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Anisotropic Hardy–Lorentz Spaces with Variable Exponents

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Abstract. In this paper we introduce Hardy–Lorentz spaces with variable exponents associated with dilations in $\mathbb{R}^n$. We establish maximal characterizations and atomic decompositions for our variable exponent anisotropic Hardy–Lorentz spaces.

1 Introduction

Fefferman and Stein's celebrated paper [26] has been crucial in the development of the real variable theory of Hardy spaces. In [26] the tempered distributions in the Hardy spaces $H^p(\mathbb{R}^n)$ were characterized as those such that certain maximal functions are in $L^p(\mathbb{R}^n)$. Coifman [10] and Latter [38] obtained atomic decompositions of the elements of the Hardy spaces $H^p(\mathbb{R}^n)$. Here, $0 < p < \infty$ and $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ provided that $1 < p < \infty$.

Many authors have investigated Hardy spaces in several settings. Some generalizations substitute the underlying domain $\mathbb{R}^n$ with other ones (see, for instance, [7, 9, 12, 44, 54, 58]). Also, Hardy spaces associated with operators have been defined (see [22, 23, 33, 34, 60], amongst others). If X is a function space, the Hardy space $H(\mathbb{R}^n, X)$ on $\mathbb{R}^n$ modelled on X consists of all those tempered distributions f on $\mathbb{R}^n$ such that the maximal function $\mathcal{M}(f)$ of f is in X. The definition of the maximal operator $\mathcal{M}$ will be shown below. The classical Hardy space $H^p(\mathbb{R}^n)$ is the Hardy space on $\mathbb{R}^n$ modelled on $L^p(\mathbb{R}^n)$. For a weight v on $\mathbb{R}^n$ and corresponding weighted Lebesgue space $L^p(\mathbb{R}^n, v)$, the Hardy space $H(\mathbb{R}^n, L^p(\mathbb{R}^n, v))$ was investigated in [28]. The Hardy space $H(\mathbb{R}^n, L^{p,q}(\mathbb{R}^n))$, where $L^{p,q}(\mathbb{R}^n)$ represents the Lorentz space, has been studied in [1,25,27,31,32]. The Hardy space $H(\mathbb{R}^n, \Lambda^p(\phi))$ on $\mathbb{R}^n$ modelled on a generalized Lorentz space $\Lambda^p(\phi)$ was studied by Almeida and Caetano [2]. The variable exponent Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$, investigated in [15,48,52,63], is the space $H(\mathbb{R}^n, L^{p(\cdot)}(\mathbb{R}^n))$ on $\mathbb{R}^n$ modelled on the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$.

By $S(\mathbb{R}^n)$, as usual, we denote the Schwartz function class on $\mathbb{R}^n$ and by $S'(\mathbb{R}^n)$ its dual space. If $\varphi \in S(\mathbb{R}^n)$, the radial maximal function $\mathcal{M} = M_\varphi$ used to characterize Hardy spaces is defined by $\mathcal{M}(f) = \sup_{t>0} |f * \varphi_t|$, $f \in S'(\mathbb{R}^n)$, where $\varphi_t(x) = t^{-n}\varphi(x/t)$, $x \in \mathbb{R}^n$ and $t > 0$. Bownik [4] studied anisotropic Hardy spaces

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on $\mathbb{R}^n$ associated with dilations in $\mathbb{R}^n$. If A is an expansive dilation matrix in $\mathbb{R}^n$, that is, a $n \times n$ real matrix such that $\min_{\lambda\in\sigma(A)} |\lambda| > 1$ where $\sigma(A)$ represents the set of eigenvalues of A, for every $k \in \mathbb{Z}$, we define

$$\varphi_{A,k}(x) = |\det A|^{-k}\varphi(A^{-k}x), \quad x \in \mathbb{R}^n,$$

and the maximal function $\mathcal{M}_A = M_{A,\varphi}$ associated with A is given by

$$\mathcal{M}_A(f) = \sup_{k\in\mathbb{Z}} |f * \varphi_{A,k}|, \quad f \in S'(\mathbb{R}^n).$$

Bownik [4] characterizes anisotropic Hardy spaces by maximal functions like $\mathcal{M}_A$. Recently, Liu, Yang, and Yuan [40] extended Bownik's results by studying anisotropic Hardy spaces on $\mathbb{R}^n$ modelled on Lorentz spaces $L^{p,q}(\mathbb{R}^n)$.

Ephremidze, Kokilashvili, and Samko [24] introduced variable exponent Lorentz spaces $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. In this paper we define anisotropic Hardy spaces on $\mathbb{R}^n$ associated with a dilation A modelled on $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. These Hardy spaces are represented by $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and they are called variable exponent anisotropic Hardy–Lorentz spaces on $\mathbb{R}^n$. We characterize the tempered distributions in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ by using anisotropic maximal function $\mathcal{M}_A$. Also, we obtain atomic decompositions for the elements of $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. Our results extend those ones in [40] to variable exponent setting.

Before establishing the results of this paper, we recall the definitions and properties about anisotropy and variable exponent Lebesgue and Lorentz spaces we will need.

An exhaustive and systematic study about variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$, where $\Omega \subset \mathbb{R}^n$, can be found in the monograph [17] and in [20]. Here, $p\colon \Omega \to (0,\infty)$ is a measurable function. We assume that $0 < p_-(\Omega) \le p_+(\Omega) < \infty$, where $p_-(\Omega) = \operatorname{ess\,inf}_{x\in\Omega} p(x)$ and $p_+(\Omega) = \operatorname{ess\,sup}_{x\in\Omega} p(x)$. The space $L^{p(\cdot)}(\Omega)$ is the collection of all measurable functions f such that, for some $\lambda > 0$, $\rho(f/\lambda) < \infty$, where

$$\rho(f) = \rho_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}dx.$$

We define $\|\cdot\|_{p(\cdot)}$ as follows:

$$\|f\|_{p(\cdot)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} dx \le 1\Big\}, \quad f \in L^{p(\cdot)}(\Omega).$$

If $p_-(\Omega) \ge 1$, then $\|\cdot\|_{p(\cdot)}$ is a norm and $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a Banach space. However, if $p_-(\Omega) < 1$, then $\|\cdot\|_{p(\cdot)}$ is a quasinorm and $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a quasi Banach space.

A crucial problem concerning to variable exponent Lebesgue spaces is to describe the exponents p for which the Hardy–Littlewood maximal function is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$ (see [13,19,49,51], amongst others). As shown in [14,16,29], the boundedness of the Hardy–Littlewood maximal function, together with extensions of Rubio de Francia's extrapolation theorem, lead to the boundedness of a wide class of operators and vector-valued inequalities on $L^{p(\cdot)}(\mathbb{R}^n)$ and the weighted $L^{p(\cdot)}(v)$. These ideas also work in the variable exponent Lorentz spaces, introduced by Ephremidze, Kokilashvili, and Samko [24], and they will play a fundamental role in the proof of some of our main results.

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The Lorentz spaces were introduced in [41] and [42] as a generalization of classical Lebesgue spaces. The theory of Lorentz spaces can be encountered in [3] and [8]. Assume that f is a measurable function. We define the distribution function $\mu_f\colon[0,\infty)\to[0,\infty]$ associated with f by

$$\mu_f(s)=|\{x\in\mathbb{R}^n:|f(x)|>s\}|,\quad s\in[0,\infty).$$

Here, $|E|$ denotes the Lebesgue measure of E, for every Lebesgue measurable set E. The non-increasing equimeasurable rearrangement $f^*\colon[0,\infty)\to[0,\infty]$ of f is defined by

$$f^*(t)=\inf\{s\geq 0:\mu_f(s)\leq t\},\quad t\in[0,\infty).$$

If $0<p,q<\infty$, the measurable function f is in the Lorentz space $L^{p,q}(\mathbb{R}^n)$ provided that

$$\|f\|_{L^{p,q}(\mathbb{R}^n)}=\Big(\int_0^\infty t^{\frac{q}{p}-1}(f^*(t))^q dt\Big)^{1/q}<\infty.$$

Then $L^{p,q}(\mathbb{R}^n)$ is complete and it is normable; that is, there exists a norm equivalent with the quasinorm $\|\cdot\|_{L^{p,q}(\mathbb{R}^n)}$ (see [8, p. 66]) for $1<p<\infty$ and $1\leq q<\infty$.

Variable exponent Lorentz spaces have been defined in two different ways: one by Ephremidze, Kokilashvili, and Samko [24] and the other by Kempka and Vybíral [36].

In this paper we consider the space defined in [24]. This election is motivated by the following fact. We need to use a vectorial inequality for the anisotropic Hardy–Littlewood maximal function (see Proposition 1.4). In order to prove this property, we use an extrapolation argument requiring us to know the associated Köthe dual space of the Lorentz space. The dual space of the variable exponent Lorentz space in [24] is known ([24, Lemma 2.7]). However, characterizations of the dual space of the variable exponent Lorentz space in [36] have not been established (see Remark 1.5).

For every $a\geq 0$ we denote by $\mathfrak{P}_a$ the set of measurable functions $p\colon(0,\infty)\to(0,\infty)$ such that $a<p_-((0,\infty))\leq p_+((0,\infty))<\infty$. By $\mathbb{P}$ we represent the class of bounded measurable functions $p\colon(0,\infty)\to(0,\infty)$ such that there exist the limits

$$p(0)=:\lim_{t\to 0^+}p(t)\quad\text{and}\quad p(\infty)=:\lim_{t\to+\infty}p(t),$$

and the following log-Hölder continuity conditions are satisfied:

$$|p(t)-p(0)|\leq\frac{C}{|\ln t|}\qquad\text{for } 0<t\leq 1/2,$$
$$|p(t)-p(\infty)|\leq\frac{C}{\ln(e+t)}\qquad\text{for}\quad t\in(0,\infty).$$

We also write $\mathbb{P}_a=\mathbb{P}\cap\mathfrak{P}_a$, for every $a\geq 0$.

Let $p,q\in\mathfrak{P}_0$. We represent by $(p(\cdot),q(\cdot))$-Lorentz space $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ the space of all those measurable functions f on $\mathbb{R}^n$ such that

$$t^{\frac{1}{p(t)}-\frac{1}{q(t)}}f^*(t)\in L^{q(\cdot)}(0,\infty).$$

We define

$$\|f\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}=\|t^{\frac{1}{p(t)}-\frac{1}{q(t)}}f^*(t)\|_{L^{q(\cdot)}(0,\infty)},\quad f\in\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n).$$

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We also consider the average f^{**} of f^* given by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds, \quad t \in (0,\infty),$$

and define

$$\|f\|^{(1)}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} = \|t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^{**}(t)\|_{L^{q(\cdot)}(0,\infty)}, \quad f \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n).$$

We note that $\|\cdot\|^{(1)}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ satisfies the triangular inequality provided that $q_-((0,\infty)) \geq 1$. It is clear that

$$\|f\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq \|f\|^{(1)}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

According to [24, Theorem 2.4], if $p \in \mathbb{P}_0$, $q \in \mathbb{P}_1$, $p(0) > 1$, and $p(\infty) > 1$, there exists $C > 0$ for which

$$\|f\|^{(1)}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}, \quad f \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n).$$

If $p, q \in \mathbb{P}_1$, then $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is a Banach function space (in the sense of [3]) and the dual space $(\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n))'$ coincides with $\mathcal{L}^{p'(\cdot),q'(\cdot)}(\mathbb{R}^n)$ [24, Lemma 2.7 and Theorem 2.8]. Here, as usual, if $r: (0,\infty) \to (1,\infty)$, $r' = \frac{r}{r-1}$. The behaviour of the anisotropic Hardy–Littlewood maximal function on $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ will be very useful in the sequel. According to [24, Theorem 3.12], the classical Hardy–Littlewood maximal operator is bounded from $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ into itself provided that $p, q \in \mathbb{P}_1$.

The main definitions and properties of the anisotropic setting we will use in this paper can be found in [4].

Suppose that A is an expansive dilation matrix in $\mathbb{R}^n$. We say that a measurable function $\rho: \mathbb{R}^n \to [0,\infty)$ is a homogeneous quasinorm associated with A when the following properties hold:

(a) $\rho(x) = 0$ if and only if $x = 0$;
(b) $\rho(Ax) = |\det A|\rho(x), \quad x \in \mathbb{R}^n$;
(c) $\rho(x+y) \leq H(\rho(x)+\rho(y)), \quad x, y \in \mathbb{R}^n$, for certain $H \geq 1$.

If P is a nondegenerate $n \times n$ matrix, the set Δ defined by

$$\Delta = \{x \in \mathbb{R}^n : |Px| < 1\}$$

is called the ellipsoid generated by P. According to [4, Lemma 2.2, p. 5], there exists an ellipsoid Δ with Lebesgue measure 1 and such that, for certain $r_0 > 1$, $\Delta \subseteq r_0\Delta \subseteq A\Delta$.

From now on, the ellipsoid Δ satisfying the above properties is fixed. For every $k \in \mathbb{Z}$, we define $B_k = A^k\Delta$, as the equivalent of the Euclidean balls in our anisotropic context, and we denote by ω the smallest integer such that $2B_0 \subset B_\omega$. We have that, for every $k \in \mathbb{Z}$, $|B_k| = b^k$, where $b = |\det A|$, and $B_k \subset r_0 B_k \subset B_{k+1}$.

The *step quasinorm* ρ_A on $\mathbb{R}^n$ is defined by

$$\rho_A(x) = \begin{cases} b^k, & x \in B_{k+1}\backslash B_k, \quad k \in \mathbb{Z}, \\ 0, & x = 0. \end{cases}$$

Thus, ρ_A is a homogeneous quasinorm associated with A.

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By [4, Lemma 2.4, p. 6] if ρ is any quasinorm associated with A, then ρ_A and ρ are equivalent; that is, for a certain $C > 0$,

$$\rho(x)/C \le \rho_A(x) \le C\rho(x), \quad x \in \mathbb{R}^n.$$

The triplet $(\mathbb{R}^n, \rho_A, |\cdot|)$, where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^n$, is a space of homogeneous type in the sense of Coifman and Weiss [11].

We now define maximal functions in our anisotropic setting. Suppose that $\varphi \in S(\mathbb{R}^n)$ and $f \in S'(\mathbb{R}^n)$. The *radial maximal function* $M^0_\varphi(f)$ of f with respect to φ is defined by

$$M^0_\varphi(f)(x) = \sup_{k\in\mathbb{Z}} |(f * \varphi_k)(x)|,$$

where $\varphi_k(x) = b^{-k}\varphi(A^{-k}x)$, $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$. Since the matrix A is fixed, we do not refer to it in the notation of maximal functions.

The nontangential maximal function $M_\varphi(f)$ with respect to φ is given by

$$M_\varphi(f)(x) = \sup_{k\in\mathbb{Z},\, y\in x+B_k} |(f * \varphi_k)(y)|, \quad x \in \mathbb{R}^n.$$

If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Let $N \in \mathbb{N}$. We consider the set

$$S_N = \Big\{ \varphi \in S(\mathbb{R}^n) : \sup_{x\in\mathbb{R}^n} (1 + \rho_A(x))^N |D^\alpha \varphi(x)| \le 1, \alpha \in \mathbb{N}^n \text{ and } |\alpha| \le N \Big\}.$$

Here,

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}},$$

when $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$.

The radial grandmaximal function $M^0_N(f)$ of f of order N is defined by

$$M^0_N(f) = \sup_{\varphi\in S_N} M^0_\varphi(f).$$

The nontangential grandmaximal function $M_N(f)$ of f of order N is given by

$$M_N(f) = \sup_{\varphi\in S_N} M_\varphi(f).$$

We now define variable exponent anisotropic Hardy–Lorentz spaces. Let $N \in \mathbb{N}$ and $p, q \in \mathfrak{P}_0$. The $(p(\cdot), q(\cdot))$-anisotropic Hardy–Lorentz space $H^{p(\cdot),q(\cdot)}_N(\mathbb{R}^n, A)$ associated with A is the set of all those $f \in S'(\mathbb{R}^n)$ such that $M_N(f) \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. On $H^{p(\cdot),q(\cdot)}_N(\mathbb{R}^n, A)$, we consider the quasinorm $\|\cdot\|_{H^{p(\cdot),q(\cdot)}_N(\mathbb{R}^n,A)}$ defined by

$$\|f\|_{H^{p(\cdot),q(\cdot)}_N(\mathbb{R}^n,A)} = \|M_N(f)\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}, \quad f \in H^{p(\cdot),q(\cdot)}_N(\mathbb{R}^n, A).$$

Our first result shows that the space $H^{p(\cdot),q(\cdot)}_N(\mathbb{R}^n, A)$ does not actually depend on N provided that N is large enough. Furthermore, we prove that $H^{p(\cdot),q(\cdot)}_N(\mathbb{R}^n, A)$ can be characterized also by using the maximal functions M^0_φ, M_φ, and M^0_N.

Theorem 1.1 *Let $f \in S'(\mathbb{R}^n)$ and $\varphi \in S(\mathbb{R}^n)$ such that $\int \varphi \ne 0$. Assume that $p, q \in \mathbb{P}_0$. Then the following assertions are equivalent.*

(i) *There exists $N_0 \in \mathbb{N}$ such that, for every $N \ge N_0$, $f \in H^{p(\cdot),q(\cdot)}_N(\mathbb{R}^n, A)$.*

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(ii) $M_\varphi(f) \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.
(iii) $M_\varphi^0(f) \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Moreover, for every $g \in S'(\mathbb{R}^n)$ the quantities

$$\|M_N(g)\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}, \quad N \geq N_0,$$
$$\|M_\varphi^0(g)\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)},$$
$$\|M_\varphi(g)\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$$

are equivalent.

According to Theorem 1.1 we let $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ denote $H_N^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$, for every $N \geq N_0$.

In order to prove this theorem, we follow the ideas developed by Bownik [4, §7] (see also [40, §4]) but we need to make some modifications due to that decreasing rearrangement and variable exponents appear.

Let $1 < r \leq \infty$, $s \in \mathbb{N}$ and $p, q \in \mathfrak{P}_0$. We say that a measurable function a on $\mathbb{R}^n$ is a $(p(\cdot), q(\cdot), r, s)$-atom associated with $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ when a satisfies

(a) $\operatorname{supp} a \subseteq x_0 + B_k$;
(b) $\|a\|_r \leq b^{k/r} \|\chi_{x_0+B_k}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ (note that $(\chi_{x_0+B_k})^* = \chi_{(0,b^k)}$);
(c) $\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0$, for every $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq s$.

Here, if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, then $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Remark 1.2 From now on, any time we write a is a $(p(\cdot), q(\cdot), r, s)$-atom associated with $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, it is understood that (a), (b), and (c) hold.

In the next result we characterize the distributions in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ by atomic decompositions.

Theorem 1.3 *Let $p, q \in \mathbb{P}_0$.*

(i) *There exist $s_0 \in \mathbb{N}$ and $C > 0$ such that if, for every $j \in \mathbb{N}$, $\lambda_j \geq 0$ and a_j is a $(p(\cdot), q(\cdot), \infty, s_0)$-atom associated with $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$, satisfying that $\sum_{j\in\mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \chi_{x_j+B_{\ell_j}} \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Then*

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A),$$

and

$$\|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} \leq C \Big\| \sum_{j\in\mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \chi_{x_j+B_{\ell_j}} \Big\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

If also $p(0) < q(0)$, then there exists $r_0 > 1$ such that for every $r_0 < r < \infty$ the above assertion is true when $(p(\cdot), q(\cdot), \infty, s_0)$-atoms are replaced by $(p(\cdot), q(\cdot), r, s_0)$-atoms.

(ii) *There exists $s_0 \in \mathbb{N}$ such that for every $s \in \mathbb{N}$, $s \geq s_0$, and $1 < r \leq \infty$, we can find $C > 0$ such that, for every $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$, there exist, for each $j \in \mathbb{N}$, $\lambda_j > 0$*

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and a $(p(\cdot),q(\cdot),r,s)$-atom a_j associated with $x_j\in\mathbb{R}^n$ and $\ell_j\in\mathbb{Z}$, satisfying that

$$\sum_{j\in\mathbb{N}}\lambda_j\|\chi_{x_j+B_{\ell_j}}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}\chi_{x_j+B_{\ell_j}}\in\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n),$$

$$f=\sum_{j\in\mathbb{N}}\lambda_j a_j \text{ in } S'(\mathbb{R}^n),$$

and

$$\Big\|\sum_{j\in\mathbb{N}}\lambda_j\|\chi_{x_j+B_{\ell_j}}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}\chi_{x_j+B_{\ell_j}}\Big\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}.$$

Let $1<r\le\infty$, $s\in\mathbb{N}$ and $p,q\in\mathfrak{P}_0$. We define the anisotropic variable exponent atomic Hardy–Lorentz space $H^{p(\cdot),q(\cdot),r,s}(\mathbb{R}^n,A)$ as follows. A distribution $f\in S'(\mathbb{R}^n)$ is in $H^{p(\cdot),q(\cdot),r,s}(\mathbb{R}^n,A)$ when, for every $j\in\mathbb{N}$ there exist $\lambda_j\ge 0$ and a $(p(\cdot),q(\cdot),r,s)$-atom a_j associated with $x_j\in\mathbb{R}^n$ and $\ell_j\in\mathbb{Z}$ such that $f=\sum_{j\in\mathbb{N}}\lambda_j a_j$, where the series converges in $S'(\mathbb{R}^n)$, and

$$\sum_{j\in\mathbb{N}}\lambda_j\|\chi_{x_j+B_{\ell_j}}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}\chi_{x_j+B_{\ell_j}}\in\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n).$$

For every $f\in H^{p(\cdot),q(\cdot),r,s}(\mathbb{R}^n,A)$ we define

$$\|f\|_{H^{p(\cdot),q(\cdot),r,s}(\mathbb{R}^n,A)}=\inf\Big\|\sum_{j\in\mathbb{N}}\lambda_j\|\chi_{x_j+B_{\ell_j}}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}\chi_{x_j+B_{\ell_j}}\Big\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)},$$

where the infimum is taken over all the sequences $(\lambda_j)_{j\in\mathbb{N}}\subset[0,\infty)$ and $(a_j)_{j\in\mathbb{N}}$ of $(p(\cdot),q(\cdot),r,s)$-atoms satisfying that $f=\sum_{j\in\mathbb{N}}\lambda_j a_j$ in $S'(\mathbb{R}^n)$ and

$$\sum_{j\in\mathbb{N}}\lambda_j\|\chi_{x_j+B_{\ell_j}}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}\chi_{x_j+B_{\ell_j}}\in\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n),$$

being a_j associated with $x_j\in\mathbb{R}^n$ and $\ell_j\in\mathbb{Z}$, for every $j\in\mathbb{N}$.

In Theorem 1.3 we state some conditions so that the inclusions $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)\subset H^{p(\cdot),q(\cdot),r,s}(\mathbb{R}^n,A)$ and $H^{p(\cdot),q(\cdot),r,s}(\mathbb{R}^n,A)\subset H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)$ hold continuously.

