admits a Lebesgue number $\delta > 0$. This guarantees that if one has $d(t.x, t'.x) < \delta$, then there is an $s \in F$ such that $t.x$ and $t'.x$ belong to $s.B(x, \varepsilon')$, and as a result, both $(t-s).x$ and $(t'-s).x$ belong to the ball $B(x, \varepsilon')$. Hence $t - t' = (t-s) - (t'-s) \in P_{\varepsilon'} - P_{\varepsilon'} \subseteq P_{\varepsilon}$, as desired. The resulting map π satisfies $\pi(x) = \mathfrak{o} \in \mathcal{Z}$, the unit element in \mathcal{Z} , and is onto, since its image is compact and contains the range of r . It is naturally a \mathbb{G} -map as one may check that $\pi(t.x) = r(t).\pi(x)$ holds for any $x \in X$ and $t \in \mathbb{G}$. Therefore, π is a factor map from (X, \mathbb{G}) to the Kronecker system $(\mathcal{Z}, \mathbb{G})$ with Kronecker action given by the morphism r , which is continuous, since it is given by $r(t) = \pi(t.x)$ for each $t \in \mathbb{G}$. It now suffices to show that x is a one-point fiber with respect to this factor map to conclude that $(\mathcal{Z}, \mathbb{G})$ must be the maximal equicontinuous factor of (X, \mathbb{G}) and that (ii) holds.

Let us show this: Suppose that $x \in X$ satisfies $\pi(x) = \pi(x)$. Since x is transitive, there is a sequence $(t_n)_n \subset \mathbb{G}$ such that $t_n.x$ converges to x , and this gives in the factor \mathcal{Z} the convergence of $r(t_n)$ towards the unit $\mathfrak{o} \in \mathcal{Z}$. Since $r: \mathbb{G} \rightarrow \mathcal{Z}$ is open when \mathbb{G} is equipped with the $(P_{\varepsilon})_{\varepsilon}$ -topology and the collection $(P_{\varepsilon})_{\varepsilon}$ forms a neighborhood basis of $0 \in \mathbb{G}$, we deduce that for any $\varepsilon > 0$ the sequence (t_n) must belong to P_{ε} for n great enough. But by construction of the sets P_{ε} , this means that $d(t_n.x, x) < \varepsilon$ for any $\varepsilon > 0$ and n great enough. This latter argument guarantees that $x = x$, giving (ii).

Assume that (ii) holds: The dynamical system (X, \mathbb{G}) admits a factor map $\pi: X \rightarrow X_{\text{eq}}$ that is 1-1 at x , where $(X_{\text{eq}}, \mathbb{G})$ is the maximal equicontinuous factor with Kronecker action given by a continuous morphism $r: \mathbb{G} \rightarrow X_{\text{eq}}$. Since π is 1-1 at x , one easily shows (by contradiction) that for any $\varepsilon > 0$, there is a neighborhood U_{ε} of $\pi(x) \in X_{\text{eq}}$ such that $\pi^{-1}(U_{\varepsilon}) \subseteq B_d(x, \varepsilon)$. From this we deduce that the sets P_{ε} are relatively dense in \mathbb{G} (*e.g.*, x is an almost periodic point). Indeed, the point $\pi(x)$ is transitive and thus almost periodic in the Kronecker system $(X_{\text{eq}}, \mathbb{G})$, so the set of $t \in \mathbb{G}$ such that $r(t) + \pi(x)$ belongs to U_{ε} is relatively dense in \mathbb{G} . Each such point admits $t.x$ as unique lift in X , which thus belong to $B_d(x, \varepsilon)$ from the very choice of U_{ε} , giving that $t \in P_{\varepsilon}$, so this latter is relatively dense. Now since X_{eq} is a topological group, one can choose for each $\varepsilon > 0$ a neighborhood U'_{ε} of the unit $\mathfrak{o} \in X_{\text{eq}}$ such that $U'_{\varepsilon} - U'_{\varepsilon} \subseteq U_{\varepsilon} - \pi(x)$. As π is continuous, there is some $\delta > 0$ such that $\pi(B_d(x, \delta)) \subseteq \pi(x) + U'_{\varepsilon}$, and therefore whenever $t, t' \in P_{\delta}$, $t.x$ and $t'.x$ have images $r(t) + \pi(x)$ and $r(t') + \pi(x)$ that belong to $\pi(x) + U'_{\varepsilon}$. Thus, $r(t - t')$ lies in $U'_{\varepsilon} - U'_{\varepsilon} \subseteq U_{\varepsilon} - \pi(x)$, and therefore $r(t - t') + \pi(x)$ belongs to U_{ε} . Consequently, its lift $(t - t').x$ lies in $B_d(x, \varepsilon)$, yielding $t - t' \in P_{\varepsilon}$. This gives (i). ■