In our proof of Theorem 1.3 a vector valued inequality, involving the Hardy-Littlewood maximal function in our anisotropic setting, plays an important role. The mentioned maximal function is defined by

$$M_{HL}(f)(x)=\sup_{k\in\mathbb{Z},y\in x+B_k}\frac{1}{b^k}\int_{y+B_k}|f(z)|dz,\quad x\in\mathbb{R}^n.$$

After proving a version of [24, Theorem 3.12] for M_{HL}, by using an extension of Rubio de Francia extrapolation theorem (see [14,16,29]), we can establish the following result.

Proposition 1.4 *Assume that $p,q\in\mathbb{P}_1$. For every $r\in(1,\infty)$, there exists $C>0$ such that*

$$\Big\|\Big(\sum_{j\in\mathbb{N}}M_{HL}(f_j)^r\Big)^{1/r}\Big\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}\le C\Big\|\Big(\sum_{j\in\mathbb{N}}|f_j|^r\Big)^{1/r}\Big\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)},$$

for each sequence $(f_j)_{j\in\mathbb{N}}$ of functions in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

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Remark 1.5 We do not know if the last vectorial inequality holds when the Lorentz space $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is replaced by the variable exponent Lorentz space $L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ introduced by Kempka and Vybíral [36]. In order to apply extrapolation technique, it is necessary to know the associated Köthe dual space (see [39, p. 25]) $(L_{p(\cdot),q(\cdot)}(\mathbb{R}^n))^*$ of $L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, but its characterization is, as far we know, an open question.

Also, in order to prove Theorem 1.3, we need to establish that $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)\cap L^1_{\rm loc}(\mathbb{R}^n)$ is a dense subspace of $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)$. At this point a careful study of Calderón–Zygmund decomposition of the distributions in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)$ must be done.

To establish boundedness of operators on Hardy spaces, atomic characterizations (as in Theorem 1.3) play an important role. Meyer [46] (see also [47, p. 513]) gave a function $f\in H^1(\mathbb{R}^n)$ whose norm is not achieved by finite atomic decomposition. More recently, Bownik [5] adapted that example to get, for every $0<p\le 1$, an atom in $H^p(\mathbb{R}^n)$ with the same property. Also, in [5, Theorem 2] it was proved that there exists a linear functional $\mathfrak{l}$ defined on the space $H^{1,\infty}_{\rm fin}(\mathbb{R}^n)$, consisting in finite linear combinations of $(1,\infty)$-atoms, such that, for a certain $C>0$, $|\mathfrak{l}(a)|\le C$, for every $(1,\infty)$-atom a, and $\mathfrak{l}$ cannot be extended to a bounded functional on the whole $H^1(\mathbb{R}^n)$.

Bownik's results have motivated some investigations of operators on Hardy spaces via atomic decompositions. Meda, Sjögren and Vallarino [45] proved that if $1<q<\infty$ and T is a linear operator defined on $H^{1,q}_{\rm fin}(\mathbb{R}^n)$, the space of finite linear combinations of $(1,q)$-atoms, into a quasi Banach space Y such that

$$\sup\{\|Ta\|_Y : a \text{ is a } (1,q)\text{-atom}\}<\infty,$$

then T can be extended to $H^1(\mathbb{R}^n)$ as a bounded operator from $H^1(\mathbb{R}^n)$ into Y. Also, it is proved that the same is true when $(1,q)$-atoms are replaced by continuous $(1,\infty)$-atoms, in contrast with the Bownik's result. Yang and Zhou [61] established the result when $0<p\le 1$ and $(p,2)$-atoms are considered. Ricci and Verdera [50] proved that, for $0<p<1$, when $H^{p,\infty}_{\rm fin}(\mathbb{R}^n)$ is endowed with the natural topology, the dual spaces of $H^{p,\infty}_{\rm fin}(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$ coincide.

Also, this type of results have been recently established for Hardy spaces in more general settings (see, for instance, [6, 15, 40, 62]).

In order to study boundedness of some singular integrals on our anisotropic Hardy–Lorentz spaces with variable exponents we consider finite atomic Hardy–Lorentz spaces in our settings.

Let $1<r<\infty$, $s\in\mathbb{N}$ and $p,q\in\mathfrak{P}_0$. The space $H^{p(\cdot),q(\cdot),r,s}_{\rm fin}(\mathbb{R}^n,A)$ consists of all those $f\in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)$ such that there exist $k\in\mathbb{N}$ and, for every $j\in\mathbb{N}$, $1\le j\le k$, $\lambda_j>0$ and a $(p(\cdot),q(\cdot),r,s)$-atom a_j such that $f=\sum_{j=1}^k\lambda_j a_j$. For every $f\in H^{p(\cdot),q(\cdot),r,s}_{\rm fin}(\mathbb{R}^n,A)$, we define

$$\|f\|_{H^{p(\cdot),q(\cdot),r,s}_{\rm fin}(\mathbb{R}^n,A)}=\inf\Big\|\sum_{j=1}^k\lambda_j\|\chi_{x_j+B_{\ell_j}}\|^{-1}_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}\chi_{x_j+B_{\ell_j}}\Big\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)},$$

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where the infimum is taken over all the finite sequences $(\lambda_j)_{j=1}^k \subset (0,\infty)$ and $(a_j)_{j=1}^k$ of $(p(\cdot), q(\cdot), r, s)$-atoms such that $f = \sum_{j=1}^k \lambda_j a_j$ and being, for every $j \in \mathbb{N}$, $j \le k$, a_j associated with $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$.

The space $H^{p(\cdot),q(\cdot),\infty,s}_{\mathrm{fin,con}}(\mathbb{R}^n, A)$ and the quasinorm $\|\cdot\|_{H^{p(\cdot),q(\cdot),\infty,s}_{\mathrm{fin,con}}(\mathbb{R}^n,A)}$ are defined in a similar way by considering continuous $(p(\cdot), q(\cdot), \infty, s)$-atoms.

In Theorem 1.6 we establish some conditions that imply that $H^{p(\cdot),q(\cdot),r,s}_{\mathrm{fin}}(\mathbb{R}^n, A)$ is dense in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.

Theorem 1.6 *Let $p, q \in \mathbb{P}_0$.*

(i) *Assume that $p(0) < q(0)$. Then there exist $s_0 \in \mathbb{N}$ and $r_0 \in (1,\infty)$ such that for every $s \in \mathbb{N}$, $s \ge s_0$, and $r \in (r_0, \infty)$,*
$$\|\cdot\|_{H^{p(\cdot),q(\cdot),r,s}_{\mathrm{fin}}(\mathbb{R}^n,A)} \quad \text{and} \quad \|\cdot\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}$$
are equivalent quasinorms in $H^{p(\cdot),q(\cdot),r,s}_{\mathrm{fin}}(\mathbb{R}^n, A)$.

(ii) *There exists $s_0 \in \mathbb{N}$ such that for every $s \ge s_0$,*
$$\|\cdot\|_{H^{p(\cdot),q(\cdot),\infty,s}_{\mathrm{fin,con}}(\mathbb{R}^n,A)} \quad \text{and} \quad \|\cdot\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}$$
are equivalent quasinorms in $H^{p(\cdot),q(\cdot),\infty,s}_{\mathrm{fin,con}}(\mathbb{R}^n, A)$.

As an application of Theorem 1.6, we prove that convolutional type Calderón–Zygmund singular integrals are bounded in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. A precise definition of the singular integral that we consider can be found in Section 6.

Theorem 1.7 *Let $p, q \in \mathbb{P}_0$. Assume that $p(0) < q(0)$. If T is a convolutional type Calderón–Zygmund singular integral of order $m \in \mathbb{N}$, $m \ge s_0$ where s_0 is as in Theorem 1.6*(i), *then*

(i) *T is bounded from $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ into $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$;*

(ii) *T is bounded from $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ into itself.*

Our results, as far as we know, are new even in the isotropic case, that is, for the Hardy–Lorentz $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ of variable exponents, extending results in [1].

The paper is organized as follows. A proof of Theorem 1.1 is presented in Section 2 where we prove the main properties of variable exponent anisotropic Hardy–Lorentz spaces. Next, in Section 3, Calderón–Zygmund decompositions in our setting are investigated. The proof of Theorem 1.3, which is presented distinguishing the cases $r = \infty$ and $r < \infty$, is included in Section 4. Finite atomic decompositions are considered in Section 5 where Theorem 1.6 is proved. In Section 6, we define the singular integral that we consider and prove Theorem 1.7 after showing some auxiliary results.

Throughout this paper, C always denotes a positive constant that can change its value from a line to another one.

2 Maximal Characterizations (Proof of Theorem 1.1)

From now on, for simplicity, we will write $\|\cdot\|_{p(\cdot),q(\cdot)}$ and $\|\cdot\|_{q(\cdot)}$ instead of $\|\cdot\|_{\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ and $\|\cdot\|_{L^{q(\cdot)}(0,\infty)}$, respectively.

First, we establish very useful boundedness results for the anisotropic maximal function M_{HL} on variable exponent Lorentz spaces.

Proposition 2.1 *Assume that $p, q \in \mathbb{P}_1(0,\infty)$. Then the maximal function M_{HL} is bounded from $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ into itself.*

Proof This property can be proved like [24, Theorem 3.12]. Indeed, it is clear that $\|M_{HL}f\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^\infty(\mathbb{R}^n)}$, $f \in L^\infty(\mathbb{R}^n)$. On the other hand, according to [4, p. 14], M_{HL} is bounded from $L^1(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$. Then, proceeding as in the proof of [3, Theorem 3.8, p. 122], we deduce that, for some $C > 0$, $(M_{HL}f)^* \le Cf^{**}$. Since $p, q \in \mathbb{P}_1(0,\infty)$, by taking $\alpha = 1/p - 1/q$ and $\nu = 0$ in [24, Theorem 2.2], we can write

$$\begin{aligned}\|M_{HL}(f)\|_{p(\cdot),q(\cdot)} &\le C\|t^{1/p(\cdot)-1/q(\cdot)}f^{**}\|_{q(\cdot)} \le C\|t^{1/p(\cdot)-1/q(\cdot)}f^{*}\|_{q(\cdot)}\\ &= C\|f\|_{p(\cdot),q(\cdot)}.\end{aligned}$$

Thus, the proof of this proposition is finished. ■

The following vectorial boundedness result for M_{HL} appears as Proposition 1.3 in the introduction.

Proposition 2.2 *Assume that $p, q \in \mathbb{P}_1$. For every $r \in (1,\infty)$, there exists $C > 0$ such that*

$$\Big\|\Big(\sum_{j\in\mathbb{N}}(M_{HL}(f_j))^r\Big)^{1/r}\Big\|_{p(\cdot),q(\cdot)} \le C\Big\|\Big(\sum_{j\in\mathbb{N}}|f_j|^r\Big)^{1/r}\Big\|_{p(\cdot),q(\cdot)}, \tag{2.1}$$

for each sequence $(f_j)_{j\in\mathbb{N}}$ of functions in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Proof According to [6, Prop. 2.6(ii)] the family of anisotropic balls $\{x + B_k\}_{x\in\mathbb{R}^n,\,k\in\mathbb{Z}}$ constitutes a Muckenhoupt basis in $\mathbb{R}^n$. For every $r > 0$, we define the r-power of the space $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, $(\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n))^r$, as follows:

$$\big(\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)\big)^r = \big\{f \text{ measurable in } \mathbb{R}^n : |f|^r \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)\big\},$$

(see [18, p. 67]). By using [15, Lemma 2.3] we deduce that, for every $r > 0$,

$$(\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n))^r = \mathcal{L}^{rp(\cdot),rq(\cdot)}(\mathbb{R}^n).$$

We choose $\beta \in (0,1)$ such that $\beta p, \beta q \in \mathbb{P}_1$. According to [24, Lemma 2.7],

$$(\mathcal{L}^{\beta p(\cdot),\beta q(\cdot)}(\mathbb{R}^n))^* = (\mathcal{L}^{\beta p(\cdot),\beta q(\cdot)}(\mathbb{R}^n))' = \mathcal{L}^{(\beta p(\cdot))',(\beta q(\cdot))'}(\mathbb{R}^n),$$

where the first space represents the associate dual space of $\mathcal{L}^{\beta p(\cdot),\beta q(\cdot)}(\mathbb{R}^n)$ in the Köthe sense (see [39, p. 25]). Since $\beta p, \beta q \in \mathbb{P}_1$, Proposition 2.1 implies that M_{HL} is bounded from $\mathcal{L}^{(\beta p(\cdot))',(\beta q(\cdot))'}(\mathbb{R}^n)$ into itself. According to [18, Corollary 4.8 and Remark 4.9], we conclude that (2.1) holds for every $r \in (1,\infty)$. ■

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As in [4, p. 44] we consider the following maximal functions that will be useful in the sequel. If $K \in \mathbb{Z}$ and $N, L \in \mathbb{N}$, we define for every $f \in S'(\mathbb{R}^n)$:

$$M_\varphi^{0,K,L}(f)(x) = \sup_{k\in\mathbb{Z}, k\le K} |(f * \varphi_k)(x)| \max(1, \rho(A^{-K}x))^{-L}(1 + b^{-k-K})^{-L}, \quad x \in \mathbb{R}^n,$$

$$M_\varphi^{K,L}(f)(x) = \sup_{k\in\mathbb{Z}, k\le K} \sup_{y\in x+B_k} |(f * \varphi_k)(y)| \max(1, \rho(A^{-K}y))^{-L}(1 + b^{-k-K})^{-L}, \quad x \in \mathbb{R}^n,$$

$$T_\varphi^{N,K,L}(f)(x) = \sup_{k\in\mathbb{Z}, k\le K} \sup_{y\in\mathbb{R}^n} \frac{|(f * \varphi_k)(y)|}{\max(1, \rho(A^{-k}(x - y)))^N} \frac{(1 + b^{-k-K})^{-L}}{\max(1, \rho(A^{-K}y))^L}, \quad x \in \mathbb{R}^n,$$

$$M_N^{0,K,L}(f) = \sup_{\varphi\in S_N} M_\varphi^{0,K,L}(f),$$

$$M_N^{K,L}(f) = \sup_{\varphi\in S_N} M_\varphi^{K,L}(f).$$

We will now establish some properties we will need later.

Lemma 2.3 *Let $K \in \mathbb{Z}$, $N, L \in \mathbb{N}$, $r > 0$, and $\varphi \in S(\mathbb{R}^n)$. Then there exists a constant $C > 0$ that does not depend neither on K, L, N, r, nor φ such that, for every $f \in S'(\mathbb{R}^n)$,*

$$\big(T_\varphi^{N,K,L}(f)(x)\big)^r \le C M_{HL}\Big(\big(M_\varphi^{K,L}(f)\big)^r\Big)(x), \quad x \in \mathbb{R}^n.$$

Proof Our proof is inspired by the ideas presented in [43, p. 10].

Let $f \in S'(\mathbb{R}^n)$, $k \in \mathbb{Z}$, $k \le K$ and $x \in \mathbb{R}^n$. Since

$$\big(|(f * \varphi_k)(y)| \max(1, \rho(A^{-K}y))^{-L}(1 + b^{-k-K})^{-L}\big)^r \le \big(M_\varphi^{K,L}(f)\big)^r(z), \quad y \in z + B_k,$$

we can write

$$\big(|(f * \varphi_k)(y)| \max(1, \rho(A^{-K}y))^{-L}(1 + b^{-k-K})^{-L}\big)^r \le \frac{1}{|y + B_k|} \int_{y+B_k} \big(M_\varphi^{K,L}(f)(z)\big)^r dz, \quad y \in \mathbb{R}^n.$$

Suppose that $z \in y + B_k$ and $y \in \mathbb{R}^n$. According to [4, p. 8], we have that

$$\rho(z - x) \le b^\omega\big(\rho(z - y) + \rho(y - x)\big) \le b^{\omega+k}\big(1 + b^{-k}\rho(y - x)\big),$$

where ω is the smallest integer so that $2B_0 \subset B_\omega$. We choose $s \in \mathbb{Z}$ such that $b^{\omega+k}(1 + b^{-k}\rho(y - x)) \in [b^s, b^{s+1})$. Then we get

$$\begin{aligned}
&\big(|(f * \varphi_k)(y)| \max(1, \rho(A^{-K}y))^{-L}(1 + b^{-k-K})^{-L}\big)^r \\
&\le b^\omega\big(1 + b^{-k}\rho(y - x)\big) \frac{1}{b^{\omega+k}(1 + b^{-k}\rho(y - x))} \int_{y+B_k} \big(M_\varphi^{K,L}(f)(z)\big)^r dz \\
&\le b^\omega\big(1 + b^{-k}\rho(y - x)\big) \frac{1}{b^s} \int_{x+B_{s+1}} \big(M_\varphi^{K,L}(f)(z)\big)^r dz \\
&\le 2b^{\omega+1}\big(1 + b^{-k}\rho(y - x)\big)^{Nr} M_{HL}\Big(\big(M_\varphi^{K,L}(f)\big)^r\Big)(x), \quad y \in x + B_k.
\end{aligned}$$

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Hence, we obtain

$$\big(T_\varphi^{N,K,L}(f)(x)\big)^r \le CM_{HL}\Big(\big(M_\varphi^{K,L}(f)\big)^r\Big)(x), \quad x\in\mathbb{R}^n. \qquad \blacksquare$$

According to [4, p. 14], for every $1<p<\infty$, the Hardy–Littlewood maximal function M_{HL} is bounded from $L^p(\mathbb{R}^n)$ into itself. So from Lemma 2.3 we deduce that, for every $1<p<\infty$, there exists $C>0$ such that

$$\|T_\varphi^{N,K,L}(f)\|_{L^p(\mathbb{R}^n)} \le C\|M_\varphi^{K,L}(f)\|_{L^p(\mathbb{R}^n)}, \quad f\in S'(\mathbb{R}^n).$$

This property was proved in [4, Lemma 7.4] by using a different procedure.

Lemma 2.4 *Let $K\in\mathbb{Z}$, $N,L\in\mathbb{N}$, and $\varphi\in S(\mathbb{R}^n)$. Assume that $p,q\in\mathbb{P}_0$. Then*

$$\|T_\varphi^{N,K,L}(f)\|_{p(\cdot),q(\cdot)} \le C\|M_\varphi^{K,L}(f)\|_{p(\cdot),q(\cdot)}, \quad f\in S'(\mathbb{R}^n),$$

where $C>0$ does not depend on (N,K,L,φ).

Proof We choose $r>0$ such that $rp,rq\in\mathbb{P}_1$. Let $f\in S'(\mathbb{R}^n)$. According to [15, Lemma 2.3] and a well-known property of the nondecreasing equimeasurable rearrangement, we get

$$\begin{aligned}\|T_\varphi^{N,K,L}(f)\|_{p(\cdot),q(\cdot)} &= \|t^{\frac{1}{rp(t)}-\frac{1}{rq(t)}}\big([T_\varphi^{N,K,L}(f)]^*(t)\big)^{1/r}\|_{rq(\cdot)}^r\\ &= \|t^{\frac{1}{rp(t)}-\frac{1}{rq(t)}}\Big[\big(T_\varphi^{N,K,L}(f)\big)^{1/r}\Big]^*(t)\|_{rq(\cdot)}^r\\ &= \|\big(T_\varphi^{N,K,L}(f)\big)^{1/r}\|_{rp(\cdot),rq(\cdot)}^r.\end{aligned}$$

From Lemma 2.3 and Proposition 2.1 it follows that

$$\|T_\varphi^{N,K,L}(f)\|_{p(\cdot),q(\cdot)} \le C\|\big(M_\varphi^{K,L}(f)\big)^{1/r}\|_{rp(\cdot),rq(\cdot)}^r = C\|M_\varphi^{K,L}(f)\|_{p(\cdot),q(\cdot)}. \qquad \blacksquare$$

The next two results were established in [4, pp. 45–47] as Lemmas 7.5 and 7.6, respectively.

Lemma 2.5 *For every $N,L\in\mathbb{N}$, there exists $M_0\in\mathbb{N}$ satisfying the following property: if $\varphi\in S(\mathbb{R}^n)$ is such that $\int\varphi(x)dx\ne0$, then there exists $C>0$ such that, for every $f\in S'(\mathbb{R}^n)$ and $K\in\mathbb{N}$,*

$$M_{M_0}^{0,K,L}(f)(x)\le CT_\varphi^{N,K,L}(f)(x), \quad x\in\mathbb{R}^n.$$

Lemma 2.6 *Let $\varphi\in S(\mathbb{R}^n)$. Then for every $M,K\in\mathbb{N}$ and $f\in S'(\mathbb{R}^n)$ there exist $L\in\mathbb{N}$ and $C>0$ such that*

$$M_\varphi^{K,L}(f)(x)\le C\max\big(1,\rho_A(x)\big)^{-M}, \quad x\in\mathbb{R}^n.$$

Actually, L does not depend on $K\in\mathbb{N}$.

Lemma 2.7 *Let $p,q\in\mathbb{P}_0$. There exists $\alpha_0>0$ such that the function g_α defined by*

$$g_\alpha(x)=\big(\max(1,\rho_A(x))\big)^{-\alpha}, \quad x\in\mathbb{R}^n,$$

is in $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, for every $\alpha\ge\alpha_0$.

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Proof Let $\alpha > 0$. According to [4, Lemma 3.2] we have that

$$g_\alpha(x) \le h_\alpha(x) = C \begin{cases} 1 & |x| \le 1, \\ |x|^{-\alpha \ln b / \ln \lambda_+} & |x| > 1, \end{cases}$$

for certain $C > 0$. Here λ_+ is greater than $\max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$ (for instance we can take $\lambda_+ = 2\max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$). N ote that $g_\alpha^* \le h_\alpha^*$.

To simplify we denote $v_n = |B(0,1)|$. We have that

$$\mu_{h_\alpha}(s) = \begin{cases} 0 & s \ge C, \\ v_n (C/s)^{n \ln(\lambda_+)/(\alpha \ln(b))} & s \in (0, C). \end{cases}$$

Then

$$h_\alpha^*(t) = C \begin{cases} 1 & t \in (0, v_n), \\ (v_n/t)^{\alpha \ln(b)/(n \ln(\lambda_+))} & t \ge v_n. \end{cases}$$

Since $q(0) > 0$ and $p(0) > 0$, we have that $\int_0^{v_n} t^{q(t)/p(t)-1} |g_\alpha^*(t)|^{q(t)} dt < \infty$. Also, there exists $\alpha_0 > 0$ such that $\int_{v_n}^{\infty} t^{q(t)/p(t)-1} |g_{\alpha_0}^*(t)|^{q(t)} dt < \infty$, because $p, q \in \mathbb{P}_0$.

Hence, $g_\alpha \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ for every $\alpha \ge \alpha_0$. ■

Lemma 2.8 *Let $p, q \in \mathfrak{P}_0$ and let D be a subset of $\mathbb{R}^n$. Then $\chi_D \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if and only if $|D| < \infty$.*

Proof We have that $(\chi_D)^* = \chi_{[0,|D|)}$. Since $p, q \in \mathfrak{P}_0$, for every $\lambda > 0$,

$$\int_0^\infty \Big(\frac{(\chi_D)^*(t) t^{\frac{1}{p(t)} - \frac{1}{q(t)}}}{\lambda} \Big)^{q(t)} dt = \int_0^{|D|} \frac{t^{-1+q(t)/p(t)}}{\lambda^{q(t)}} dt < \infty,$$

if and only if $|D| < \infty$. ■

Proof of Theorem 1.1 We recall that we are taking $f \in S'(\mathbb{R}^n)$ and $\varphi \in S(\mathbb{R}^n)$ such that $\int \varphi(x) dx \ne 0$. It is clear that for every $N \in \mathbb{N}$,

$$\|M_\varphi^0(f)\|_{p(\cdot),q(\cdot)} \le \|M_\varphi(f)\|_{p(\cdot),q(\cdot)} \le C \|f\|_{H_N^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)}.$$

Hence, (i)⇒ (ii) ⇒ (iii).

Now, we are going to complete the proof. Let M_0 be the value in Lemma 2.5 for $N = L = 0$. Then for a certain $C > 0$,

$$\|M_M^0(g)\|_{p(\cdot),q(\cdot)} \le C \|M_\varphi(g)\|_{p(\cdot),q(\cdot)}, \quad g \in S'(\mathbb{R}^n), \quad \text{and} \quad M \ge M_0. \tag{2.2}$$

Indeed, by Lemma 2.5, there exists $C > 0$ such that

$$M_M^{0,K,0}(g)(x) \le C T_\varphi^{0,K,0}(g)(x), \quad x \in \mathbb{R}^n, \quad g \in S'(\mathbb{R}^n), K \in \mathbb{N}, \quad \text{and} \quad M \ge M_0.$$

Then Lemma 2.4 leads to

$$\|M_M^{0,K,0}(g)\|_{p(\cdot),q(\cdot)} \le C \|M_\varphi^{K,0}(g)\|_{p(\cdot),q(\cdot)}, \quad g \in S'(\mathbb{R}^n), \ K \in \mathbb{N}, \text{ and } M \ge M_0.$$

By using monotone convergence theorem in $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ (see [24, Definition 2.5 v)]) jointly with [15, Lemma 2.3] and by letting $K \to \infty$, we conclude that (2.2) holds.

Our next objective is to see that, for a certain $C > 0$,

$$\|M_\varphi(f)\|_{p(\cdot),q(\cdot)} \le C \|M_\varphi^0(f)\|_{p(\cdot),q(\cdot)}. \tag{2.3}$$

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Note that by combining (2.2), (2.3), and [4, Proposition 3.10], we conclude that (iii) ⇒ (ii) ⇒ (i).

In order to show (2.3), we first note that there exists $L_0 \in \mathbb{N}$ such that $M_\varphi^{K,L_0}(f) \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, for every $K \in \mathbb{N}$. Indeed, we denote by α_0 the constant appearing in Lemma 2.7. According to Lemma 2.6, we can find $L_0 \in \mathbb{N}$ such that, for every $K \in \mathbb{N}$, there exists $C > 0$ for which

$$M_\varphi^{K,L_0}(f)(x) \le C \max\big(1, \rho(x)\big)^{-\alpha_0}, \quad x \in \mathbb{R}^n.$$

Then Lemma 2.7 leads to $M_\varphi^{K,L_0}(f) \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, for each $K \in \mathbb{N}$.

From Lemmas 2.4 and 2.5, we infer that there exist $M_0 \in \mathbb{N}$ and $C_0 > 0$ such that

$$\|M_{M_0}^{0,K,L_0}(f)\|_{p(\cdot),q(\cdot)} \le C_0 \|M_\varphi^{K,L_0}(f)\|_{p(\cdot),q(\cdot)}, \tag{2.4}$$

for every $K \in \mathbb{N}$.

Fix $K_0 \in \mathbb{N}$. We define the set Ω_0 by

$$\Omega_0 = \big\{ x \in \mathbb{R}^n : M_{M_0}^{0,K_0,L_0}(f)(x) \le C_2 M_\varphi^{K_0,L_0}(f)(x) \big\},$$

where $C_2 > 0$ will be specified later.

By using (2.4), [15, Lemma 2.3], and [24, Theorem 2.4] and choosing $r > 1$ such that $rp, rq \in \mathbb{P}_1$, we get

$$\begin{aligned}
&\|M_\varphi^{K_0,L_0}(f)\|_{p(\cdot),q(\cdot)}\\
&\quad= \|t^{\frac{1}{p(t)}-\frac{1}{q(t)}} \big(M_\varphi^{K_0,L_0}(f)\big)^*(t)\|_{q(\cdot)}\\
&\quad= \|t^{\frac{1}{rp(t)}-\frac{1}{rq(t)}} \big([M_\varphi^{K_0,L_0}(f)]^*(t)\big)^{1/r}\|_{rq(\cdot)}^r\\
&\quad= \|t^{\frac{1}{rp(t)}-\frac{1}{rq(t)}} \big([M_\varphi^{K_0,L_0}(f)]^{1/r}\big)^*(t)\|_{rq(\cdot)}^r = \|[M_\varphi^{K_0,L_0}(f)]^{1/r}\|_{rp(\cdot),rq(\cdot)}^r\\
&\quad\le \big(\|[M_\varphi^{K_0,L_0}(f)]^{1/r}\|_{rp(\cdot),rq(\cdot)}^{(1)}\big)^r\\
&\quad\le A_1\Big\{\big(\|[M_\varphi^{K_0,L_0}(f)\chi_{\Omega_0}]^{1/r}\|_{rp(\cdot),rq(\cdot)}^{(1)}\big)^r + \big(\|[M_\varphi^{K_0,L_0}(f)\chi_{\Omega_0^c}]^{1/r}\|_{rp(\cdot),rq(\cdot)}^{(1)}\big)^r\Big\}\\
&\quad\le A_1\Big\{\big(\|[M_\varphi^{K_0,L_0}(f)\chi_{\Omega_0}]^{1/r}\|_{rp(\cdot),rq(\cdot)}^{(1)}\big)^r + \frac{1}{C_2}\big(\|[M_{M_0}^{0,K_0,L_0}(f)]^{1/r}\|_{rp(\cdot),rq(\cdot)}^{(1)}\big)^r\Big\}\\
&\quad\le A_2\Big\{\big(\|[M_\varphi^{K_0,L_0}(f)\chi_{\Omega_0}]^{1/r}\|_{rp(\cdot),rq(\cdot)}\big)^r + \frac{1}{C_2}\big(\|[M_{M_0}^{0,K_0,L_0}(f)]^{1/r}\|_{rp(\cdot),rq(\cdot)}\big)^r\Big\}\\
&\quad\le A_2\Big(\|M_\varphi^{K_0,L_0}(f)\chi_{\Omega_0}\|_{p(\cdot),q(\cdot)} + \frac{1}{C_2}\|M_{M_0}^{0,K_0,L_0}(f)\|_{p(\cdot),q(\cdot)}\Big)\\
&\quad\le A_2\Big(\|M_\varphi^{K_0,L_0}(f)\chi_{\Omega_0}\|_{p(\cdot),q(\cdot)} + \frac{C_0}{C_2}\|M_\varphi^{K_0,L_0}(f)\|_{p(\cdot),q(\cdot)}\Big),
\end{aligned}$$

where $A_1, A_2 > 0$ depend only on p, q, and r. Hence, by taking $C_2 \ge 2C_0A_2$, we obtain

$$\|M_\varphi^{K_0,L_0}(f)\|_{p(\cdot),q(\cdot)} \le 2A_2 \|M_\varphi^{K_0,L_0}(f)\chi_{\Omega_0}\|_{p(\cdot),q(\cdot)},$$

because $M_\varphi^{K_0,L_0}(f) \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

According to [4, (7.16)], we have that

$$M_\varphi^{K_0,L_0}(f)(x) \le C\big[M_{HL}\big(M_\varphi^{0,K_0,L_0}(f)^{1/r}\big)(x)\big]^r, \quad x \in \Omega_0. \tag{2.5}$$

The constant C>0 does not depend on K_0, but it does depend on L_0.

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From (2.5), Proposition 2.1, and [15, Lemma 2.3], we obtain

$$\begin{aligned}\|M_\varphi^{K_0,L_0}(f)\chi_{\Omega_0}\|_{p(\cdot),q(\cdot)} &\le C\Big\|\Big(M_{HL}(M_\varphi^{0,K_0,L_0}(f)^{1/r})\Big)^r\Big\|_{p(\cdot),q(\cdot)}\\ &= C\|M_{HL}(M_\varphi^{0,K_0,L_0}(f)^{1/r})\|_{rp(\cdot),rq(\cdot)}^r\\ &\le C\|M_\varphi^{0,K_0,L_0}(f)^{1/r}\|_{rp(\cdot),rq(\cdot)}^r\\ &= C\|M_\varphi^{0,K_0,L_0}(f)\|_{p(\cdot),q(\cdot)}.\end{aligned}$$

We conclude that

$$\|M_\varphi^{K_0,L_0}(f)\|_{p(\cdot),q(\cdot)} \le C\|M_\varphi^{0,K_0,L_0}(f)\|_{p(\cdot),q(\cdot)}.$$

Again, note that this constant $C>0$ does not depend on K_0 and it depends on L_0.

We have that $M_\varphi^{K,L_0}(f)(x) \uparrow M_\varphi(f)(x)$, as $K\to\infty$, for every $x\in\mathbb{R}^n$, and $M_\varphi^{0,K,L_0}(f)(x)\uparrow M_\varphi^0(f)(x)$, as $K\to\infty$, for every $x\in\mathbb{R}^n$. Hence, the monotone convergence theorem in the $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$-setting ([24, Theorem 2.8 and Definition 2.5, v)], jointly with [15, Lemma 2.3]), leads to

$$\|M_\varphi(f)\|_{p(\cdot),q(\cdot)} \le C\|M_\varphi^0(f)\|_{p(\cdot),q(\cdot)}.$$

Observe that the last inequality says that $M_\varphi(f)\in\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, but the constant $C>0$ depends on f, because L_0 depends also on f.

On the other hand, since $M_\varphi(f)\in\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, $M_\varphi^{K,0}(f)\in\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, for every $K\in\mathbb{N}$. Hence, we can take $L_0=0$ at the beginning of the proof of this part. By proceeding as above we concluded that

$$\|M_\varphi(f)\|_{p(\cdot),q(\cdot)} \le C\|M_\varphi^0(f)\|_{p(\cdot),q(\cdot)},$$

where $C>0$ does not depend on f.

Thus, the proof of the theorem is finished. ■

The last part of this section is dedicated to establishing some properties of the space $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)$.

Proposition 2.9 *Let $p,q\in\mathbb{P}_0$. Then $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)$ is continuously contained in $S'(\mathbb{R}^n)$.*

Proof Let $f\in S'(\mathbb{R}^n)$ and $\varphi\in S(\mathbb{R}^n)$. We define $\lambda_0=|\langle f,\varphi\rangle|$. We can write

$$\lambda_0 = |(f*\varphi)(0)| \le \sup_{z\in x+B_0}|(f*\varphi)(z)| \le M_\varphi(f)(x),\qquad x\in B_0.$$

Then

$$|\{x\in\mathbb{R}^n : M_\varphi(f)(x)>\lambda_0/2\}|\ge 1\quad\text{and}\quad (M_\varphi(f))^*(t)\ge\lambda_0/2,\qquad t\in(0,1).$$

Hence, we get

$$\begin{aligned}\|M_\varphi(f)\|_{p(\cdot),q(\cdot)} &\ge \|t^{\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)}}(M_\varphi(f))^*(t)\chi_{(1/2,1)}(t)\|_{q(\cdot)}\\ &\ge \frac{\lambda_0}{2}\|t^{\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)}}\chi_{(1/2,1)}(t)\|_{q(\cdot)}.\end{aligned}$$

Since $\|t^{\frac{1}{p(\cdot)}-\frac{1}{q(\cdot)}}\chi_{(1/2,1)}(t)\|_{q(\cdot)}>0$, we conclude the desired result. ■

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Proposition 2.10 *Let $p, q \in \mathbb{P}_0$. If $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$, then f is a bounded distribution in $S'(\mathbb{R}^n)$.*

Proof Let $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and $\varphi \in S(\mathbb{R}^n)$. For every $x \in \mathbb{R}^n$, we have that

$$|(f * \varphi)(x)| \le \sup_{z \in y + B_0} |(f * \varphi)(z)| \le M_\varphi(f)(y), \quad y \in x + B_0.$$

By proceeding as in the proof of Proposition 2.9, we deduce that for a certain $C > 0$,

$$|(f * \varphi)(x)| \le C \|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)}, \quad x \in \mathbb{R}^n.$$

Thus, we prove that f is a bounded distribution in $S'(\mathbb{R}^n)$. ∎

Proposition 2.11 *Assume that $p, q \in \mathbb{P}_0$. Then $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ is complete.*

Proof We choose $r \in (0, 1]$ such that $p(\cdot)/r, q(\cdot)/r \in \mathbb{P}_1$. In order to see that $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ is complete, it is sufficient to prove that if $(f_k)_{k\in\mathbb{N}}$ is a sequence in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ such that $\sum_{k\in\mathbb{N}} \|f_k\|^r_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} < \infty$, then the series $\sum_{k\in\mathbb{N}} f_k$ converges in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ (see, for instance, [3, Theorem 1.6, p. 5]). Assume that $(f_k)_{k\in\mathbb{N}}$ is a sequence in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ such that

$$\sum_{k\in\mathbb{N}} \|f_k\|^r_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} < \infty.$$

For every $j \in \mathbb{N}$, we define $F_j = \sum_{k=0}^{j} f_k$. According to [15, Lemma 2.3] and [24, Theorem 2.4], if $j, \ell \in \mathbb{N}$, $j < \ell$, we get

$$\begin{aligned}
&\|F_\ell - F_j\|^r_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} \\
&\quad = \Big\| \sum_{k=j+1}^{\ell} f_k \Big\|^r_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} \le \Big\| \sum_{k=j+1}^{\ell} M_N(f_k) \Big\|^r_{p(\cdot),q(\cdot)} \\
&\quad = \Big\| \Big(\sum_{k=j+1}^{\ell} M_N(f_k) \Big)^r \Big\|_{p(\cdot)/r,q(\cdot)/r} \le \Big\| \sum_{k=j+1}^{\ell} (M_N(f_k))^r \Big\|_{p(\cdot)/r,q(\cdot)/r} \\
&\quad \le \Big\| \sum_{k=j+1}^{\ell} (M_N(f_k))^r \Big\|^{(1)}_{p(\cdot)/r,q(\cdot)/r} \le \sum_{k=j+1}^{\ell} \|(M_N(f_k))^r\|^{(1)}_{p(\cdot)/r,q(\cdot)/r} \\
&\quad \le C \sum_{k=j+1}^{\ell} \|(M_N(f_k))^r\|_{p(\cdot)/r,q(\cdot)/r} = C \sum_{k=j+1}^{\ell} \|f_k\|^r_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}.
\end{aligned}$$

Hence, $(F_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. By Proposition 2.9, $(F_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $S'(\mathbb{R}^n)$. Then there exists $F \in S'(\mathbb{R}^n)$ such that $F_j \to F$, as $j \to \infty$, in $S'(\mathbb{R}^n)$. We have that

$$M_N(F) \le \lim_{j\to\infty} \sum_{k=0}^{j} M_N(f_k).$$

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According to [24, Theorem 2.8 and Definition 2.5 v)], by proceeding as above, we obtain

$$\|M_N(F)\|^r_{p(\cdot),q(\cdot)} \le \Big\|\lim_{j\to\infty}\sum_{k=0}^{j} M_N(f_k)\Big\|^r_{p(\cdot),q(\cdot)} = \lim_{j\to\infty}\Big\|\sum_{k=0}^{j} M_N(f_k)\Big\|^r_{p(\cdot),q(\cdot)}$$
$$\le \sum_{k\in\mathbb{N}} \|(M_N(f_k))^r\|_{p(\cdot)/r,q(\cdot)/r} = C\sum_{k\in\mathbb{N}}\|f_k\|^r_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}.$$

Then $F \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. Also, we have that

$$\|F - \sum_{k=0}^{j} f_k\|^r_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} \le C\sum_{k=j+1}^{\infty}\|f_k\|^r_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}, \quad j\in\mathbb{N}.$$

Hence, $F = \sum_{k\in\mathbb{N}} f_k$ in the sense of convergence in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. ∎

3 A Calderón–Zygmund Decomposition

In this section we study a Calderón–Zygmund decomposition for our anisotropic setting (associated with the matrix dilation A) for a distribution $f \in S'(\mathbb{R}^n)$ satisfying that $|\{x \in \mathbb{R}^n : M_N f(x) > \lambda\}| < \infty$, where $N \in \mathbb{N}$, $N \ge 2$ and $\lambda > 0$. We will use the ideas and results established in [4, Section 5, Chapter I]. Also, we prove new properties involving variable exponent Hardy–Lorentz norms that will be useful in the sequel.

Let $\lambda > 0$, $N \in \mathbb{N}$, $N \ge 2$, and $f \in S'(\mathbb{R}^n)$ such that $|\Omega_\lambda| < \infty$, where

$$\Omega_\lambda = \left\{x \in \mathbb{R}^n : M_N(f)(x) > \lambda\right\}.$$

By the Whitney Lemma ([4, Lemma 2.7]), there exist sequences $(x_j)_{j\in\mathbb{N}} \subset \Omega_\lambda$ and $(\ell_j)_{j\in\mathbb{N}} \subset \mathbb{Z}$ satisfying the following

$$\Omega_\lambda = \bigcup_{j\in\mathbb{N}}(x_j + B_{\ell_j}); \tag{3.1}$$

$$(x_i + B_{\ell_i-\omega}) \cap (x_j + B_{\ell_j-\omega}) = \varnothing, \quad i,j\in\mathbb{N}, i \ne j; \tag{3.2}$$
$$(x_j + B_{\ell_j+4\omega}) \cap {\Omega_\lambda}^c = \varnothing, \quad (x_j + B_{\ell_j+4\omega+1}) \cap {\Omega_\lambda}^c \ne \varnothing, \quad j\in\mathbb{N};$$
$$\text{if } i,j\in\mathbb{N} \text{ and } (x_i + B_{\ell_i+2\omega}) \cap (x_j + B_{\ell_j+2\omega}) \ne \varnothing, \quad \text{then} \quad |\ell_i - \ell_j| \le \omega;$$

$$\sharp\{j\in\mathbb{N} : (x_i + B_{\ell_i+2\omega}) \cap (x_j + B_{\ell_j+2\omega}) \ne \varnothing\} \le L, \quad i\in\mathbb{N}. \tag{3.3}$$

Here, L denotes a nonnegative integer that does not depend on Ω_λ. If $E \subset \mathbb{R}^n$ by $\sharp E$ we represent the cardinality of E.

Assume now that $\theta \in C^\infty(\mathbb{R}^n)$ satisfies that $\operatorname{supp}\theta \subset B_\omega$, $0 \le \theta \le 1$, and $\theta = 1$ on B_0. For every $j \in \mathbb{N}$, we define

$$\theta_j(x) = \theta(A^{-\ell_j}(x - x_j)), \quad x \in \mathbb{R}^n,$$

and, for every $i \in \mathbb{N}$,

$$\zeta_i(x) = \begin{cases}\theta_i(x)/(\sum_{j\in\mathbb{N}}\theta_j(x)) & x \in \Omega_\lambda,\\ 0 & x \in \Omega_\lambda^c.\end{cases}$$

The sequence $\{\zeta_i\}_{i\in\mathbb{N}}$ is a smooth partition of unity associated with the covering $\{x_i + B_{\ell_i+\omega}\}_{i\in\mathbb{N}}$ of Ω_λ.

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Let $i, s \in \mathbb{N}$. By $\mathcal{P}_s$ we denote the linear space of polynomials in $\mathbb{R}^n$ with degree at most s. $\mathcal{P}_s$ is endowed with the norm $\|\cdot\|_{i,s}$ defined by

$$\|P\|_{i,s} = \Big(\frac{1}{\int \zeta_i} \int_{\mathbb{R}^n} |P(x)|^2 \zeta_i(x)dx \Big)^{1/2}, \quad P \in \mathcal{P}_s.$$

Thus, $(\mathcal{P}_s, \|\cdot\|_{i,s})$ is a Hilbert space. We consider the functional $T_{f,i,s}$ on $\mathcal{P}_s$ given by

$$T_{f,i,s}(Q) = \frac{1}{\int \zeta_i} \langle f, Q\zeta_i \rangle, \quad Q \in \mathcal{P}_s.$$

Then $T_{f,i,s}$ is continuous in $(\mathcal{P}_s, \|\cdot\|_{i,s})$, and there exists $P_{f,i,s} \in \mathcal{P}_s$ such that

$$T_{f,i,s}(Q) = \frac{1}{\int \zeta_i} \int_{\mathbb{R}^n} P_{f,i,s}(x)Q(x)\zeta_i(x)dx, \quad Q \in \mathcal{P}_s.$$

To simplify, we write P_i to refer to $P_{f,i,s}$. We define $b_i = (f - P_i)\zeta_i$.

We will find values of s and N for which the series $\sum_{i\in\mathbb{N}} b_i$ converges in $S'(\mathbb{R}^n)$ provided that $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. Then, we define $g = f - \sum_{i\in\mathbb{N}} b_i$.

The representation $f = g + \sum_{i\in\mathbb{N}} b_i$ is known as the Calderón–Zygmund decomposition of f of degree s and height λ associated with $M_N(f)$.

First, note that if $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and $N \in \mathbb{N}$, $N \geq N_0$, then

$$\|\chi_{\{x\in\mathbb{R}^n : M_N(f)(x)>\mu\}}\|_{p(\cdot),q(\cdot)} < \infty$$

for every $\mu > 0$, and by Lemma 2.8, $|\{x \in \mathbb{R}^n : M_N(f)(x) > \mu\}| < \infty$, for every $\mu > 0$. Here, N_0 is the one defined in Theorem 1.1.

Our next objective is to prove that $L^1_{\text{loc}}(\mathbb{R}^n) \cap H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ is a dense subspace of $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. This property will be useful to deal with the proof that every element of $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ can be represented as a sum of a special kind of distributions, so called atoms, which will be developed in the next section.

We need to establish some auxiliary results. First, we prove the absolute continuity of the norm $\|\cdot\|_{p(.),q(.)}$.

Proposition 3.1 *Let $(E_k)_{k\in\mathbb{N}}$ be a sequence of measurable sets satisfying that $E_k \supset E_{k+1}$, $k \in \mathbb{N}$, $|E_1| < \infty$, and $|\cap_{k\in\mathbb{N}} E_k| = 0$. Assume that $p, q \in \mathbb{P}_0$. If $f \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, then*

$$\|f\chi_{E_k}\|_{p(\cdot),q(\cdot)} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty.$$

Proof Let $f \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $k \in \mathbb{N}$. We have that $(f\chi_{E_k})^* \leq f^*$. Then $f\chi_{E_k} \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Moreover, since $|\cap_{k\in\mathbb{N}} E_k| = \lim_{k\to\infty} |E_k| = 0$, for every $t > 0$ there exists $k_0 \in \mathbb{N}$ such that $(f\chi_{E_k})^*(t) = 0$, $k \in \mathbb{N}$, $k \geq k_0$. Hence, for every $t > 0$,

$$t^{\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)}} (f\chi_{E_k})^*(t) \longrightarrow 0, \quad \text{as } k \longrightarrow \infty.$$

By using dominated convergence theorem ([20, Lemma 3.2.8]) jointly with [15, Lemma 2.3] and by taking into account that $q \in \mathbb{P}_0$ and that $f \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, we obtain

$$\|f\chi_{E_k}\|_{p(\cdot),q(\cdot)} \to 0, \quad \text{as } k \to \infty. \qquad \blacksquare$$

Note that the last property also holds by more general exponent functions p and q.