Now, the proof of the Main Theorem can directly be given as follows.

Proof of the Main Theorem A repetitive model set of \mathbb{G} is, by our definition, a repetitive Meyer set whose hull X contains a non-singular model set Λ . Hence, from Theorem 4.1, Proposition 4.2, and Theorem 5.1 the dynamical system (X, \mathbb{G}) is almost automorphic, with Λ an almost automorphic point in X . This shows the first implication. If, conversely, we are given a repetitive Meyer set Λ whose associated dynamical system is almost automorphic, then it contains an almost automorphic element, which again from Theorem 5.1, Proposition 4.2, and Theorem 4.1 must be a

non-singular Model set. Since Λ belongs to the hull of this model set, it must then itself be a repetitive model set, as desired. ■

Remark 5.2 It is possible to set an elementary proof of Theorem 5.1 by invoking a much more general (and deeper) result due to Veech [22], which can be stated as follows. First, since both statements in Theorem 5.1 imply minimality of the system (X, \mathbb{G}) , we can assume this condition. Then consider a metric minimal system (X, \mathbb{G}) , with factor map $\pi: X \rightarrow X_{\text{eq}}$ over its maximal equicontinuous factor, and a point $x \in X$ with associated family of sets $(P_\varepsilon)_{\varepsilon>0}$. The result of [22] then describes the set of points in X belonging in the fiber of x over X_{eq} as

$$\pi^{-1}(\pi(x)) = \bigcap_{\varepsilon>0} \overline{(P_\varepsilon - P_\varepsilon).x}.$$

Using this result, one easily shows Theorem 5.1. Indeed, if (i) holds then the fiber $\pi^{-1}(\pi(x))$ is equally given by the intersection of the sets $\overline{P_\varepsilon.x}$, which is nothing but the singleton $\{x\}$, so that (ii) holds. Conversely, if (ii) holds, then for any $\varepsilon > 0$ there is a $\delta > 0$ such that $(P_\delta - P_\delta).x$ must belong to $B(x, \varepsilon)$, giving $P_\delta - P_\delta \subseteq P_\varepsilon$, and thus (i) holds.

As a result, the characterization of model sets by dynamical systems as presented here is mainly the combination of Schlottmann's characterization and of Veech's result on the equicontinuous structure relation for minimal systems. Despite the existence of this very direct argument, we decided to keep the proof of Theorem 5.1 as presented above, first for self-containment of the note, and second because it involves a completion process of topological groups that is very similar to the one appearing in the proof of Schlottmann's characterization given in [18].