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Proposition 3.2 *Assume that $p, q \in \mathbb{P}_0$. There exists $s_0 \in \mathbb{N}$, such that for every $s \in \mathbb{N}$, $s \geq s_0$, and each $N \in \mathbb{N}$, $N > \max\{N_0, s\}$, where N_0 is defined in Theorem 1.1, the following two properties holds.*

(i) *Let $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and $\lambda > 0$. If $f = g + \sum_{i\in\mathbb{N}} b_i$ is the anisotropic Calderón–Zygmund decomposition of f associated with $M_N f$ of height λ and degree s, then the series $\sum_{i\in\mathbb{N}} b_i$ converges in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.*

(ii) *Suppose that $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and that, for every $j \in \mathbb{Z}$, $f = g_j + \sum_{i\in\mathbb{N}} b_{i,j}$ is the anisotropic Calderón–Zygmund decomposition of f associated with $M_N f$ of height 2^j and degree s. Then $(g_j)_{j\in\mathbb{Z}} \subset H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and $(g_j)_{j\in\mathbb{Z}}$ converges to f, as $j \to +\infty$, in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.*

Proof (i) Let $s, N \in \mathbb{N}$, $N > \max\{N_0, s\}$. The Calderón–Zygmund decomposition of f associated with $M_N f$ of height $\lambda > 0$ and degree s is $f = g + \sum_{i\in\mathbb{N}} b_i$. We are going to specify s and N in order that the series $\sum_{i\in\mathbb{N}} b_i$ converges in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.

By using [4, Lemmas 5.4 and 5.6], we get that there exists $C > 0$ so that, for every $i \in \mathbb{N}$,

$$\begin{aligned}
&M_N(b_i)(x)\\
&\quad \leq C\Big(M_N f(x)\chi_{x_i+B_{\ell_i+2\omega}}(x) + \lambda \sum_{k\in\mathbb{N}} \lambda_-^{-k(s+1)} \chi_{x_i+(B_{\ell_i+2\omega+1+k}\setminus B_{\ell_i+2\omega+k})}(x)\Big), \quad x \in \mathbb{R}^n.
\end{aligned}$$

Let $j, m \in \mathbb{N}$, $j < m$. For every $x \in \mathbb{R}^n$, we infer

$$\begin{aligned}
&M_N\Big(\sum_{i=j}^m b_i\Big)(x)\\
&\quad \leq \sum_{i=j}^m M_N(b_i)(x)\\
&\quad \leq C\Big(M_N f(x) \sum_{i=j}^m \chi_{x_i+B_{\ell_i+2\omega}}(x) + \lambda \sum_{i=j}^m \sum_{k\in\mathbb{N}} \lambda_-^{-k(s+1)} \chi_{x_i+(B_{\ell_i+2\omega+1+k}\setminus B_{\ell_i+2\omega+k})}(x)\Big).
\end{aligned}$$

We also have that, for every $x \in x_i + (B_{\ell_i+2\omega+1+k}\setminus B_{\ell_i+2\omega+k})$, with $i, k \in \mathbb{N}$, $i \leq m$,

$$M_{HL}(\chi_{x_i+B_{\ell_i+2\omega}})(x) \geq \frac{1}{|x_i + B_{\ell_i+2\omega+1+k}|} \int_{x_i+B_{\ell_i+2\omega+1+k}} \chi_{x_i+B_{\ell_i+2\omega}}(y)dy = b^{-k-1}.$$

We choose $r > 1$ such that $rp, rq \in \mathbb{P}_1$. Then we take $s \in \mathbb{N}$ such that $\lambda_-^{-s} b^r \leq 1$ and $N_0 < s$. For every $i \in \mathbb{N}$, $i \leq m$, we get

$$\begin{aligned}
&\sum_{k=0}^{\infty} \lambda_-^{-k(s+1)} \chi_{x_i+(B_{\ell_i+2\omega+1+k}\setminus B_{\ell_i+2\omega+k})}(x)\\
&\quad \leq C \max_{k\in\mathbb{N}} (\lambda_-^{-s-1} b^r)^k \Big(M_{HL}(\chi_{x_i+B_{\ell_i+2\omega}})(x)\Big)^r\\
&\quad \leq C\Big(M_{HL}(\chi_{x_i+B_{\ell_i+2\omega}})(x)\Big)^r, \quad x \in (x_i + B_{\ell_i+2\omega})^c.
\end{aligned}$$

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Hence, we obtain

$$M_N\Big(\sum_{i=j}^{m} b_i\Big)(x) \le C_0\Big(M_N f(x) \sum_{i=j}^{m} \chi_{x_i+B_{\ell_i+2\omega}}(x) + \lambda \sum_{i=j}^{m} \Big(M_{HL}(\chi_{x_i+B_{\ell_i+2\omega}})(x)\Big)^r\Big), \quad x \in \mathbb{R}^n.$$

By using [15, Lemma 2.3], since $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is a quasi Banach space, we obtain

$$(3.4)\qquad \Big\| M_N\Big(\sum_{i=j}^{m} b_i\Big)\Big\|_{p(\cdot),q(\cdot)} \le C\Big(\Big\| M_N(f) \sum_{i=j}^{m} \chi_{x_i+B_{\ell_i+2\omega}}\Big\|_{p(\cdot),q(\cdot)} + \lambda \Big\| \sum_{i=j}^{m} \Big(M_{HL}(\chi_{x_i+B_{\ell_i+2\omega}})\Big)^r \Big\|_{p(\cdot),q(\cdot)}\Big)$$
$$= C\Big(\Big\| M_N(f) \sum_{i=j}^{m} \chi_{x_i+B_{\ell_i+2\omega}}\Big\|_{p(\cdot),q(\cdot)} + \lambda \Big\| \Big(\sum_{i=j}^{m} \Big(M_{HL}(\chi_{x_i+B_{\ell_i+2\omega}})\Big)^r\Big)^{1/r} \Big\|^r_{rp(\cdot),rq(\cdot)}\Big)$$

By using Proposition 2.2, we get

$$\Big\| \Big(\sum_{i=j}^{m} \Big(M_{HL}(\chi_{x_i+B_{\ell_i+2\omega}})\Big)^r\Big)^{1/r} \Big\|^r_{rp(\cdot),rq(\cdot)} \le C \Big\| \Big(\sum_{i=j}^{m} \chi_{x_i+B_{\ell_i+2\omega}}\Big)^{1/r}\Big\|^r_{rp(\cdot),rq(\cdot)} = C \Big\| \sum_{i=j}^{m} \chi_{x_i+B_{\ell_i+2\omega}}\Big\|_{p(\cdot),q(\cdot)}.$$

From (3.3) and (3.4), it follows that

$$\Big\| M_N\Big(\sum_{i=j}^{m} b_i\Big)\Big\|_{p(\cdot),q(\cdot)} \le C\Big(\Big\| M_N(f) \sum_{i=j}^{m} \chi_{x_i+B_{\ell_i+2\omega}}\Big\|_{p(\cdot),q(\cdot)} + \lambda \Big\| \sum_{i=j}^{m} \chi_{x_i+B_{\ell_i+2\omega}}\Big\|_{p(\cdot),q(\cdot)}\Big)$$
$$\le C \Big\| M_N(f) \sum_{i=j}^{m} \chi_{x_i+B_{\ell_i+2\omega}}\Big\|_{p(\cdot),q(\cdot)} \le C \Big\| M_N(f) \chi_{\bigcup_{i=j}^{\infty}(x_i+B_{\ell_i+2\omega})}\Big\|_{p(\cdot),q(\cdot)}.$$

For every $k \in \mathbb{N}$, we define $E_k = \bigcup_{i=k}^{\infty}(x_i + B_{\ell_i+2\omega})$. By (3.3) there exists $C > 0$ such that $\sum_{i=k}^{\infty} \chi_{x_i+B_{\ell_i+2\omega}} \le C\chi_{E_k}$, $k \in \mathbb{N}$. By (3.1) and (3.2), $\bigcup_{i\in\mathbb{N}}(x_i+B_{\ell_i-\omega}) \subset \Omega_\lambda$, and then $\sum_{i\in\mathbb{N}} |x_i + B_{\ell_i-\omega}| = b^{-\omega} \sum_{i\in\mathbb{N}} b^{\ell_i} \le |\Omega_\lambda| < \infty$, where $\Omega_\lambda = \{x \in \mathbb{R}^n : M_N(f)(x) > \lambda\}$. We deduce that

$$|E_k| \le \sum_{i=k}^{\infty} |x_i + B_{\ell_i+2\omega}| = b^{2\omega} \sum_{i=k}^{\infty} b^{\ell_i}, \quad k \in \mathbb{N}.$$

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Proposition 3.1 implies that

$$\lim_{k\to\infty} \|M_N(f)\chi_{E_k}\|_{p(\cdot),q(\cdot)} = 0.$$

Hence, the sequence $\{\sum_{i=0}^{k} b_i\}_{k\in\mathbb{N}}$ is Cauchy in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. Since $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ is complete (Proposition 2.11), the series $\sum_{i\in\mathbb{N}} b_i$ converges in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.

(ii) In order to prove this property, we can proceed as in the proof of (i). Assume that $j \in \mathbb{Z}$. We define $\Omega_j = \{x \in \mathbb{R}^n : M_N f(x) > 2^j\}$. By putting $b_j = \sum_{i\in\mathbb{N}} b_{i,j}$, since, as we have just proved in (i), the last series converges in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and then in $S'(\mathbb{R}^n)$, we obtain, for a chosen $r > 1$ verifying that $rp, rq \in \mathbb{P}_1$,

$$M_N(b_j)(x) \le C_0\Big(M_N f(x)\chi_{\Omega_j}(x) + 2^j \sum_{i\in\mathbb{N}} \big(M_{HL}(\chi_{x_i+B_{\ell_i+2\omega}})(x)\big)^r\Big), \quad x \in \mathbb{R}^n.$$

It follows that

$$\begin{aligned}(3.5)\quad &\big\| M_N(b_j)\big\|_{p(\cdot),q(\cdot)} \\ &\le C\Big(\|M_N(f)\chi_{\Omega_j}\|_{p(\cdot),q(\cdot)} + 2^j \Big\| \Big(\sum_{i\in\mathbb{N}} \big(M_{HL}(\chi_{x_i+B_{\ell_i+2\omega}})\big)^r\Big)^{1/r} \Big\|^r_{rp(\cdot),rq(\cdot)}\Big).\end{aligned}$$

From Proposition 2.2, we get

$$\begin{aligned}\Big\| \Big(\sum_{i\in\mathbb{N}} \big(M_{HL}(\chi_{\{x_i+B_{\ell_i+2\omega}\}})\big)^r\Big)^{1/r} \Big\|^r_{rp(\cdot),rq(\cdot)} &\le C\Big\| \Big(\sum_{i\in\mathbb{N}} \chi_{x_i+B_{\ell_i+2\omega}}\Big)^{1/r}\Big\|^r_{rp(\cdot),rq(\cdot)} \\ &= C\Big\| \sum_{i\in\mathbb{N}} \chi_{x_i+B_{\ell_i+2\omega}}\Big\|_{p(\cdot),q(\cdot)} \\ &\le C\|\chi_{\Omega_j}\|_{p(\cdot),q(\cdot)}.\end{aligned}$$

From (3.5) it follows that

$$\begin{aligned}\|M_N(b_j)\|_{p(\cdot),q(\cdot)} &\le C\big(\|M_N(f)\chi_{\Omega_j}\|_{p(\cdot),q(\cdot)} + 2^j\|\chi_{\Omega_j}\|_{p(\cdot),q(\cdot)}\big) \\ &\le C\|M_N(f)\chi_{\Omega_j}\|_{p(\cdot),q(\cdot)}.\end{aligned}$$

Since $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$, by again invoking [15, Lemma 2.3], we have that

$$\|M_N(f)\|^{1/r}_{p(\cdot),q(\cdot)} = \|(M_N(f))^{1/r}\|_{rp(\cdot),rq(\cdot)} < \infty.$$

Then by [24, Theorem 2.8] (see [24, Definition 2.5 vii)]), $M_N(f)(x) < \infty$, a.e. $x \in \mathbb{R}^n$. Hence, $M_N(f)\chi_{\Omega_j} \downarrow 0$, as $j \to +\infty$, for a.e. $x \in \mathbb{R}^n$. According to Proposition 3.1 we conclude that $\|M_N(f)\chi_{\Omega_j}\|_{p(\cdot),q(\cdot)} \to 0$, as $j \to +\infty$. Hence, $\|M_N(b_j)\|_{p(\cdot),q(\cdot)} \to 0$, as $j \to +\infty$, and $\|f - g_j\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} \to 0$, as $j \to +\infty$. ∎

By $C_c^\infty(\mathbb{R}^n)$ we denote the space of smooth functions with compact support in $\mathbb{R}^n$. We say that a distribution $h \in S'(\mathbb{R}^n)$ is in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ when there exists a (unique) $H \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\langle h, \phi\rangle = \int_{\mathbb{R}^n} H(x)\phi(x)dx, \quad \phi \in C_c^\infty(\mathbb{R}^n).$$

The space $S'(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is also sometimes denoted by $S_r(\mathbb{R}^n)$ and it was studied, for instance, in [21, 56, 57].

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Proposition 3.3 *If $f \in S'(\mathbb{R}^n)$, $\lambda > 0$, $s, N \in \mathbb{N}$, $N \geq 2$, and $s < N$, and $f = g + \sum_{i\in\mathbb{N}} b_i$ is the anisotropic Calderón–Zygmund decomposition of f associated with $M_N(f)$ of height λ and degree s, then $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.*

Proof Let $\lambda > 0$, $N \in \mathbb{N}$, $N \geq 2$, $s \in \mathbb{N}$, $s < N$, and $f \in S'(\mathbb{R}^n)$ such that $|\Omega_\lambda| < \infty$, where $\Omega_\lambda = \{x \in \mathbb{R}^n : M_N(f)(x) > \lambda\}$. We write $f = g + \sum_{i\in\mathbb{N}} b_i$ the Calderón–Zygmund decomposition of f associated with $M_N(f)$ of height λ and degree s.

According to [4, Lemma 5.9] we have that

$$M_N(g)(x) \leq C\lambda \sum_{i\in\mathbb{N}} \lambda_-^{-t_i(s+1)} + M_N(f)(x)\chi_{\Omega_\lambda^c}(x), \quad x \in \mathbb{R}^n,$$

where

$$t_i = t_i(x) = \begin{cases} t & \text{if } x \in x_i + (B_{\ell_i+2\omega+t+1} \backslash B_{\ell_i+2\omega+t}), \text{ for some } t \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

As shown in the proof of [4, Lemma 5.10 (i), p. 34], we get

$$\int_{\mathbb{R}^n} \sum_{i\in\mathbb{N}} \lambda_-^{-t_i(x)(s+1)} dx \leq C|\Omega_\lambda|.$$

Then, since $M_N(f)(x) \leq \lambda$, $x \in \Omega_\lambda^c$, we obtain that $M_N(g) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Let $\varphi \in S(\mathbb{R}^n)$. Since for a certain $C > 0$, we have $g * \varphi_k \leq CM_N(g)$, $k \in \mathbb{N}$, by proceeding as in the proof of [4, Theorem 3.9] we can prove that for every compact subset F of $\mathbb{R}^n$ there exists a sequence $\{k_j\}_{j\in\mathbb{N}} \subset \mathbb{Z}$ such that $k_j \to -\infty$, as $j \to \infty$, and $g * \varphi_{k_j} \to G_F$, as $j \to \infty$, in the weak topology of $L^1(F)$ for a certain $G_F \in L^1(F)$. A diagonal argument allows us to get a sequence $\{k_j\}_{j\in\mathbb{N}} \subset \mathbb{Z}$ such that $k_j \to -\infty$, as $j \to \infty$, and $g * \varphi_{k_j} \to G$, in the weak * topology of $\mathcal{M}(K)$ (the space of complex measures supported in K) for every compact subset K of $\mathbb{R}^n$, being $G \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. According to [4, Lemma 3 .8], $g * \varphi_{k_j} \to g$, as $j \to \infty$ in $S'(\mathbb{R}^n)$. If $\phi \in C_c^\infty(\mathbb{R}^n)$, we have that

$$\langle g, \phi \rangle = \lim_{j\to\infty} \int_{\mathbb{R}^n} (g * \varphi_{k_j})(x)\phi(x)dx = \int_{\mathbb{R}^n} G(x)\phi(x)dx. \tag{3.6}$$

Since $C_c^\infty(\mathbb{R}^n)$ is a dense subspace of $S(\mathbb{R}^n)$, g is characterized by (3.6). ■

Corollary 3.4 *Assume that $p, q \in \mathbb{P}_0$. Then $L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ is a dense subspace of $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.*

Proof This property is a consequence of Propositions 3.2 and 3.3. ■

We finish this section with a convergence property for the good parts of Calderón–Zygmund decomposition of distributions in $L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$, which we will use in the proof of atomic decompositions of the elements of $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.

Proposition 3.5 *Assume that $p, q \in \mathbb{P}_0$, and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. For every $j \in \mathbb{N}$, $f = g_j + \sum_{i\in\mathbb{N}} b_{i,j}$ is the anisotropic Calderón–Zygmund decomposition of f associated with $M_N(f)$ of height 2^j and degree s, with $s, N \in \mathbb{N}$, $s \geq s_0$, and $N > \max\{s, N_0\}$, where N_0 is as in Theorem 1.1 and s_0 is as in Proposition 3.2. Then $g_j \to 0$, as $j \to -\infty$, in $S'(\mathbb{R}^n)$.*

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Proof Since $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, there exists a unique $F \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} F(x)\phi(x)dx, \quad \phi \in C_c^\infty(\mathbb{R}^n).$$

According to Proposition 3.3, for every $j \in \mathbb{Z}$, there exists a unique $G_j \in L^1_{\text{loc}}(\mathbb{R}^n)$ for which

$$\langle g_j, \phi \rangle = \int_{\mathbb{R}^n} G_j(x)\phi(x)dx, \quad \phi \in C_c^\infty(\mathbb{R}^n). \tag{3.7}$$

Let $j \in \mathbb{Z}$ and $\phi \in C_c^\infty(\mathbb{R}^n)$. We are going to see that

$$\sum_{i \in \mathbb{N}} \int_{\mathbb{R}^n} \Big| \big(F(x) - P_{i,j}(x) \big) \zeta_{i,j}(x) \Big| |\phi(x)| dx < \infty.$$

For every $i \in \mathbb{N}$, by [4, Lemma 5.3], we have that

$$\int_{\mathbb{R}^n} |(F(x) - P_{i,j}(x))\zeta_{i,j}(x)||\phi(x)|dx \\ \leq C\Big(\int_{(x_{i,j}+B_{l_{i,j}+\omega}) \cap \operatorname{supp}(\phi)} |F(x)|dx + 2^j |(x_{i,j} + B_{l_{i,j}+\omega}) \cap \operatorname{supp}(\phi)| \Big).$$

Then

$$\sum_{i \in \mathbb{N}} \int_{\mathbb{R}^n} |(F(x) - P_{i,j}(x))\zeta_{i,j}(x)||\phi(x)|dx \leq C\Big(\int_{\operatorname{supp}(\phi)} |F(x)|dx + 2^j |\operatorname{supp}(\phi)| \Big).$$

Hence, from Proposition 3.2(i), we get

$$\begin{aligned} \int_{\mathbb{R}^n} \sum_{i \in \mathbb{N}} (F(x) - P_{i,j}(x))\zeta_{i,j}(x)\phi(x)dx &= \sum_{i \in \mathbb{N}} \int_{\mathbb{R}^n} (F(x) - P_{i,j}(x))\zeta_{i,j}(x)\phi(x)dx \\ &= \sum_{i \in \mathbb{N}} \big\langle (f - P_{i,j})\zeta_{i,j}, \phi \big\rangle \\ &= \Big\langle \sum_{i \in \mathbb{N}} (f - P_{i,j})\zeta_{i,j}, \phi \Big\rangle. \end{aligned}$$

Then there exists a measurable subset $E \subset \mathbb{R}^n$ such that $|\mathbb{R}^n \setminus E| = 0$, and

$$G_j(x) = F(x) - \sum_{i \in \mathbb{N}} (F(x) - P_{i,j}(x))\zeta_{i,j}(x), \quad x \in E \quad \text{and} \quad j \in \mathbb{Z},$$

for a suitable sense of the convergence of series. Note that we have used a diagonal argument to justify the convergence for every $j \in \mathbb{Z}$.

We can write

$$G_j(x) = F(x)\chi_{\Omega_j^c}(x) - \sum_{i \in \mathbb{N}} P_{i,j}(x)\zeta_{i,j}(x), \quad x \in E \quad \text{and} \quad j \in \mathbb{Z},$$

where $\Omega_j = \{x \in \mathbb{R}^n : M_N(f)(x) > 2^j\}$, $j \in \mathbb{Z}$. Note that the last series is actually a finite sum for every $x \in \mathbb{R}^n$.

Let $j \in \mathbb{Z}$. According to [4, Lemma 5.3] we obtain

$$|G_j(x)| \leq C2^j, \quad \text{a.e.} \quad x \in \Omega_j.$$

On the other hand, $G_j(x) = F(x)$, a.e. $x \in \Omega_j^c$. Also, we have that

$$|F| \leq \sup_{k \in \mathbb{Z}, \varphi \in C_c^\infty(\mathbb{R}^n) \cap S_N} |f * \varphi_k| \leq M_N(f).$$

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Then $|G_j(x)| \le C2^j$, a.e. $x \in \Omega_j^c$. Hence, we conclude that

$$|G_j(x)| \le C2^j, \quad \text{a.e.} \quad x \in \mathbb{R}^n. \tag{3.8}$$

We consider the functional T_j defined on $S(\mathbb{R}^n)$ by

$$T_j(\phi) = \int_{\mathbb{R}^n} G_j(x)\phi(x)dx, \quad \phi \in S(\mathbb{R}^n).$$

From (3.8) we deduce that $T_j \in S'(\mathbb{R}^n)$. By (3.7), $T_j(\phi) = \langle g_j, \phi \rangle$, $\phi \in C_c^\infty(\mathbb{R}^n)$. Then

$$\langle g_j, \phi \rangle = \int_{\mathbb{R}^n} G_j(x)\phi(x)dx, \quad \phi \in S(\mathbb{R}^n),$$

and, again from (3.8), it follows that $g_j \to 0$, as $j \to -\infty$, in $S'(\mathbb{R}^n)$. ∎

4 Atomic Characterization (Proof of Theorem 1.3)

As we mentioned in the introduction, we are going to prove Theorem 1.3 in two steps, first in the case where $r = \infty$ and then when $r < \infty$.

4.1 Proof of Theorem 1.3 when $r = \infty$.

(i) Suppose that for every $j \in \mathbb{N}$, a_j is a $(p(\cdot), q(\cdot), \infty, s)$-atom associated with $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$. Here, $s \in \mathbb{N}$ will be fixed later. Assume also that $(\lambda_j)_{j\in\mathbb{N}} \subset (0,\infty)$ and that

$$\Big\| \sum_{j\in\mathbb{N}} \frac{\lambda_j}{\|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}} \chi_{x_j+B_{\ell_j}} \Big\|_{p(\cdot),q(\cdot)} < \infty.$$

We are going to show that the series $\sum_{j\in\mathbb{N}} \lambda_j a_j$ converges in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. Let $\ell, m \in \mathbb{N}$, $\ell < m$. We define $f_{\ell,m} = \sum_{j=\ell}^m \lambda_j a_j$, and we take $\varphi \in S(\mathbb{R}^n)$. We have that

$$\begin{aligned} &\|M_\varphi(f_{\ell,m})\|_{p(\cdot),q(\cdot)} \\ &\quad\le \Big\| \sum_{j=\ell}^m \lambda_j M_\varphi(a_j) \Big\|_{p(\cdot),q(\cdot)} \\ &\quad\le C\bigg(\Big\| \sum_{j=\ell}^m \lambda_j M_\varphi(a_j)\chi_{x_j+B_{\ell_j+\omega}} \Big\|_{p(\cdot),q(\cdot)} + \Big\| \sum_{j=\ell}^m \lambda_j M_\varphi(a_j)\chi_{x_j+B^c_{\ell_j+\omega}} \Big\|_{p(\cdot),q(\cdot)} \bigg) \\ &\quad= I_1 + I_2. \end{aligned} \tag{4.1}$$

We now estimate I_i, $i = 1,2$. We first study I_1. Let $j \in \mathbb{N}$. Since a_j is a $(p(\cdot), q(\cdot), \infty, s)$-atom, we can write

$$M_\varphi(a_j)(x) \le \|a_j\|_\infty \|\varphi\|_1 \le C\|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)}, \quad x \in \mathbb{R}^n.$$

By defining $g_j = \chi_{x_j+B_{\ell_j}} (\|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)}\lambda_j)^\alpha$, it follows that

$$\begin{aligned} M_{HL} g_j(x) &\ge \big(\|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)}\lambda_j \big)^\alpha \frac{1}{|B_{\ell_j+\omega}|} \int_{x_j+B_{\ell_j+\omega}} \chi_{x_j+B_{\ell_j}}(y)dy \\ &= b^{-\omega}\big(\|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)}\lambda_j \big)^\alpha, \quad x \in x_j + B_{\ell_j+\omega}, \end{aligned}$$

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where $\alpha \in (0,1)$ is such that $p(\cdot)/\alpha, q(\cdot)/\alpha \in \mathbb{P}_1$. According to Proposition 2.2 and [15, Lemma 2.3], we have that

$$I_1 \le C\Big\| \sum_{j=\ell}^{m} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)} \chi_{x_j+B_{\ell_j+\omega}} \Big\|_{p(\cdot),q(\cdot)} \tag{4.2}$$
$$\le C\Big\| \sum_{j=\ell}^{m} (M_{HL} g_j)^{1/\alpha} \Big\|_{p(\cdot),q(\cdot)}$$
$$\le C\Big\| \Big(\sum_{j=\ell}^{m} (M_{HL} g_j)^{1/\alpha} \Big)^{\alpha} \Big\|^{1/\alpha}_{p(\cdot)/\alpha,q(\cdot)/\alpha}$$
$$\le C\Big\| \Big(\sum_{j=\ell}^{m} g_j^{1/\alpha} \Big)^{\alpha} \Big\|^{1/\alpha}_{p(\cdot)/\alpha,q(\cdot)/\alpha}$$
$$= C\Big\| \sum_{j=\ell}^{m} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)} \chi_{x_j+B_{\ell_j}} \Big\|_{p(\cdot),q(\cdot)}.$$

Suppose now that a is a $(p(\cdot),q(\cdot),\infty,s)$-atom associated with $z \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. Let $m \in \mathbb{N}$. By proceeding as in [4, pp. 19–20] we obtain

$$M_\varphi(a)(x)$$
$$\le C \frac{1}{\|\chi_{z+B_k}\|_{p(\cdot),q(\cdot)}} (b\lambda_-^{s+1})^{-m}$$
$$\le C \frac{1}{\|\chi_{z+B_k}\|_{p(\cdot),q(\cdot)}} b^{m(\gamma-1)} \lambda_-^{-m(s+1)} \Big(\frac{1}{|B_{k+m+\omega+1}|} \int_{z+B_{k+m+\omega+1}} \chi_{z+B_k}(y)\,dy \Big)^{\gamma}$$
$$\le C \frac{b^{m(\gamma-1)} \lambda_-^{-m(s+1)}}{\|\chi_{z+B_k}\|_{p(\cdot),q(\cdot)}} \big(M_{HL}(\chi_{z+B_k})(x) \big)^{\gamma}, \quad x \in z + (B_{k+m+\omega+1} \backslash B_{k+m+\omega}).$$

Here γ is chosen such that $\gamma p(\cdot), \gamma q(\cdot) \in \mathbb{P}_1$. We now take $s \in \mathbb{N}$, satisfying that $b^{\gamma-1}\lambda_-^{-(s+1)} \le 1$. We obtain

$$M_\varphi(a)(x) \le C \frac{1}{\|\chi_{z+B_k}\|_{p(\cdot),q(\cdot)}} \big(M_{HL}(\chi_{z+B_k})(x) \big)^{\gamma}, \quad x \notin z + B_{k+\omega}.$$

By proceeding as above, we get

$$I_2 \le C\Big\| \sum_{j=\ell}^{m} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)} \big(M_{HL}(\chi_{x_j+B_{\ell_j}}) \big)^{\gamma} \Big\|_{p(\cdot),q(\cdot)} \tag{4.3}$$
$$= C\Big\| \Big(\sum_{j=\ell}^{m} \big(\lambda_j^{1/\gamma} \|\chi_{x_j+B_{\ell_j}}\|^{-1/\gamma}_{p(\cdot),q(\cdot)} M_{HL}(\chi_{x_j+B_{\ell_j}}) \big)^{\gamma} \Big)^{1/\gamma} \Big\|^{\gamma}_{\gamma p(\cdot),\gamma q(\cdot)}$$
$$\le C\Big\| \sum_{j=\ell}^{m} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)} \chi_{x_j+B_{\ell_j}} \Big\|_{p(\cdot),q(\cdot)}.$$

By combining (4.1), (4.2), and (4.3), we infer that the sequence $(\sum_{j=0}^{k} \lambda_j a_j)_{k\in\mathbb{N}}$ is Cauchy in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. Since $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ is complete (Proposition 2.11),

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the series $\sum_{j\in\mathbb{N}} \lambda_j a_j$ converges in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. Moreover, we get

$$\Big\| \sum_{j\in\mathbb{N}} \lambda_j a_j \Big\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} \leq C \Big\| \sum_{j\in\mathbb{N}} \frac{\lambda_j}{\|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}} \chi_{x_j+B_{\ell_j}} \Big\|_{p(\cdot),q(\cdot)}.$$

(ii) Assume that $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A) \cap L^1_{\rm loc}(\mathbb{R}^n)$, $s \geq s_0$ (s_0 was defined in Proposition 3.2), and $N > \max\{N_0, s\}$ (N_0 was defined in Theorem 1.1). We recall that $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A) \cap L^1_{loc}(\mathbb{R}^n)$ is a dense subspace of $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ (Corollary 3.4). Let $j \in \mathbb{Z}$. We define $\Omega_j = \{x \in \mathbb{R}^n : M_N(f)(x) > 2^j\}$. According to [4, Chapter 1, Section 5] we can write $f = g_j + \sum_{k\in\mathbb{N}} b_{j,k}$, that is, the Calderón–Zygmund decomposition of degree s and height 2^j associated with $M_N f$. The properties of g_j and $b_{j,k}$ will be specified when we need each of them.

As proved in Proposition 3.2(ii), $g_j \to f$, as $j \to +\infty$, in both $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and $S'(\mathbb{R}^n)$, and in Proposition 3.5, $g_j \to 0$, as $j \to -\infty$, in $S'(\mathbb{R}^n)$. We have that

$$f = \sum_{j\in\mathbb{Z}} (g_{j+1} - g_j), \quad \text{in} \quad S'(\mathbb{R}^n).$$

As in [4, p. 38] we can write, for every $j \in \mathbb{Z}$,

$$g_{j+1} - g_j = \sum_{i\in\mathbb{N}} h_{i,j}, \quad \text{in} \quad S'(\mathbb{R}^n),$$

where

$$h_{i,j} = (f - P_i^j)\zeta_i^j - \sum_{k\in\mathbb{N}} \Big((f - P_k^{j+1})\zeta_i^j - P_{i,k}^{j+1} \Big) \zeta_k^{j+1}, \quad i \in \mathbb{N}.$$

According to the properties of the polynomials P's and the functions ζ's it follows that, for every $P \in \mathcal{P}_s$,

$$\int_{\mathbb{R}^n} h_{i,j}(x) P(x)\, dx = 0, \quad i, j \in \mathbb{N}.$$

We also have that, for certain $C_0 > 0$, $\|h_{i,j}\|_\infty \leq C_0 2^j$ and $\operatorname{supp} h_{i,j} \subset x_{i,j} + B_{\ell_{i,j}+4\omega}$, for every $i, j \in \mathbb{N}$ ([4, (6.12) and (6.13), p. 38]). Hence, for every $i, j \in \mathbb{N}$, the function $a_{i,j} = h_{i,j} 2^{-j} C_0^{-1} \|\chi_{x_{i,j}+B_{\ell_{i,j}+4\omega}}\|^{-1}_{p(\cdot),q(\cdot)}$ is a $(p(\cdot), q(\cdot), \infty, s)$-atom. Moreover,

$$f = \sum_{i\in\mathbb{N}, j\in\mathbb{Z}} \lambda_{i,j} a_{i,j} \quad \text{in} \quad S'(\mathbb{R}^n), \tag{4.4}$$

where $\lambda_{i,j} = 2^j C_0 \|\chi_{x_{i,j}+B_{\ell_{i,j}+4\omega}}\|_{p(\cdot),q(\cdot)}$, for every $i \in \mathbb{N}$, $j \in \mathbb{Z}$.

We are going to explain the convergence of the double series in (4.4).

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We now choose $\beta > 1$ such that $\beta p, \beta q \in \mathbb{P}_1$. Assume that $\pi = (\pi_1, \pi_2): \mathbb{N} \to \mathbb{N} \times \mathbb{Z}$ is a bijection. By proceeding as before we get, for every $k \in \mathbb{N}$,

$$\begin{aligned}
&\Big\| \sum_{m=0}^{k} \frac{\lambda_{\pi(m)}}{\|\chi_{x_{\pi(m)}+B_{\ell_{\pi(m)}+4\omega}}\|_{p(\cdot),q(\cdot)}} \chi_{x_{\pi(m)}+B_{\ell_{\pi(m)}+4\omega}} \Big\|_{p(\cdot),q(\cdot)} \\
&\quad \le C \Big\| \sum_{m=0}^{k} 2^{\pi_2(m)} \chi_{x_{\pi(m)}+B_{\ell_{\pi(m)}+4\omega}} \Big\|_{p(\cdot),q(\cdot)} \\
&\quad \le C \Big\| \sum_{m=0}^{k} \big(2^{\pi_2(m)/\beta} \chi_{x_{\pi(m)}+B_{\ell_{\pi(m)}+4\omega}} \big)^{\beta} \Big\|_{p(\cdot),q(\cdot)} \\
&\quad \le C \Big\| \sum_{m=0}^{k} \big(2^{\pi_2(m)/\beta} M_{HL}(\chi_{x_{\pi(m)}+B_{\ell_{\pi(m)}+2\omega}}) \big)^{\beta} \Big\|_{p(\cdot),q(\cdot)} \\
&\quad = C \Big\| \Big(\sum_{m=0}^{k} \big(M_{HL}(2^{\pi_2(m)/\beta} \chi_{x_{\pi(m)}+B_{\ell_{\pi(m)}+2\omega}}) \big)^{\beta} \Big)^{1/\beta} \Big\|_{\beta p(\cdot),\beta q(\cdot)}^{\beta} \\
&\quad \le C \Big\| \Big(\sum_{m=0}^{k} 2^{\pi_2(m)} \chi_{x_{\pi(m)}+B_{\ell_{\pi(m)}+2\omega}} \Big)^{1/\beta} \Big\|_{\beta p(\cdot),\beta q(\cdot)}^{\beta} \\
&\quad \le C \Big\| \sum_{j\in\mathbb{Z}} 2^{j} \sum_{i\in\mathbb{N}} \chi_{x_{i,j}+B_{\ell_{i,j}+2\omega}} \Big\|_{p(\cdot),q(\cdot)} \\
&\quad \le C \Big\| \sum_{j\in\mathbb{Z}} 2^{j} \chi_{\Omega_j} \Big\|_{p(\cdot),q(\cdot)}.
\end{aligned}$$

Since $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$, by [24, Thm. 2.8 and Def. 2.5 vii)], $M_N(f)(x) < \infty$, a.e. $x \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ such that $M_N(f)(x) < \infty$. There exists $j_0 \in \mathbb{Z}$ such that $2^{j_0} < M_N(f)(x) \le 2^{j_0+1}$. We have that

$$\sum_{j\in\mathbb{Z}} 2^{j} \chi_{\Omega_j}(x) = \sum_{j\le j_0} 2^{j} = 2^{j_0+1} \le 2M_N(f)(x).$$

We conclude that

$$\begin{aligned}
&\Big\| \sum_{m=0}^{k} \frac{\lambda_{\pi(m)}}{\|\chi_{x_{\pi(m)}+B_{\ell_{\pi(m)}+4\omega}}\|_{p(\cdot),q(\cdot)}} \chi_{x_{\pi(m)}+B_{\ell_{\pi(m)}+4\omega}} \Big\|_{p(\cdot),q(\cdot)} \\
&\quad = \Big\| \Big(\sum_{m=0}^{k} \frac{\lambda_{\pi(m)}}{\|\chi_{x_{\pi(m)}+B_{\ell_{\pi(m)}+4\omega}}\|_{p(\cdot),q(\cdot)}} \chi_{x_{\pi(m)}+B_{\ell_{\pi(m)}+4\omega}} \Big)^{1/\beta} \Big\|_{\beta p(\cdot),\beta q(\cdot)}^{\beta} \\
&\quad \le C \|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)},
\end{aligned}$$

where $C > 0$ does not depend on (k, π).

According to [24, Theorem 2.8 and Definition 2.5, v)], we deduce that

$$\Big\| \sum_{m\in\mathbb{N}} \frac{\lambda_{\pi(m)}}{\|\chi_{x_{\pi(m)}+B_{\ell_{\pi(m)}+4\omega}}\|_{p(\cdot),q(\cdot)}} \chi_{x_{\pi(m)}+B_{\ell_{\pi(m)}+4\omega}} \Big\|_{p(\cdot),q(\cdot)} \le C \|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}.$$

From the property we have just established in part (i) of this proof, we deduce that the series $\sum_{m\in\mathbb{N}} \lambda_{\pi(m)} a_{\pi(m)}$ converges both in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and $S'(\mathbb{R}^n)$. Hence, for every $\phi \in S(\mathbb{R}^n)$, the series $\sum_{m\in\mathbb{N}} \lambda_{\pi(m)} \langle a_{\pi(m)}, \phi \rangle$ converges in $\mathbb{C}$.

Also, we have that if $\Lambda\colon \mathbb{N}\times\mathbb{Z}\to\mathbb{N}\times\mathbb{Z}$ is a bijection, then the series

$$\sum_{m\in\mathbb{N}} \lambda_{\Lambda\circ\pi(m)}\langle a_{\Lambda\circ\pi(m)},\phi\rangle$$

converges in $\mathbb{C}$, for every $\phi\in S(\mathbb{R}^n)$. In other words, the series $\sum_{m\in\mathbb{N}}\lambda_{\pi(m)}\langle a_{\pi(m)},\phi\rangle$ converges unconditionally in $\mathbb{C}$, for every $\phi\in S(\mathbb{R}^n)$. Hence,

$$\sum_{m\in\mathbb{N}} \lambda_{\pi(m)}|\langle a_{\pi(m)},\phi\rangle|<\infty,$$

for every $\phi\in S(\mathbb{R}^n)$.

Let $\phi\in S(\mathbb{R}^n)$. Since $\sum_{m\in\mathbb{N}}\lambda_{\pi(m)}|\langle a_{\pi(m)},\phi\rangle|<\infty$, the double series

$$\sum_{(i,j)\in\mathbb{N}\times\mathbb{Z}} \lambda_{i,j}\langle a_{i,j},\phi\rangle$$

is summable, that is, $\sup_{m\in\mathbb{N}}\sum_{1\le i\le m,\,|j|\le m}\lambda_{i,j}|\langle a_{i,j},\phi\rangle|<\infty$. Then for every bijection $\pi\colon\mathbb{N}\to\mathbb{N}\times\mathbb{Z}$, we have that

$$\langle f,\phi\rangle=\sum_{i\in\mathbb{N}}\Big(\sum_{j\in\mathbb{Z}}\lambda_{i,j}\langle a_{i,j},\phi\rangle\Big)=\sum_{m\in\mathbb{N}}\lambda_{\pi(m)}\langle a_{\pi(m)},\phi\rangle.$$

Suppose now that $f\in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)$. Then there exists a sequence $\{f_j\}_{j\in\mathbb{N}}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)\cap H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)$ such that $f_1=0$, $f_j\to f$, as $j\to\infty$, in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)$, and $\|f_{j+1}-f_j\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}<2^{-j}\|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}$, for every $j\in\mathbb{N}$. Then we can write

$$f=\sum_{j\in\mathbb{N}}(f_{j+1}-f_j),$$

in the sense of convergence in both $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)$ and $S'(\mathbb{R}^n)$. For every $j\in\mathbb{N}$, there exist a sequence $\{\lambda_{i,j}\}_{i\in\mathbb{N}}\subset(0,\infty)$ and a sequence $\{a_{i,j}\}_{i\in\mathbb{N}}$ of $(p(\cdot),q(\cdot),\infty,s)$-atoms, being for every $i\in\mathbb{N}$, $a_{i,j}$ associated with $x_{i,j}\in\mathbb{R}^n$ and $\ell_{i,j}\in\mathbb{Z}$, satisfying that

$$f_{j+1}-f_j=\sum_{i\in\mathbb{N}}\lambda_{i,j}a_{i,j},\quad \text{in } S'(\mathbb{R}^n),$$

and

$$\Big\|\sum_{i\in\mathbb{N}}\frac{\lambda_{i,j}}{\|\chi_{x_{i,j}+B_{\ell_{i,j}}}\|_{p(\cdot),q(\cdot)}}\chi_{x_{i,j}+B_{\ell_{i,j}}}\Big\|_{p(\cdot),q(\cdot)}\le C2^{-j}\|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}.$$

Here, $C>0$ does not depend on f.

We have that

$$\begin{aligned}&\Big\|\sum_{i\in\mathbb{N},j\in\mathbb{Z}}\frac{\lambda_{i,j}}{\|\chi_{x_{i,j}+B_{\ell_{i,j}}}\|_{p(\cdot),q(\cdot)}}\chi_{x_{i,j}+B_{\ell_{i,j}}}\Big\|_{p(\cdot),q(\cdot)}\\&\quad\le\sum_{j\in\mathbb{Z}}\Big\|\sum_{i\in\mathbb{N}}\frac{\lambda_{i,j}}{\|\chi_{x_{i,j}+B_{\ell_{i,j}}}\|_{p(\cdot),q(\cdot)}}\chi_{x_{i,j}+B_{\ell_{i,j}}}\Big\|_{p(\cdot),q(\cdot)}\\&\quad\le C\|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}.\end{aligned}$$

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By proceeding as above we can write

$$f = \sum_{m \in \mathbb{N}} \lambda_{\pi(m)} a_{\pi(m)}, \quad \text{in } S'(\mathbb{R}^n),$$

for every bijection $\pi \colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

Thus, the proof of this case is completed. ∎

4.2 Proof of Theorem 1.3 when $r < \infty$.

In order to prove this property we proceed in a series of steps establishing auxiliary and partial results.

Proposition 4.1 *Let $1 < r < \infty$ and let $p, q \in \mathbb{P}_0$. There exists $s_0 \in \mathbb{N}$ satisfying that if $s \in \mathbb{N}$, $s \geq s_0$, we can find $C > 0$ for which, for every $f \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$, there exist, for each $j \in \mathbb{N}$, $\lambda_j > 0$ and a $(p(\cdot), q(\cdot), r, s)$-atom a_j associated with some $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$, such that*

$$\Big\| \sum_{j \in \mathbb{N}} \lambda_j \| \chi_{x_j + B_{\ell_j}} \|^{-1}_{p(\cdot),q(\cdot)} \chi_{x_j + B_{\ell_j}} \Big\|_{p(\cdot),q(\cdot)} \leq C \|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)}$$

and $f = \sum_{j \in \mathbb{N}} \lambda_j a_j$ in $S'(\mathbb{R}^n)$.

Proof Suppose that a is a $(p(\cdot), q(\cdot), \infty, s)$-atom associated with $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. We have that

$$\|a\|_r = \Big(\int_{x_0 + B_k} |a(x)|^r dx \Big)^{1/r} \leq b^{k/r} \|a\|_\infty \leq b^{k/r} \| \chi_{x_0 + B_k} \|^{-1}_{p(\cdot),q(\cdot)}.$$

Hence, a is a $(p(\cdot), q(\cdot), r, s)$-atom associated with $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. Then this property follows from the previous case $r = \infty$. ∎

We are going to see that the $(p(\cdot), q(\cdot), r, s)$-atoms are in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.