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References

- [1] J.-B. Aujogue, *On embedding of repetitive Meyer multiple sets into model multiple sets*. Ergodic Theory Dynam. Systems 36(2016), no. 6, 1679–1702. <http://dx.doi.org/10.1017/etds.2014.133>
- [2] J.-B. Aujogue, M. Barge, J. Kellendonk, and D. Lenz. *Equicontinuous factors, proximality and Ellis semigroup for Delone sets*. In: Mathematics of aperiodic order, Progr. Math., 309, Birkhäuser/Springer, Basel, 2015, pp. 137–1494.
- [3] J. Auslander, *Minimal flows and their extensions*. North-Holland Mathematics Studies, 153, Notas de Matemática, 122, North-Holland Publishing Co., Amsterdam, 1988.
- [4] M. Baake and U. Grimm, *Aperiodic order. Vol. 1. A mathematical invitation*. Encyclopedia of Mathematics and its Applications, 149, Cambridge University Press, Cambridge, 2013.
- [5] M. Baake, J. Hermisson, and P. A. B. Pleasants, *The torus parametrization of quasiperiodic LI-classes*. J. Phys. A 30(1997), 3029–3056, <http://dx.doi.org/10.1088/0305-4470/30/9/016>
- [6] M. Baake, D. Lenz, and R. V. Moody, *Characterization of model sets by dynamical systems*. Ergodic Theory Dynam. Systems 27(2007), 341–382. <http://dx.doi.org/10.1017/S0143385706000800>
- [7] M. Baake and R. V. Moody, *Diffraction point sets with entropy*. J. Physics A 31(1998), no. 45, 9023–9039. <http://dx.doi.org/10.1088/0305-4470/31/45/003>
- [8] A. Forrest, J. Hutton, and J. Kellendonk, *Topological invariants for projection method patterns*. Mem. Amer. Math. Soc. 159(2002), no. 758.

- [9] A. Hof, *On diffraction by aperiodic structures*. *Comm. Math. Phys.* 169(1995), no. 1, 25–43.
<http://dx.doi.org/10.1007/BF02101595>
- [10] J. L. Kelley, *General topology*. Graduate Texts in Mathematics, 27, Springer-Verlag, New York-Berlin, 1975.
- [11] J.-Y. Lee and R. V. Moody, *A characterization of model multi-colour sets*. *Ann. Henri Poincaré* 7(2006), no. 1, 125–143. <http://dx.doi.org/10.1007/s00023-005-0244-6>
- [12] D. Lenz and C. Richard, *Pure point diffraction and cut and project schemes for measures: the smooth case*. *Math. Z.* 256(2007), no. 2, 347–378. <http://dx.doi.org/10.1007/s00209-006-0077-0>
- [13] A. L. Mackay, *Crystallography and the Penrose pattern*. In: Proceedings of the Tenth International Colloquium on Group-Theoretical Methods in Physics (Canterbury, 1981). *Phys. A* 114(1982), no. 1–3, 609–613. [http://dx.doi.org/10.1016/0378-4371\(82\)90359-4](http://dx.doi.org/10.1016/0378-4371(82)90359-4)
- [14] Y. Meyer, *Algebraic numbers and harmonic analysis*. North-Holland Mathematical Library, Vol. 2, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1972.
- [15] R. V. Moody, *Meyer sets and their duals*. In: The mathematics of long-range aperiodic order (Waterloo, ON, 1995), *Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, 489, Kluwer Acad. Publ., Dordrecht, 1997, pp. 403–442.
- [16] ———, *Uniform distribution in model sets*. *Canad. Math. Bull.* 45(2002), no. 1, 123–130.
<http://dx.doi.org/10.4153/CMB-2002-015-3>
- [17] E. A. Robinson Jr., *The dynamical properties of Penrose tilings*. *Trans. Amer. Math. Soc.* 348(1996), no. 11, 4447–4464. <http://dx.doi.org/10.1090/S0002-9947-96-01640-6>
- [18] M. Schlottmann, *Cut-and-project sets in locally compact abelian groups*. In: Quasicrystals and discrete geometry (Toronto, ON, 1995), *Fields Inst. Monogr.*, 10, American Mathematical Society, Providence, RI, 1998.
- [19] ———, *Generalized model sets and dynamical systems*. In: Directions in mathematical quasicrystals, CRM Monogr. Ser., 13, American Mathematic Society, Providence, RI, 2000.
- [20] N. Strungaru, *Almost periodic measures and Meyer sets*. [arxiv:1501.00945](https://arxiv.org/abs/1501.00945)
- [21] W. A. Veech, *Almost automorphic functions on groups*. *Amer. J. Math.* 87(1965), 719–751.
<http://dx.doi.org/10.2307/2373071>
- [22] ———, *The equicontinuous structure relation for minimal Abelian transformation groups*. *Amer. J. Math.* 90(1968), 723–732. <http://dx.doi.org/10.2307/2373480>

Department of Mathematics, Universidad de Santiago de Chile, Santiago, Chile
e-mail: jean.baptiste@usach.cl