Proposition 4.2 *Let $p, q \in \mathbb{P}_0$ such that $p(0) < q(0)$. Assume that $\max\{1, q_+\} < r < \infty$. There exists $s_0 \in \mathbb{N}$ such that if a is a $(p(\cdot), q(\cdot), r, s_0)$-atom, then $a \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.*

Proof Let $\varphi \in S(\mathbb{R}^n)$. Assume that a is a $(p(\cdot), q(\cdot), r, s)$-atom associated with $x_0 \in \mathbb{R}^n$ and $\ell_0 \in \mathbb{Z}$, where $s \in \mathbb{N}$ will be specified later. We have that

$$\begin{aligned} \|M_\varphi(a)\|_{p(\cdot),q(\cdot)} &\leq C \big(\|M_\varphi(a) \chi_{x_0 + B_{\ell_0 + w}}\|_{p(\cdot),q(\cdot)} \\ &\qquad + \|M_\varphi(a) \chi_{(x_0 + B_{\ell_0 + w})^c}\|_{p(\cdot),q(\cdot)} \big) \\ &= I_1 + I_2. \end{aligned}$$

It is clear that

$$(M_\varphi(a) \chi_{x_0 + B_{\ell_0 + w}})^*(t) = 0 \quad \text{for} \quad t \geq |x_0 + B_{\ell_0 + w}| = b^{\ell_0 + w}.$$

Then since $0 < p(0) = \lim_{t \to 0^+} p(t) < q(0) = \lim_{t \to 0^+} q(t)$, we can write

$$I_1 \leq C \Big\| t^{1/p(t) - 1/q(t)} (M_\varphi(a))^* \chi_{(0, b^{\ell_0 + w})} \Big\|_{q(\cdot)} \leq C \Big\| (M_\varphi(a))^* \chi_{(0, b^{\ell_0 + w})} \Big\|_{q(\cdot)}.$$

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By using [15, Lemma 2.2] and since $r > \max\{1, q_+\}$, we obtain

$$I_1 \le C\|(M_\varphi(a))^*\|_{L^r(0,\infty)} = C\|M_\varphi(a)\|_{L^r(\mathbb{R}^n)} \le C\|a\|_{L^r(\mathbb{R}^n)} \le Cb^{\ell_0/r}\|\chi_{x_0+B_{\ell_0}}\|^{-1}_{p(\cdot),q(\cdot)} < \infty.$$

By proceeding as in the proof of the case $r = \infty$ (see [4, pp. 19–21]) we get

$$M_\varphi(a)(x) \le \frac{C}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot),q(\cdot)}}(M_{HL}(\chi_{x_0+B_{\ell_0}})(x))^\gamma, \quad x \notin x_0 + B_{\ell_0+w},$$

provided that $s \ge \frac{\gamma-1}{\log_b(\lambda_-)} - 1$, where $\gamma > 1$ is such that $\gamma p, \gamma q \in \mathbb{P}_1$. Then Proposition 2.1 implies that

$$I_2 \le C\frac{\|(M_{HL}(\chi_{x_0+B_{\ell_0}}))^\gamma\|_{p(\cdot),q(\cdot)}}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot),q(\cdot)}} = C\frac{\|M_{HL}(\chi_{x_0+B_{\ell_0}})\|^\gamma_{\gamma p(\cdot),\gamma q(\cdot)}}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot),q(\cdot)}} \le C.$$

Thus, we have shown that $a \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$. ∎

Note that the constant C in the proof of the last proposition depends on the atom a. This fact indicates that this next result cannot be a consequence of Proposition 4.2. We need a more involved argument to show the following property.

Proposition 4.3 *Let $p, q \in \mathbb{P}_0$ with $p(0) < q(0)$. There exist $s_0 \in \mathbb{N}$ and $r_0 > 1$ such that, for every $r \ge r_0$ we can find $C > 0$ satisfying that if, for every $j \in \mathbb{N}$, $\lambda_j > 0$ and a_j is a $(p(\cdot), q(\cdot), r, s_0)$-atom associated with $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$ such that*

$$\sum_{j\in\mathbb{N}} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)}\chi_{x_j+B_{\ell_j}} \in \mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n),$$

then $f = \sum_{j\in\mathbb{N}} \lambda_j a_j \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ and

$$\|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} \le C\Big\|\sum_{j\in\mathbb{N}} \lambda_j\|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)}\chi_{x_j+B_{\ell_j}}\Big\|_{p(\cdot),q(\cdot)}.$$

In order to prove this proposition we need to establish some preliminary properties.

Lemma 4.4 *Assume that $(\lambda_k)_{k\in\mathbb{N}}$ is a sequence in $(0,\infty)$, $(\ell_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{Z}$, $(x_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{R}^n$, ν is a doubling weight, that is, νdx is a doubling measure, (with respect to the anisotropic balls), $\ell \in \mathbb{N}$, $\ell \ge 1$, and $0 < p < \infty$. Then*

$$\Big\|\sum_{k\in\mathbb{N}} \lambda_k\chi_{x_k+B_{\ell_k+\ell}}\Big\|_{L^p(\mathbb{R}^n,\nu)} \le Cb^{\ell\delta}\Big\|\sum_{k\in\mathbb{N}} \lambda_k\chi_{x_k+B_{\ell_k}}\Big\|_{L^p(\mathbb{R}^n,\nu)}. \tag{4.5}$$

Here, $C, \delta > 0$ depends only on ν.

Proof Suppose first that $p > 1$. We follow the ideas in the proof of [55, Theorem 2, p. 53]. We take $0 \le g \in L^{p'}(\mathbb{R}^n, \nu)$, where p' is the exponent conjugated to p, that is, $p' = p/(p-1)$. Let $y \in \mathbb{R}^n$ and $k \in \mathbb{Z}$. We define the maximal operator M_ν by

$$M_\nu(h)(z) = \sup_{m\in\mathbb{Z},\, y\in z+B_m} \frac{1}{\nu(y+B_m)}\int_{y+B_m} |h(x)|\nu(x)dx, \quad z \in \mathbb{R}^n.$$

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Since v is doubling with respect to the anisotropic balls, for a certain $\delta > 0$, we have that

$$\int_{y+B_{k+\ell}} g(x)v(x)dx \le b^{\ell\delta} \frac{v(y+B_k)}{v(y+B_{k+\ell})} \int_{y+B_{k+\ell}} g(x)v(x)dx \\ \le b^{\ell\delta} \int_{y+B_k} M_v(g)(x)v(x)dx, \quad y \in \mathbb{R}^n,\ k \in \mathbb{Z}.$$

We have taken into account that

$$M_v(g)(z) \ge \frac{1}{v(y+B_{\ell+k})} \int_{y+B_{\ell+k}} g(x)v(x)dx, \quad z \in y+B_k.$$

Let $m \in \mathbb{N}$. We can write

$$\int_{\mathbb{R}^n} \sum_{k=0}^m \lambda_k \chi_{x_k+B_{\ell+\ell_k}}(x)g(x)v(x)dx = \sum_{k=0}^m \lambda_k \int_{x_k+B_{\ell+\ell_k}} g(x)v(x)dx \\ \le b^{\ell\delta} \sum_{k=0}^m \lambda_k \int_{x_k+B_{\ell_k}} M_v(g)(x)v(x)dx.$$

Hence, the maximal theorem [55, Theorem 3, p. 3] leads to

$$\Big| \int_{\mathbb{R}^n} \sum_{k=0}^m \lambda_k \chi_{x_k+B_{\ell+\ell_k}}(x)g(x)v(x)dx \Big| \\ \le b^{\ell\delta} \Big\| \sum_{k=0}^m \lambda_k \chi_{x_k+B_{\ell_k}} \Big\|_{L^p(\mathbb{R}^n,v)} \|M_v(g)\|_{L^{p'}(\mathbb{R}^n,v)} \\ \le Cb^{\ell\delta} \Big\| \sum_{k=0}^m \lambda_k \chi_{x_k+B_{\ell_k}} \Big\|_{L^p(\mathbb{R}^n,v)} \|g\|_{L^{p'}(\mathbb{R}^n,v)}.$$

We conclude that

$$\Big\| \sum_{k=0}^m \lambda_k \chi_{x_k+B_{\ell+\ell_k}}(x) \Big\|_{L^p(\mathbb{R}^n,v)} \le Cb^{\ell\delta} \Big\| \sum_{k=0}^m \lambda_k \chi_{x_k+B_{\ell_k}} \Big\|_{L^p(\mathbb{R}^n,v)}.$$

By taking $m \to \infty$, the monotone convergence theorem allows us to establish (4.5) in this case.

Assume now that $0 < p \le 1$. For every $x_0 \in \mathbb{R}^n$ and $k_0 \in \mathbb{Z}$, we denote by $\delta_{(x_0,k_0)}$ the Dirac measure in $\mathbb{R}^{n+1}$ supported in (x_0, k_0). Let $m \in \mathbb{N}$. We have that

$$\int_{x\in y+B_{\ell+j}} \sum_{k=0}^m \lambda_k \delta_{(x_k,\ell_k)}(y,j) \\ = \sum_{k=0}^m \lambda_k \int_{\mathbb{R}^{n+1}} \chi_{\{(y,j):x\in y+B_{\ell+j}\}}(y,j)\delta_{(x_k,\ell_k)}(y,j) \\ = \sum_{k=0}^m \lambda_k \chi_{\{(y,j):x\in y+B_{\ell+j}\}}(x_k,\ell_k) = \sum_{k=0}^m \lambda_k \chi_{x_k+B_{\ell+\ell_k}}(x), \quad x \in \mathbb{R}^n.$$

Also, we can write

$$\int_{x\in y+B_j} \sum_{k=0}^m \lambda_k \delta_{(x_k,\ell_k)}(y,j) = \sum_{k=0}^m \lambda_k \chi_{x_k+B_{\ell_k}}(x), \quad x \in \mathbb{R}^n.$$

By arguing as in the proof of [55, Theorem 1, p. 52], replacing the area Littlewood–Paley functions by our area integrals, we can prove that

$$\Big\| \int_{x\in y+B_{\ell+j}} \sum_{k=0}^{m} \lambda_k \delta_{(x_k,\ell_k)}(y,j) \Big\|_{L^p(\mathbb{R}^n,v)} \leq Cb^{\ell\delta} \Big\| \int_{x\in y+B_j} \sum_{k=0}^{m} \lambda_k \delta_{(x_k,\ell_k)}(y,j) \Big\|_{L^p(\mathbb{R}^n,v)}.$$

By letting $m\to\infty$ and using monotone convergence theorem we conclude (4.5). ∎

We now recall definitions of anisotropic $\mathcal{A}_r$-weights and anisotropic weighted Hardy spaces (see [6,55]).

Let $r\in(1,\infty)$ and v be a nonnegative measurable function on $\mathbb{R}^n$. The function v is said to be a weight in the anisotropic Muckenhoupt class $\mathcal{A}_r(\mathbb{R}^n,A)$ when

$$[v]_{\mathcal{A}_r(\mathbb{R}^n,A)} =: \sup_{x\in\mathbb{R}^n,\,k\in\mathbb{Z}} \Big(\frac{1}{|B_k|}\int_{x+B_k} v(y)dy\Big)\Big(\frac{1}{|B_k|}\int_{x+B_k}(v(y))^{-1/(r-1)}dy\Big)^{r-1} < \infty.$$

We say that v belongs to the anisotropic Muckenhoupt class $\mathcal{A}_1(\mathbb{R}^n,A)$ when

$$[v]_{\mathcal{A}_1(\mathbb{R}^n,A)} =: \sup_{x\in\mathbb{R}^n,\,k\in\mathbb{Z}} \Big(\frac{1}{|B_k|}\int_{x+B_k} v(y)dy\Big) \sup_{y\in x+B_k}\big(v(y)\big)^{-1} < \infty.$$

We define $\mathcal{A}_\infty(\mathbb{R}^n,A) = \bigcup_{1\leq r<\infty}\mathcal{A}_r(\mathbb{R}^n,A)$.

The weight v satisfies the reverse Hölder condition $RH_r(\mathbb{R}^n,A)$ (in short, $v\in RH_r(\mathbb{R}^n,A)$) if there exists $C>0$ such that

$$\Big(\frac{1}{|B_k|}\int_{x+B_k}(v(y))^r dy\Big)^{1/r} \leq C\frac{1}{|B_k|}\int_{x+B_k} v(y)dy, \quad x\in\mathbb{R}^n, \quad \text{and} \quad k\in\mathbb{Z}.$$

The classes $\mathcal{A}_r(\mathbb{R}^n,A)$ and $RH_\alpha(\mathbb{R}^n,A)$ are closely connected. In particular, if $v\in\mathcal{A}_1(\mathbb{R}^n,A)$, there exists $\alpha\in(1,\infty)$ such that $v\in RH_\alpha(\mathbb{R}^n,A)$ ([37, Theorem 1.3]).

Let $1\leq r<\infty$ and $v\in\mathcal{A}_r(\mathbb{R}^n,A)$. For every $N\in\mathbb{N}$, the anisotropic Hardy space $H^r_N(\mathbb{R}^n,v,A)$ consists of all those $f\in S'(\mathbb{R}^n)$ such that $M_N(f)\in L^r(\mathbb{R}^n,v)$. There exists $N_{r,v}\in\mathbb{N}$ satisfying that $H^r_N(\mathbb{R}^n,v,A) = H^r_{N_{r,v}}(\mathbb{R}^n,v,A)$, for every $N\geq N_{r,v}$. Moreover, when $N\geq N_{r,v}$, the quantities $\|M_N(f)\|_{L^r(\mathbb{R}^n,v)}$ and $\|M_{N_{r,v}}(f)\|_{L^r(\mathbb{R}^n,v)}$ are equivalent, for every $f\in H^r_{N_{r,v}}(\mathbb{R}^n,v,A)$. We denote by $H^r(\mathbb{R}^n,v,A)$ to the space $H^r_{N_{r,v}}(\mathbb{R}^n,v,A)$.

By proceeding as in the proof of [55, Lemma 5, p. 116], we can obtain the following property.

Lemma 4.5 *Let $p\in(0,\infty)$ and $q>\max\{1,p\}$. Assume that $v\in RH_{(q/p)'}(\mathbb{R}^n,A)$. Then there exists $C>0$ such that if, for every $k\in\mathbb{N}$, the measurable function a_k has its support contained in the ball $x_k+B_{\ell_k}$, where $x_k\in\mathbb{R}^n$, $\ell_k\in\mathbb{Z}$, $\|a_k\|_q\leq\|\chi_{x_k+B_{\ell_k}}\|_q$, and $\lambda_k>0$, we have that*

$$\Big\|\sum_{k\in\mathbb{N}}\lambda_k a_k\Big\|_{L^p(\mathbb{R}^n,v)} \leq C\Big\|\sum_{k\in\mathbb{N}}\lambda_k\chi_{x_k+B_{\ell_k}}\Big\|_{L^p(\mathbb{R}^n,v)}.$$

If $1<r\leq\infty$ and $N\in\mathbb{N}$, we say that a function $a\in L^r(\mathbb{R}^n)$ is a (r,N)-atom associated with $x_0\in\mathbb{R}^n$ and $j_0\in\mathbb{Z}$, when a satisfies the following properties:

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(i) $\operatorname{supp} a \subset x_0 + B_{j_0}$,
(ii) $\|a\|_r \le b^{j_0/r}$,
(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$, for all $|\alpha| \le N$, $\alpha \in \mathbb{N}^n$.

The next result is an anisotropic version of the second part of [55, Theorem 1, p. 112].

Lemma 4.6 *Let $0 < p < \infty$. Assume that $v \in RH_{(q/p)'}(\mathbb{R}^n, A)$ where $q > \max\{1, p\}$. There exists $N_1 \in \mathbb{N}$ and $C > 0$ such that if, for every $k \in \mathbb{N}$, a_k is a (q, N_1)-atom associated with $x_k \in \mathbb{R}^n$ and $\ell_k \in \mathbb{Z}$, and $\lambda_k > 0$, satisfying that*

$$\Big\| \sum_{k=1}^{\infty} \lambda_k \chi_{x_k + B_{\ell_k}} \Big\|_{L^p(\mathbb{R}^n, v)} < \infty,$$

the series $\sum_{k=1}^{\infty} \lambda_k a_k$ converges in both $S'(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n, v, A)$ to an element $f \in H^p(\mathbb{R}^n, v, A)$ such that

$$\|f\|_{H^p(\mathbb{R}^n, v, A)} \le C \Big\| \sum_{k=1}^{\infty} \lambda_k \chi_{x_k + B_{\ell_k}} \Big\|_{L^p(\mathbb{R}^n, v)}.$$

Proof Suppose that a is a (q, N)-atom associated with $x_0 \in \mathbb{R}^n$ and $\ell_0 \in \mathbb{Z}$. Here $N \in \mathbb{N}$ will be specified later.

We choose $\varphi \in S(\mathbb{R}^n)$. We now estimate $\|M_\varphi(a)\|_{L^q(\mathbb{R}^n)}$ by considering in a separate way the regions $x_0 + B_{\ell_0 + w}$ and $(x_0 + B_{\ell_0 + w})^c$.

Since $q > 1$ the maximal theorem ([4, Theorem 3.6]) implies that

$$\Big(\int_{\mathbb{R}^n} \chi_{x_0 + B_{\ell_0 + w}}(x) |M_\varphi(a)(x)|^q dx \Big)^{1/q} \le \|M_\varphi(a)\|_{L^q(\mathbb{R}^n)} \le C b^{\ell_0/q} \le C b^{(\ell_0 + w)/q}.$$

Hence, the function $\beta_0 = \frac{1}{C} \chi_{x_0 + B_{\ell_0 + w}} M_\varphi(a)$ is a $(q, -1)$-atom associated with x_0 and $\ell_0 + w$. The index -1 means that no null moment condition needs to be satisfied.

By proceeding as in [4, p. 20] we get, for every $m \in \mathbb{N}$,

$$M_\varphi(a)(x) \le C (b \lambda_-^{N+1})^{-m}, \quad x \in x_0 + (B_{\ell_0 + w + m + 1} \setminus B_{\ell_0 + w + m}).$$

We define $\rho_m = \chi_{x_0 + B_{\ell_0 + w + m + 1}}$, $m \in \mathbb{N}$. It is clear that ρ_m is a $(q, -1)$-atom associated with x_0 and $\ell_0 + w + m + 1$, for every $m \in \mathbb{N}$, and that

$$\chi_{(x_0 + B_{\ell_0 + w})^c} M_\varphi(a) \le C \sum_{m \in \mathbb{N}} (b \lambda_-^{N+1})^{-m} \rho_m.$$

Hence, we obtain

$$M_\varphi(a) \le C \Big(\beta_0 + \sum_{m \in \mathbb{N}} (b \lambda_-^{N+1})^{-m} \rho_m \Big). \tag{4.6}$$

Here, $C > 0$ does not depend on a.

Suppose that $k \in \mathbb{N}$ and, for every $j \in \mathbb{N}$, $j \le k$, $\lambda_j > 0$ and a_j is a (q, N)-atom associated with $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$. According to (4.6) we get

$$M_\varphi \Big(\sum_{j=0}^{k} \lambda_j a_j \Big) \le C \Big(\sum_{j=0}^{k} \lambda_j \big(\beta_{0,j} + \sum_{m=0}^{\infty} (b \lambda_-^{N+1})^{-m} \rho_{m,j} \big) \Big),$$

where $\beta_{0,j}$ and $\rho_{m,j}$, $j = 1, \dots, k$, and $m \in \mathbb{N}$ have the obvious meaning and are $(q, -1)$-atoms. By using Lemmas 4.4 and 4.5, and by taking $p_1 = \min\{1, p\}$ we have

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that

$$\begin{aligned}
&\Big\|\sum_{j=0}^{k}\lambda_j a_j\Big\|_{H^p(\mathbb{R}^n,\nu,A)}^{p_1}\\
&\quad\le C\Big\|\sum_{j=0}^{k}\lambda_j\Big(\beta_{0,j}+\sum_{m\in\mathbb{N}}(b\lambda_-^{N+1})^{-m}\rho_{m,j}\Big)\Big\|_{L^p(\mathbb{R}^n,\nu)}^{p_1}\\
&\quad\le C\Big(\sum_{m\in\mathbb{N}}(b\lambda_-^{N+1})^{-mp_1}\Big\|\sum_{j=0}^{k}\lambda_j\rho_{m,j}\Big\|_{L^p(\mathbb{R}^n,\nu)}^{p_1}+\Big\|\sum_{j=0}^{k}\lambda_j\beta_{0,j}\Big\|_{L^p(\mathbb{R}^n,\nu)}^{p_1}\Big)\\
&\quad\le C\Big(\sum_{m\in\mathbb{N}}(b\lambda_-^{N+1})^{-mp_1}\Big\|\sum_{j=0}^{k}\lambda_j\chi_{x_j+B_{\ell_{j+w+m+1}}}\Big\|_{L^p(\mathbb{R}^n,\nu)}^{p_1}+\Big\|\sum_{j=0}^{k}\lambda_j\chi_{x_j+B_{\ell_j}}\Big\|_{L^p(\mathbb{R}^n,\nu)}^{p_1}\Big)\\
&\quad\le C\Big(\sum_{m\in\mathbb{N}}(b\lambda_-^{N+1})^{-mp_1}b^{\delta mp_1}+1\Big)\Big\|\sum_{j=0}^{k}\lambda_j\chi_{x_j+B_{\ell_{j+w}}}\Big\|_{L^p(\mathbb{R}^n,\nu)}^{p_1},
\end{aligned}$$

for a certain $\delta>0$. Hence, if $(\delta-1)\ln b/\ln(\lambda_-)<N+1$, we conclude that

$$\Big\|\sum_{j=0}^{k}\lambda_j a_j\Big\|_{H^p(\mathbb{R}^n,\nu,A)}\le C\Big\|\sum_{j=0}^{k}\lambda_j\chi_{x_j+B_{\ell_j}}\Big\|_{L^p(\mathbb{R}^n,\nu)}.$$

Standard arguments allow us to finish the proof of this property. ∎

From Lemma 4.6 we can deduce the following.

Lemma 4.7 *Assume that $p,q\in\mathbb{P}_0$, $p_0\in(0,\infty)$, $q_0>\max\{1,p_0\}$ and $\nu\in\mathcal{A}_1(\mathbb{R}^n,A)\bigcap RH_{(q_0/p_0)'}(\mathbb{R}^n,A)$. Suppose that, for every $k\in\mathbb{N}$, $\lambda_k>0$ and a_k is a $(p(\cdot),q(\cdot),q_0,N_1)$-atom associated with $x_k\in\mathbb{R}^n$ and $\ell_k\in\mathbb{Z}$, satisfying that*

$$\Big\|\sum_{k\in\mathbb{N}}\lambda_k\|\chi_{x_k+B_{\ell_k}}\|_{p(\cdot),q(\cdot)}^{-1}\chi_{x_k+B_{\ell_k}}\Big\|_{L^{p_0}(\mathbb{R}^n,\nu)}<\infty.$$

Here, N_1 is the one defined in Lemma 4.6.

Then the series $f=\sum_{k\in\mathbb{N}}\lambda_k a_k$ converges in $H^{p_0}(\mathbb{R}^n,\nu,A)$ and

$$\|f\|_{H^{p_0}(\mathbb{R}^n,\nu,A)}\le C\Big\|\sum_{k\in\mathbb{N}}\lambda_k\|\chi_{x_k+B_{\ell_k}}\|_{p(\cdot),q(\cdot)}^{-1}\chi_{x_k+B_{\ell_k}}\Big\|_{L^{p_0}(\mathbb{R}^n,\nu)}.$$

Here C does not depend on $\{\lambda_k\}_{k\in\mathbb{N}}$ and $\{a_k\}_{k\in\mathbb{N}}$.

Proof It is sufficient to note that, for every $k\in\mathbb{N}$, $a_k\|\chi_{x_k+B_{\ell_k}}\|_{p(\cdot),q(\cdot)}$ is a (q_0,N_1)-atom and ν is doubling with respect to anisotropic balls. ∎

Proof of Proposition 4.3 We choose $\alpha>1$ such that $\alpha p,\alpha q\in\mathbb{P}_1$, so we have $(\alpha p)',(\alpha q)'\in\mathbb{P}_1$. We recall that the dual space $(\mathcal{L}^{\alpha p(\cdot),\alpha q(\cdot)}(\mathbb{R}^n))^*$ of $\mathcal{L}^{\alpha p(\cdot),\alpha q(\cdot)}(\mathbb{R}^n)$ is $\mathcal{L}^{(\alpha p(\cdot))',(\alpha q(\cdot))'}(\mathbb{R}^n)$ and the maximal operator M_{HL} is bounded from $\mathcal{L}^{(\alpha p(\cdot))',(\alpha q(\cdot))'}(\mathbb{R}^n)$ into itself (Proposition 2.1).

In the sequel our argument is (as in [15]) supported in Rubio de Francia iteration algorithm. Given a function h we define $M_{HL}^0(h)=|h|$ and, for every $i\in\mathbb{N}$, $i\ge 1$,

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$M^i_{HL}(h) = M_{HL} \circ M^{i-1}_{HL}(h)$. We consider

$$R(h) = \sum_{i=0}^{\infty} \frac{M^i_{HL}(h)}{2^i \|M_{HL}\|^i_{(\alpha p(\cdot))',(\alpha q(\cdot))'}}.$$

We have that

(i) $|h| \le R(h)$;

(ii) R is bounded from $\mathcal{L}^{(\alpha p(\cdot))',(\alpha q(\cdot))'}(\mathbb{R}^n)$ into itself and

$$\|R(h)\|_{(\alpha p(\cdot))',(\alpha q(\cdot))'} \le 2\|h\|_{(\alpha p(\cdot))',(\alpha q(\cdot))'};$$

(iii) $R(h) \in \mathcal{A}_1(\mathbb{R}^n, A)$ and $[R(h)]_{\mathcal{A}_1(\mathbb{R}^n,A)} \le 2\|M_{HL}\|_{(\alpha p(\cdot))',(\alpha q(\cdot))'}$. Hence, there exists $\beta_0 > 1$ such that $R(h) \in RH_{\beta_0}(\mathbb{R}^n, A)$.

We choose $r > \max\{1, q_+\}$ such that $R(h) \in RH_{(r\alpha)'}(\mathbb{R}^n, A)$. It is sufficient to take $r > \max\{1, q_+, \beta_0/(\alpha(\beta_0 - 1))\}$.

Suppose that $k \in \mathbb{N}$ and, for every $j \in \mathbb{N}$, $j \le k$, $\lambda_j > 0$ and a_j is a $(p(\cdot), q(\cdot), r, N_1)$-atom associated with $x_j \in \mathbb{R}^n$ and $\ell_j \in \mathbb{Z}$. Here, N_1 is the one defined in Lemma 4.6. We define $f_k = \sum_{j=0}^k \lambda_j a_j$. According to Proposition 4.2, $f_k \in H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$.

By $R(h) \in \mathcal{A}_1(\mathbb{R}^n, A) \cap RH_{(r\alpha)'}(\mathbb{R}^n, A)$ and Lemma 4.7, $f_k \in H^{1/\alpha}(\mathbb{R}^n, R(h), A)$ and

$$\|f_k\|_{H^{1/\alpha}(\mathbb{R}^n,R(h),A)} \le C\Big\| \sum_{j=0}^{k} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)} \chi_{x_j+B_{\ell_j}} \Big\|_{L^{1/\alpha}(\mathbb{R}^n,R(h))}. \tag{4.7}$$

Let $\varphi \in S(\mathbb{R}^n)$. By using [15, Lemma 2.3] and [24, Lemma 2.7], we can write

$$\begin{aligned}\|M_\varphi(f_k)\|^{1/\alpha}_{p(\cdot),q(\cdot)} &= \|(M_\varphi(f_k))^{1/\alpha}\|_{\alpha p(\cdot),\alpha q(\cdot)} \\ &\le C \sup_h \int_{\mathbb{R}^n} (M_\varphi(f_k)(x))^{1/\alpha} h(x)dx,\end{aligned}$$

where the supremum is taken over all the functions $0 \le h \in \mathcal{L}^{(\alpha p(\cdot))',(\alpha q(\cdot))'}(\mathbb{R}^n)$ such that $\|h\|_{(\alpha p(\cdot))',(\alpha q(\cdot))'} \le 1$.

By the above properties (i), (ii), and (iii) and (4.7), for every

$$0 \le h \in \mathcal{L}^{(\alpha p(\cdot))',(\alpha q(\cdot))'}(\mathbb{R}^n)$$

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such that $\|h\|_{(\alpha p(\cdot))',(\alpha q(\cdot))'} \le 1$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} (M_\varphi(f_k)(x))^{1/\alpha} h(x)dx \\
&\quad \le \int_{\mathbb{R}^n} (M_\varphi(f_k)(x))^{1/\alpha} R(h)(x)dx \\
&\quad \le C \int_{\mathbb{R}^n} \Big(\sum_{j=0}^{k} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)} \chi_{x_j+B_{\ell_j}}(x) \Big)^{1/\alpha} R(h)(x)dx \\
&\quad \le C \Big\| \Big(\sum_{j=0}^{k} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)} \chi_{x_j+B_{\ell_j}} \Big)^{1/\alpha} \Big\|_{\alpha p(\cdot),\alpha q(\cdot)} \|R(h)\|_{(\alpha p(\cdot))',(\alpha q(\cdot))'} \\
&\quad \le C \Big\| \sum_{j=0}^{k} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)} \chi_{x_j+B_{\ell_j}} \Big\|^{1/\alpha}_{p(\cdot),q(\cdot)} \|h\|_{(\alpha p(\cdot))',(\alpha q(\cdot))'} \\
&\quad \le C \Big\| \sum_{j=0}^{k} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)} \chi_{x_j+B_{\ell_j}} \Big\|^{1/\alpha}_{p(\cdot),q(\cdot)}.
\end{aligned}$$

Hence, we obtain

$$\|f_k\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} \le C \Big\| \sum_{j=0}^{k} \lambda_j \|\chi_{x_j+B_{\ell_j}}\|^{-1}_{p(\cdot),q(\cdot)} \chi_{x_j+B_{\ell_j}} \Big\|_{p(\cdot),q(\cdot)}.$$

We finish the proof by using standard arguments. ∎

5 Finite Atomic Decomposition (Proof of Theorem 1.6)

The proof of this result follows the ideas developed in [6,45]. Here we only show those points where a variable exponent Lorentz space norm appears.

(i) Assume that $r_0 < r < \infty$ and $s \in \mathbb{N}$, $s \ge s_0$, r_0 and s_0 being the parameters appearing in Theorem 1.3(i). By using this result, we get that

$$H^{p(\cdot),q(\cdot),r,s}_{\text{fin}}(\mathbb{R}^n, A) \subset H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$$

and, for every $f \in H^{p(\cdot),q(\cdot),r,s}_{\text{fin}}(\mathbb{R}^n, A)$,

$$\|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} \le C\|f\|_{H^{p(\cdot),q(\cdot),r,s}_{\text{fin}}(\mathbb{R}^n,A)}.$$

We now prove that there exists $C > 0$ such that $\|f\|_{H^{p(\cdot),q(\cdot),r,s}_{\text{fin}}(\mathbb{R}^n,A)} \le C$, provided that

$$f \in H^{p(\cdot),q(\cdot),r,s}_{\text{fin}}(\mathbb{R}^n, A) \quad \text{and} \quad \|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} = 1.$$

Let $f \in H^{p(\cdot),q(\cdot),r,s}_{\text{fin}}(\mathbb{R}^n, A)$ such that $\|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)} = 1$. We have that $f \in L^r(\mathbb{R}^n)$ and $\operatorname{supp} f \subset B_{m_0}$ for some $m_0 \in \mathbb{Z}$. For every $j \in \mathbb{Z}$, we define the set $\Omega_j = \{x \in \mathbb{R}^n : M_N(f)(x) > 2^j\}$, where $N \in \mathbb{N}$, $N > \max\{N_0, s\}$ (here N_0 is as in Theorem 1.1). According to the proof of Theorem 1.3 and [6, p. 3088], for every $i \in \mathbb{N}$ and $j \in \mathbb{Z}$ there exist $\lambda_{i,j} > 0$ and a $(p(\cdot), q(\cdot), \infty, s)$-atom $a_{i,j}$ satisfying the following properties:

(a) $f = \sum_{i,j} \lambda_{i,j} a_{i,j}$, where the series converges unconditionally in $S'(\mathbb{R}^n)$.

(b) $|\lambda_{i,j} a_{i,j}| \le C2^j$, $i \in \mathbb{N}$ and $j \in \mathbb{Z}$;

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and for certain sequences $\{x_{i,j}\}_{i\in\mathbb{N},j\in\mathbb{Z}}\subset\mathbb{R}^n$ and $\{\ell_{i,j}\}_{i\in\mathbb{N},j\in\mathbb{Z}}\subset\mathbb{Z}$,

(c) $\operatorname{supp}(a_{i,j})\subset x_{i,j}+B_{\ell_{i,j}+4\omega}$;
(d) $\Omega_j=\bigcup_{i\in\mathbb{N}}(x_{i,j}+B_{\ell_{i,j}+4\omega})$;
(e) $(x_{i,j}+B_{\ell_{i,j}-2\omega})\cap(x_{k,j}+B_{\ell_{k,j}-2\omega})=\varnothing$, $j\in\mathbb{Z}$, $i,k\in\mathbb{N}$, $i\neq k$;
(f) $$\Big\|\sum_{i\in\mathbb{N},j\in\mathbb{Z}}\lambda_{i,j}\|\chi_{x_{i,j}+B_{\ell_{i,j}+4\omega}}\|^{-1}_{p(\cdot),q(\cdot)}\chi_{x_{i,j}+B_{\ell_{i,j}+4\omega}}\Big\|_{p(\cdot),q(\cdot)}\le C\|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}=C.$$

The constants C in (b) and (f) do not depend on f.

By using (b), (c), (d), and (e), we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{N}}|\lambda_{i,j}a_{i,j}(x)|dx
&\le C\sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{N}}2^j|x_{i,j}+B_{\ell_{i,j}+4\omega}|\\
&\le C\sum_{j\in\mathbb{Z}}2^j\sum_{i\in\mathbb{N}}|x_{i,j}+B_{\ell_{i,j}-2\omega}|=C\sum_{j\in\mathbb{Z}}2^j\Big|\bigcup_{i\in\mathbb{N}}(x_{i,j}+B_{\ell_{i,j}+4\omega})\Big|\\
&\le C\sum_{j\in\mathbb{Z}}2^j|\Omega_j|=C\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}2^j\chi_{\Omega_j}(x)dx\le C\int_{\mathbb{R}^n}M_N(f)(x)dx.\end{aligned}$$

Note that $M_N(f)\in L^1(\mathbb{R}^n)$, because f is a multiple of a $(1,r,s)$-atom. Let $\pi=(\pi_1,\pi_2):\mathbb{N}\to\mathbb{N}\times\mathbb{Z}$ be a bijection. We have that $\int_{\mathbb{R}^n}\sum_{m\in\mathbb{N}}|\lambda_{\pi(m)}a_{\pi(m)}(x)|dx<\infty$. Then there exist a monotone function $\mu:\mathbb{N}\to\mathbb{N}$ and a subset $E\subset\mathbb{R}^n$ such that $\sum_{m\in\mathbb{N}}|\lambda_{\pi(\mu(m))}a_{\pi(\mu(m))}(x)|<\infty$, for every $x\in E$ and $|\mathbb{R}^n\backslash E|=0$. Hence, $\sum_{m\in\mathbb{N}}|\lambda_{\pi(m)}a_{\pi(m)}(x)|<\infty$, for every $x\in E$. Since the last series has positive terms, we conclude that the series $\sum_{m\in\mathbb{N}}\lambda_{\pi(m)}a_{\pi(m)}(x)$ is unconditionally convergent, for every $x\in E$, and $\sum_{m\in\mathbb{N}}\lambda_{\pi(m)}a_{\pi(m)}(x)=\sum_{j\in\mathbb{Z}}(\sum_{i\in\mathbb{N}}\lambda_{i,j}a_{i,j}(x))$, $x\in E$. Moreover, the arguments in the proof of Theorem 1.3(ii) (see also [6, pp. 3088–3089]) lead us to

$$f(x)=\sum_{j\in\mathbb{Z}}\Big(\sum_{i\in\mathbb{N}}\lambda_{i,j}a_{i,j}(x)\Big),\quad x\in E.$$

We have that

$$M_N(f)(x)\le C_1\|\chi_{B_{m_0}}\|^{-1}_{p(\cdot),q(\cdot)},\quad x\in(B_{m_0+4\omega})^c. \tag{5.1}$$

Indeed, let $x\in(B_{m_0+4\omega})^c$. It was proved in [6, pp. 3092–3093] that

$$M_N(f)(x)\le C\inf_{u\in B_{m_0}}M_N(f)(u).$$

Then we obtain

$$\begin{aligned}M_N(f)(x)&\le\frac{C}{\|\chi_{B_{m_0}}\|_{p(\cdot),q(\cdot)}}\Big\|\inf_{u\in B_{m_0}}[M_N(f)(u)]\chi_{B_{m_0}}\Big\|_{p(\cdot),q(\cdot)}\\
&\le\frac{C}{\|\chi_{B_{m_0}}\|_{p(\cdot),q(\cdot)}}\|M_N(f)\chi_{B_{m_0}}\|_{p(\cdot),q(\cdot)}\\
&\le\frac{C}{\|\chi_{B_{m_0}}\|_{p(\cdot),q(\cdot)}}\|M_N(f)\|_{p(\cdot),q(\cdot)}\\
&\le C_1\|\chi_{B_{m_0}}\|^{-1}_{p(\cdot),q(\cdot)}.\end{aligned}$$

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Thus, (5.1) is established.

We now choose $j_0 \in \mathbb{Z}$ such that $2^{j_0} < C_1 \|\chi_{B_{m_0+4\omega}}\|^{-1}_{p(\cdot),q(\cdot)} \le 2^{j_0+1}$, where C_1 is the constant appearing in (5.1). We have that

$$\Omega_j \subset B_{m_0+4\omega}, \qquad j > j_0.$$

By following the ideas developed in [45] (see also [6]), we define

$$h = \sum_{j\le j_0} \sum_{i\in\mathbb{N}} \lambda_{i,j} a_{i,j} \quad \text{and} \quad \mathfrak{l} = \sum_{j>j_0} \sum_{i\in\mathbb{N}} \lambda_{i,j} a_{i,j}.$$

Note that the series converges unconditionally in $S'(\mathbb{R}^n)$ and almost everywhere. We have that $\bigcup_{j>j_0} \Omega_j \subset B_{m_0+4\omega}$. Then $\operatorname{supp} \mathfrak{l} \subset B_{m_0+4\omega}$. Since $\operatorname{supp} f \subset B_{m_0+4\omega}$, we also have that $\operatorname{supp} h \subset B_{m_0+4\omega}$. As above, we can see that

$$\int_{\mathbb{R}^n} \sum_{j>j_0} \sum_{i\in\mathbb{N}} |\lambda_{i,j} a_{i,j}(x) x^\alpha| dx = \int_{B_{m_0+4\omega}} \sum_{j>j_0} \sum_{i\in\mathbb{N}} |\lambda_{i,j} a_{i,j}(x) x^\alpha| dx < \infty.$$

Then $\int_{\mathbb{R}^n} \mathfrak{l}(x) x^\alpha dx = 0$, for every $\alpha \in \mathbb{N}^n$ and $|\alpha| \le s$. Since $\int_{\mathbb{R}^n} f(x) x^\alpha dx = 0$, for every $\alpha \in \mathbb{N}^n$ and $|\alpha| \le s$, we also have that $\int_{\mathbb{R}^n} h(x) x^\alpha dx = 0$, for every $\alpha \in \mathbb{N}^n$ and $|\alpha| \le s$.

Moreover, by using (3.3) we get

$$|h(x)| \le C \sum_{j\le j_0} 2^j \le C 2^{j_0} \le C_2 \|\chi_{B_{m_0+4\omega}}\|^{-1}_{p(\cdot),q(\cdot)}.$$

Here C_2 does not depend on f. Hence, h/C_2 is a $(p(\cdot), q(\cdot), \infty, s)$-atom associated with the ball $B_{m_0+4\omega}$.

As in [6, Step 4, p. 3094], we can see that if

$$F_J = \{(i,j) : i \in \mathbb{N},\ j \in \mathbb{Z},\ j > j_0 \text{ and } i + |j| \le J\} \quad \text{and} \quad \mathfrak{l}_J = \sum_{(i,j)\in F_J} \lambda_{i,j} a_{i,j},$$

for every $J \in \mathbb{N}$ such that $J > |j_0|$, then $\lim_{J\to+\infty} \mathfrak{l}_J = \mathfrak{l}$, in $L^r(\mathbb{R}^n)$. Moreover, we can find J large enough such that $\mathfrak{l} - \mathfrak{l}_J$ is a $(p(\cdot), q(\cdot), r, s)$-atom associated with the ball $B_{m_0+4\omega}$. We have that $f = h + \mathfrak{l}_J + (\mathfrak{l} - \mathfrak{l}_J)$ and

$$\begin{aligned}
&\|f\|_{H^{p(\cdot),q(\cdot),r,s}(\mathbb{R}^n,A)} \\
&\quad \le \Big\| C_2 \frac{\chi_{B_{m_0+4\omega}}}{\|\chi_{B_{m_0+4\omega}}\|_{p(\cdot),q(\cdot)}} \\
&\qquad + \sum_{(i,j)\in F_J} \lambda_{i,j} \frac{\chi_{x_{i,j}+B_{\ell_{i,j}+4\omega}}}{\|\chi_{x_{i,j}+B_{\ell_{i,j}+4\omega}}\|_{p(\cdot),q(\cdot)}} + \frac{\chi_{B_{m_0+4\omega}}}{\|\chi_{B_{m_0+4\omega}}\|_{p(\cdot),q(\cdot)}} \Big\|_{p(\cdot),q(\cdot)} \\
&\quad \le C\Big(C_2 + 1 + \Big\| \sum_{(i,j)\in F_J} \lambda_{i,j} \frac{\chi_{x_{i,j}+B_{\ell_{i,j}+4\omega}}}{\|\chi_{x_{i,j}+B_{\ell_{i,j}+4\omega}}\|_{p(\cdot),q(\cdot)}} \Big\|_{p(\cdot),q(\cdot)} \Big) \le C.
\end{aligned}$$

Thus, (i) is established.

(ii) This assertion can be proved by using Theorem 1.3 and by proceeding as in [6, Steps 5 and 6, pp. 3094 and 3095] (see also [45, pp. 2926–2927]).

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6 Applications (Proof of Theorem 1.7)

In this section, we present a proof of Theorem 1.7. We have been inspired by some ideas due to Cruz-Uribe and Wang [15], but their arguments have to be modified to adapt them to the anisotropic setting and variable exponent Lorentz spaces.

First of all we formulate the type of operator that we are working with. We consider an operator $T: S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ that commutes with translations. It is well known that this commuting property is equivalent to both the fact that T commutes with convolutions and that there exists $L \in S'(\mathbb{R}^n)$ such that

$$T(\phi) = L * \phi, \quad \phi \in S(\mathbb{R}^n).$$

Assume that:

(i) The Fourier transform $\widehat{L}$ of L is in $L^\infty(\mathbb{R}^n)$.

This property is equivalent to that the operator T can be extended to $L^2(\mathbb{R}^n)$ as a bounded operator from $L^2(\mathbb{R}^n)$ into itself.

We say that T is associated with a measurable function $K: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ when, for every $\phi \in L_c^\infty(\mathbb{R}^n)$, the space of $L^\infty(\mathbb{R}^n)$-functions with compact support,

$$T(\phi)(x) = \int_{\mathbb{R}^n} K(x-y)\phi(y)\,dy, \quad x \notin \operatorname{supp} \phi. \tag{6.1}$$

We assume that K satisfies the following properties: there exists $C_K > 0$ such that

(ii) $|K(x)| \le \frac{C_K}{\rho(x)}$, $x \in \mathbb{R}^n \setminus \{0\}$,

(iii) for some $\gamma > 0$,

$$|K(x-y) - K(x)| \le C_K \frac{\rho(y)^\gamma}{\rho(x)^{\gamma+1}}, \quad b^{2\omega}\rho(y) \le \rho(x).$$

An operator T satisfying the above properties is usually called a *Calderón–Zygmund singular integral* in our anisotropic context. These operators and other ones related with them have been studied, for instance, in [4, 40, 59]. In [59] some sufficient conditions are given in order that a measurable function $K: \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ defines by (6.1) a principal value integral tempered distribution having a Fourier transform in $L^\infty(\mathbb{R}^n)$.

If T is an anisotropic Calderón–Zygmund singular integral, T can be extended from $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n, v)$ to $L^p(\mathbb{R}^n, v)$ as a bounded operator from $L^p(\mathbb{R}^n, v)$ into itself, for every $1 < p < \infty$ and $v \in \mathcal{A}_p(\mathbb{R}^n, A)$, and as a bounded operator from $L^1(\mathbb{R}^n, v)$ into $L^{1,\infty}(\mathbb{R}^n, v)$, for every $v \in \mathcal{A}_1(\mathbb{R}^n, A)$. Also, anisotropic Calderón–Zygmund singular integrals satisfy the following Kolmogorov type inequality.

Proposition 6.1 *Let T be an anisotropic Calderón–Zygmund singular integral. If $v \in \mathcal{A}_1(\mathbb{R}^n, A)$ and $0 < r < 1$, there exists $C > 0$ such that, for every $x_0 \in \mathbb{R}^n$ and $\ell \in \mathbb{Z}$,*

$$\int_{x_0+B_\ell} |Tf(x)|^r v(x)\,dx \le C v(x_0 + B_\ell)^{1-r} \Big(\int_{\mathbb{R}^n} |f(x)| v(x)\,dx \Big)^r, \quad f \in L^1(\mathbb{R}^n, v).$$

Here, $C = C([v]_{\mathcal{A}_1(\mathbb{R}^n,A)}, r)$.

Proof This property can be proved by taking into account that the operator T is bounded from $L^1(\mathbb{R}^n, v)$ into $L^{1,\infty}(\mathbb{R}^n, v)$, provided that $v \in \mathcal{A}_1(\mathbb{R}^n, A)$. Indeed, let

$v \in \mathcal{A}_1(\mathbb{R}^n, A)$ and $0 < r < 1$. For every $f \in L^1(\mathbb{R}^n, v)$, we have that

$$\begin{aligned}
&\int_{a+B_\ell} |Tf(x)|^r v(x)dx \\
&\quad = r\int_0^\infty \lambda^{r-1} v(\{x \in a+B_\ell : |Tf(x)| > \lambda\})d\lambda \\
&\quad \leq C\int_0^\infty \lambda^{r-1} \min\Big\{ v(a+B_\ell), \frac{\|f\|_{L^1(\mathbb{R}^n,v)}}{\lambda}\Big\} d\lambda \\
&\quad \leq C\Big(v(a+B_\ell)\int_0^{\frac{\|f\|_{L^1(\mathbb{R}^n,v)}}{v(a+B_\ell)}} \lambda^{r-1}d\lambda + \|f\|_{L^1(\mathbb{R}^n,v)} \int_{\frac{\|f\|_{L^1(\mathbb{R}^n,v)}}{v(a+B_\ell)}}^\infty \lambda^{r-2}d\lambda\Big) \\
&\quad \leq Cv(a+B_\ell)^{1-r}\Big(\int_{\mathbb{R}^n} |f(x)|v(x)dx\Big)^r. \qquad \blacksquare
\end{aligned}$$

In order to study Calderón–Zygmund singular integrals in Hardy spaces it is usual to require on the kernel K more restrictive regularity conditions than the above ones (ii) and (iii).

As in [4, p. 61] (see also [35]) we say that the anisotropic Calderón–Zygmund singular integral T associated with the kernel K is of order m when $K \in C^m(\mathbb{R}^n \backslash \{0\})$ and there exists $C_{K,m} > 0$ such that for every $x, y \in \mathbb{R}^n$, $x \neq y$,

$$(6.2) \qquad |(\partial_y^\alpha \widetilde{K})(x, A^{-k}y)| \leq \frac{C_{K,m}}{\rho(x-y)} = C_{K,m}b^{-k}, \quad \alpha \in \mathbb{N}^n, \ |\alpha| \leq m,$$

where k is the unique integer such that $x - y \in B_{k+1} \backslash B_k$. Here $\widetilde{K}$ is defined by

$$\widetilde{K}(x,y) = K(x - A^k y), \quad x, y \in \mathbb{R}^n, \ x - y \in B_{k+1}\backslash B_k.$$

As it can be seen in [4, p. 61] this property reduces to the usual condition in the isotropic setting.

In order to prove Theorem 1.7 we need to consider weighted finite atomic anisotropic Hardy spaces as follows.

Let $p, q \in \mathbb{P}_0$, $r > 1$, $s \in \mathbb{N}$, $p_0 \in (0,1)$ and $v \in \mathcal{A}_1(\mathbb{R}^n, A)$. The space

$$H^{p(\cdot),q(\cdot),r,s}_{p_0,v,fin}(\mathbb{R}^n, A)$$

consists of all finite sums of multiple of $(p(\cdot), q(\cdot), r, s)$-atoms and it is endowed with the norm $\|\cdot\|_{H^{p(\cdot),q(\cdot),r,s}_{p_0,v,fin}(\mathbb{R}^n,A)}$ defined as follows: for every

$$f \in H^{p(\cdot),q(\cdot),r,s}_{p_0,v,fin}(\mathbb{R}^n, A),$$

$$\begin{aligned}
&\|f\|_{H^{p(\cdot),q(\cdot),r,s}_{p_0,v,fin}(\mathbb{R}^n,A)} \\
&\qquad = \inf\Big\{ \Big\| \sum_{j=1}^k \lambda_j^{p_0} \|\chi_{x_j+B_{\ell_j}}\|^{-p_0}_{p(\cdot),q(\cdot)} \chi_{x_j+B_{\ell_j}} \Big\|^{1/p_0}_{L^1(\mathbb{R}^n,v)} : f = \sum_{j=1}^k \lambda_j a_j \Big\},
\end{aligned}$$

where the infimum, as usual, is taken over all the possible finite decompositions. Note that according to Proposition 4.2, if $\max\{1, q_+\} < r < \infty$ and $s \geq s_0$, being s_0 the same as in Proposition 4.2, then $H^{p(\cdot),q(\cdot),r,s}_{p_0,v,fin}(\mathbb{R}^n, A) = H^{p(\cdot),q(\cdot),r,s}_{\mathrm{fin}}(\mathbb{R}^n, A)$ as sets.

The following property will be useful in the sequel.

Lemma 6.2 *Let* $p, q \in \mathbb{P}_0$, $\max\{1, q_+\} < r < \infty$ *and* $s \in \mathbb{N}$. *There exists* $s_0 \in \mathbb{N}$ *such that if* $s \geq s_0$, $p_0 < \min\{p_-, q_-\}$ *and* $v \in \mathcal{A}_1(\mathbb{R}^n, A) \cap \mathcal{L}^{(p(\cdot)/p_0)', (q(\cdot)/p_0)'}(\mathbb{R}^n)$ *we can find* $C > 0$ *such that, for every* $f \in H^{p(\cdot),q(\cdot),r,s}_{p_0,v,fin}(\mathbb{R}^n, A)$,

$$\|f\|_{H^{p(\cdot),q(\cdot),r,s}_{p_0,v,fin}(\mathbb{R}^n,A)} \leq C\|f\|_{H^{p_0}(\mathbb{R}^n,v,A)}.$$

Proof The proof of this property follows the same ideas as in the proof of Theorem 1.6. Let s_0 be as in Proposition 4.2 and let $f \in H^{p(\cdot),q(\cdot),r,s}_{p_0,v,fin}(\mathbb{R}^n, A)$, with $s \geq s_0$. Then $f \in H^{p(\cdot),q(\cdot),r,s}_{\text{fin}}(\mathbb{R}^n, A)$ and there exists $m_0 \in \mathbb{Z}$ such that $\operatorname{supp} f \subset B_{m_0}$. Also, $f \in L^r(\mathbb{R}^n)$ and, as we proved in (5.1), $M_N(f)(x) \leq C_1\|\chi_{B_{m_0}}\|^{-1}_{p(\cdot),q(\cdot)}$ when $x \in (B_{m_0+4\omega})^c$.

Assume that $\|f\|_{H^{p_0}(\mathbb{R}^n,v,A)} = 1$. Our objective is to see that

$$\|f\|_{H^{p(\cdot),q(\cdot),r,s}_{p_0,v,fin}(\mathbb{R}^n,A)} \leq C,$$

for some $C > 0$ that does not depend on f.

A careful reading of the proof of [6, Lemma 5.4] allows us to see that there exist a sequence $\{x_{i,k}\}_{i\in\mathbb{N},k\in\mathbb{Z}} \subset \mathbb{R}^n$, a sequence $\{\ell_{i,k}\}_{i\in\mathbb{N},k\in\mathbb{Z}} \subset \mathbb{Z}$, and a bounded sequence $\{b_{i,k}\}_{i\in\mathbb{N},k\in\mathbb{Z}}$ such that

(i) $f = \sum_{k\in\mathbb{Z}}(\sum_{i\in\mathbb{N}} 2^k b_{i,k})$,

where the convergence is unconditional in $S'(\mathbb{R}^n)$ and almost everywhere of $\mathbb{R}^n$;

(ii) for a certain $s_1 \in \mathbb{N}$, $\int_{\mathbb{R}^n} b_{i,k}(x)x^\alpha dx = 0$, $\alpha \in \mathbb{N}^n$ and $|\alpha| \leq s_1$;

(iii) $\operatorname{supp}(b_{i,k}) \subset x_{i,k} + B_{\ell_{i,k}+4\omega}$, $k \in \mathbb{Z}$ and $i \in \mathbb{N}$;

(iv) $\Omega_k := \{x \in \mathbb{R}^n : M_N(f)(x) > 2^k\} = \bigcup_{i\in\mathbb{N}}(x_{i,k} + B_{\ell_{i,k}+4\omega})$, $k \in \mathbb{Z}$;

(v) there exists $L \in \mathbb{N}$ for which $\sharp\{j \in \mathbb{N} : (x_{i,k}+B_{\ell_{i,k}+2\omega}) \cap (x_{j,k}+B_{\ell_{j,k}+2\omega}) \neq \varnothing\} \leq L$, $i \in \mathbb{N}$ and $k \in \mathbb{Z}$.

We define, for every $k \in \mathbb{Z}$ and $i \in \mathbb{N}$,

$$\lambda_{i,k} = 2^k\|\chi_{x_{i,k}+B_{\ell_{i,k}}}\|_{p(\cdot),q(\cdot)} \quad \text{and} \quad a_{i,k} = b_{i,k}\|\chi_{x_{i,k}+B_{\ell_{i,k}}}\|^{-1}_{p(\cdot),q(\cdot)}.$$

There exists $C_0 > 0$ such that $C_0 a_{i,k}$ is a $(p(\cdot), q(\cdot), \infty, s_1)$-atom, for every $k \in \mathbb{Z}$ and $i \in \mathbb{N}$.

We have that

$$\sum_{i\in\mathbb{N}} \lambda^{p_0}_{i,k} \frac{\chi_{x_{i,k}+B_{\ell_{i,k}}}(x)}{\|\chi_{x_{i,k}+B_{\ell_{i,k}}}\|^{p_0}_{p(\cdot),q(\cdot)}} \leq C2^{kp_0}\chi_{\Omega_k}(x), \quad k \in \mathbb{Z} \text{ and } x \in \mathbb{R}^n.$$

Then

$$\begin{aligned}\Big\|\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} \lambda^{p_0}_{i,k} \frac{\chi_{x_{i,k}+B_{\ell_{i,k}}}}{\|\chi_{x_{i,k}+B_{\ell_{i,k}}}\|^{p_0}_{p(\cdot),q(\cdot)}}\Big\|_{L^1(\mathbb{R}^n,v)} &\leq C\Big\|\sum_{k\in\mathbb{Z}} 2^{kp_0}\chi_{\Omega_k}\Big\|_{L^1(\mathbb{R}^n,v)} \leq C\|M_N(f)^{p_0}\|_{L^1(\mathbb{R}^n,v)} \\ &= C\|M_N(f)\|^{p_0}_{L^{p_0}(\mathbb{R}^n,v)} = C\|f\|^{p_0}_{H^{p_0}(\mathbb{R}^n,v,A)}.\end{aligned}$$

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We now choose $k_0 \in \mathbb{Z}$ such that $2^{k_0} \leq \|\chi_{B_{m_0}}\|^{-1}_{p(\cdot),q(\cdot)}$. We define, as in the proof of Theorem 1.6,

$$h = \sum_{k \leq k_0} \sum_{i \in \mathbb{N}} \lambda_{i,k} a_{i,k} \quad \text{and} \quad \mathfrak{l} = \sum_{k > k_0} \sum_{i \in \mathbb{N}} \lambda_{i,k} a_{i,k},$$

where the convergence of the two series is unconditional in $S'(\mathbb{R}^n)$ and almost everywhere of $\mathbb{R}^n$. We have that:

(i) There exists $C_1 > 0$ independent of f such that h/C_1 is a $(p(\cdot), q(\cdot), \infty, s_1)$-atom;

(ii) By defining, for every $J \in \mathbb{N}$, F_J and $\mathfrak{l}_J$ as in the proof of Theorem 1.6, there exists $J_1 \in \mathbb{N}$ such that $\mathfrak{l} - \mathfrak{l}_{J_1}$ is a $(p(\cdot), q(\cdot), r, s_1)$-atom;

(iii) $f = C_1 \frac{h}{C_1} + (\mathfrak{l} - \mathfrak{l}_{J_1}) + \mathfrak{l}_{J_1}$.

Then for $s \geq \max\{s_0, s_1\}$, we can write

$$\begin{aligned}
&\|f\|_{H^{p(\cdot),q(\cdot),r,s}_{p_0,\nu,fin}(\mathbb{R}^n,A)} \\
&\leq \Big\| C_1^{p_0} \frac{\chi_{B_{m_0}}}{\|\chi_{B_{m_0}}\|^{p_0}_{p(\cdot),q(\cdot)}} + \frac{\chi_{B_{m_0}}}{\|\chi_{B_{m_0}}\|^{p_0}_{p(\cdot),q(\cdot)}} \\
&\qquad + \sum_{(i,k) \in F_{J_1}} \lambda^{p_0}_{i,k} \frac{\chi_{x_{i,k}+B_{\ell_{i,k}+4\omega}}}{\|\chi_{x_{i,k}+B_{\ell_{i,k}+4\omega}}\|^{p_0}_{p(\cdot),q(\cdot)}} \Big\|_{L^1(\mathbb{R}^n,\nu)} \\
&\leq C\Big(1 + \frac{\nu(B_{m_0})}{\|\chi_{B_{m_0}}\|^{p_0}_{p(\cdot),q(\cdot)}}\Big) = C\Big(1 + \frac{\nu(B_{m_0})}{\|\chi_{B_{m_0}}\|_{p(\cdot)/p_0,q(\cdot)/p_0}}\Big) \\
&\leq C\big(1 + \|\nu\|_{(p(\cdot)/p_0)',(q(\cdot)/p_0)'}\big).
\end{aligned}$$

The last inequality follows, because $\mathcal{L}^{(p(\cdot)/p_0)',(q(\cdot)/p_0)'} = (\mathcal{L}^{p(\cdot)/p_0,q(\cdot)/p_0})'$ (see [24]), since $p_0 < \min\{p_-, q_-\}$, and then

$$\nu(B_{m_0}) = \int_{\mathbb{R}^n} \chi_{B_{m_0}}(x)\nu(x)dx \leq \|\chi_{B_{m_0}}\|_{p(\cdot)/p_0,q(\cdot)/p_0} \|\nu\|_{(p(\cdot)/p_0)',(q(\cdot)/p_0)'}.$$

Hence, $\|f\|_{H^{p(\cdot),q(\cdot),r,s}_{p_0,\nu,fin}(\mathbb{R}^n,A)} \leq C$, where C does not depend on f. ■

We now prove a general boundedness result for sublinear operators.

Proposition 6.3 *Assume that $p, q \in \mathbb{P}_0$, $p(0) < q(0)$, $0 < p_0 < \min\{p_-, q_-, 1\}$, $\max\{1, q_+,\} < r$, and $s \in \mathbb{N}$. There exist $s_0 \in \mathbb{N}$ and $r_0 > 1$ such that if $s \geq s_0$, $r > r_0$ and T is a sublinear operator defined on* span$\{a : a$ *is a* $(p(\cdot), q(\cdot), r, s)$*-atom*$\}$, *then the following hold.*

(i) *T has a (unique) extension on $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ as a bounded operator from $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ into $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, provided that for each*

$$\nu \in \mathcal{A}_1(\mathbb{R}^n, A) \cap RH_{(r/p_0)'}(\mathbb{R}^n, A)$$

there exists $C = C([\nu]_{\mathcal{A}_1(\mathbb{R}^n,A)}, [\nu]_{RH_{(r/p_0)'}(\mathbb{R}^n,A)}) > 0$ such that

$$\|Ta\|_{L^{p_0}(\mathbb{R}^n,\nu)} \leq C \frac{\nu(x_0 + B_{\ell_0})^{1/p_0}}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot),q(\cdot)}}$$

for every $(p(\cdot), q(\cdot), r/p_0, s)$-atom a associated with $x_0 \in \mathbb{R}^n$ and $\ell_0 \in \mathbb{Z}$.

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(ii) *T has a (unique) extension on $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)$ as a bounded operator from $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)$ into itself, provided that for each $v\in\mathcal{A}_1(\mathbb{R}^n,A)\cap RH_{(r/p_0)'}(\mathbb{R}^n,A)$ there exists $C=C([v]_{\mathcal{A}_1(\mathbb{R}^n,A)},[v]_{RH_{(r/p_0)'}(\mathbb{R}^n,A)})>0$ such that*

$$\|Ta\|_{H^{p_0}(\mathbb{R}^n,v,A)}\le C\frac{v(x_0+B_{\ell_0})^{1/p_0}}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot),q(\cdot)}},$$

for every $(p(\cdot),q(\cdot),r/p_0,s)$-atom a associated with $x_0\in\mathbb{R}^n$ and $\ell_0\in\mathbb{Z}$.

Proof (i) Suppose that for every $v\in\mathcal{A}_1(\mathbb{R}^n,A)\cap RH_{(r/p_0)'}(\mathbb{R}^n,A)$ there exists $C>0$ such that, for every $(p(\cdot),q(\cdot),r/p_0,s)$-atom a associated with $x_0\in\mathbb{R}^n$ and $\ell_0\in\mathbb{Z}$,

$$\|Ta\|_{L^{p_0}(\mathbb{R}^n,v)}\le C\frac{v(x_0+B_{\ell_0})^{1/p_0}}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot),q(\cdot)}}. \tag{6.3}$$

Here, C can depend on $[v]_{\mathcal{A}_1(\mathbb{R}^n,A)}$ and $[v]_{RH_{(r/p_0)'}(\mathbb{R}^n,A)}$.

The set $H_{\mathrm{fin}}^{p(\cdot),q(\cdot),r/p_0,s}(\mathbb{R}^n,A)$ is dense in $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)$ (see Theorem 1.6). Hence, in order to see that there exists an extension $\widetilde{T}$ of T to $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)$ as a bounded operator from $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)$ into $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, it is sufficient to prove that, there exists $C>0$ such that

$$\|T(f)\|_{p(\cdot),q(\cdot)}\le C\|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)},\quad f\in H_{\mathrm{fin}}^{p(\cdot),q(\cdot),r/p_0,s}(\mathbb{R}^n,A).$$

Let $f\in H_{\mathrm{fin}}^{p(\cdot),q(\cdot),r/p_0,s}(\mathbb{R}^n,A)$. As in the proof of Proposition 4.3, Rubio de Francia's iteration algorithm allows us to write,

$$\|T(f)\|_{p(\cdot),q(\cdot)}^{p_0}=\|(Tf)^{p_0}\|_{p(\cdot)/p_0,q(\cdot)/p_0}\le\sup\int_{\mathbb{R}^n}|Tf(x)|^{p_0}Rh(x)dx,$$

where the supremum is taken over all the functions $h\in\mathcal{L}^{(p(\cdot)/p_0)',(q(\cdot)/p_0)'}(\mathbb{R}^n)$ such that $\|h\|_{(p(\cdot)/p_0)',(q(\cdot)/p_0)'}\le 1$. Also, there exists $r_1>1$ such that if $r>r_1$, we can find $C>0$ such that for every $h\in\mathcal{L}^{(p(\cdot)/p_0)',(q(\cdot)/p_0)'}(\mathbb{R}^n)$,

$$Rh\in\mathcal{A}_1(\mathbb{R}^n,A)\cap RH_{(r/p_0)'}(\mathbb{R}^n,A)\quad\text{and}\quad[Rh]_{\mathcal{A}_1(\mathbb{R}^n,A)}+[Rh]_{RH_{(r/p_0)'}(\mathbb{R}^n,A)}\le C.$$

Let $h\in\mathcal{L}^{(p(\cdot)/p_0)',(q(\cdot)/p_0)'}(\mathbb{R}^n)$ such that $\|h\|_{(p(\cdot)/p_0)',(q(\cdot)/p_0)'}\le 1$. We are going to estimate $\|T(f)\|_{L^{p_0}(\mathbb{R}^n,Rh)}$. As it was mentioned above

$$H_{\mathrm{fin}}^{p(\cdot),q(\cdot),r/p_0,s}(\mathbb{R}^n,A)=H_{p_0,Rh,fin}^{p(\cdot),q(\cdot),r/p_0,s}(\mathbb{R}^n,A).$$

We write $f=\sum_{j=1}^k\lambda_ja_j$, where for every $j\in\mathbb{N}$, $j\le k$, $\lambda_j>0$ and a_j is a $(p(\cdot),q(\cdot),r/p_0,s)$-atom associated with $x_j\in\mathbb{R}^n$ and $\ell_j\in\mathbb{Z}$. Since $0<p_0<1$

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and T is sublinear, from (6.3) we deduce that

$$\begin{aligned}\|T(f)\|^{p_0}_{L^{p_0}(\mathbb{R}^n,Rh)} &= \int_{\mathbb{R}^n}|T(f)(x)|^{p_0}Rh(x)dx \le \sum_{j=1}^k \lambda_j^{p_0}\int_{\mathbb{R}^n}|Ta_j(x)|^{p_0}Rh(x)dx\\ &\le C\sum_{j=1}^k \lambda_j^{p_0}\frac{Rh(x_j+B_{\ell_j})}{\|\chi_{x_j+B_{\ell_j}}\|^{p_0}_{p(\cdot),q(\cdot)}}\\ &= C\Big\|\sum_{j=1}^k \lambda_j^{p_0}\frac{\chi_{x_j+B_{\ell_j}}}{\|\chi_{x_j+B_{\ell_j}}\|^{p_0}_{p(\cdot),q(\cdot)}}\Big\|_{L^1(\mathbb{R}^n,Rh)}.\end{aligned}$$

As established in the proof of Proposition 4.3,

$$Rh\in\mathcal{A}_1(\mathbb{R}^n,A)\cap\mathcal{L}^{(p(\cdot)/p_0)',(q(\cdot)/p_0)'}(\mathbb{R}^n).$$

According to Lemma 6.2, the arbitrariness of the representation of f leads to

$$\|T(f)\|_{L^{p_0}(\mathbb{R}^n,Rh)}\le C\|f\|_{H^{p_0}(\mathbb{R}^n,Rh,A)}.$$

Since R is bounded from $\mathcal{L}^{(p(\cdot)/p_0)',(q(\cdot)/p_0)'}(\mathbb{R}^n)$ into itself, we can write

$$\begin{aligned}&\|T(f)\|_{L^{p_0}(\mathbb{R}^n,Rh)}\\ &\quad\le C\|f\|_{H^{p_0}(\mathbb{R}^n,Rh,A)}\le C\int_{\mathbb{R}^n}(M_N(f)(x))^{p_0}Rh(x)dx\\ &\quad\le C\|(M_N(f))^{p_0}\|_{p(\cdot)/p_0,q(\cdot)/p_0}\|Rh\|_{(p(\cdot)/p_0)',(q(\cdot)/p_0)'}\\ &\quad\le C\|(M_N(f))^{p_0}\|_{p(\cdot)/p_0,q(\cdot)/p_0}=C\|M_N(f)\|^{p_0}_{p(\cdot),q(\cdot)}=C\|f\|^{p_0}_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)},\end{aligned}$$

provided that $h\in\mathcal{L}^{(p(\cdot)/p_0)',(q(\cdot)/p_0)'}(\mathbb{R}^n)$ and $\|h\|_{(p(\cdot)/p_0)',(q(\cdot)/p_0)'}\le 1$.

We conclude that

$$\|T(f)\|_{p(\cdot),q(\cdot)}\le C\|f\|_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)},$$

and the proof of (i) is finished.

(ii) We proceed in a similar way as in the proof of (i). Assume that $\varphi\in S(\mathbb{R}^n)$ such that $\int\varphi dx\ne 0$. Let $f\in H^{p(\cdot),q(\cdot),r/p_0,s}_{\text{fin}}(\mathbb{R}^n,A)$, with $s\ge s_0$, and s_0 as before. We have that

$$\begin{aligned}\|T(f)\|^{p_0}_{H^{p(\cdot),q(\cdot)}(\mathbb{R}^n,A)}&\le C\|M^0_\varphi(Tf)\|^{p_0}_{p(\cdot),q(\cdot)}=C\|(M^0_\varphi(Tf))^{p_0}\|_{p(\cdot)/p_0,q(\cdot)/p_0}\\ &\le C\sup\int_{\mathbb{R}^n}(M^0_\varphi(Tf)(x))^{p_0}Rh(x)dx\\ &\le C\sup\|T(f)\|^{p_0}_{H^{p_0}(\mathbb{R}^n,Rh,A)},\end{aligned}$$

where the supremum is taken over all the functions $h\in\mathcal{L}^{(p(\cdot)/p_0)',(q(\cdot)/p_0)'}(\mathbb{R}^n)$ such that $\|h\|_{(p(\cdot)/p_0)',(q(\cdot)/p_0)'}\le 1$.

We now finish the proof in the same way as (i) provided that, for every $v\in\mathcal{A}_1(\mathbb{R}^n,A)\cap RH_{(r/p_0)'}(\mathbb{R}^n,A)$ there exists $C>0$ such that

$$\|Ta\|_{H^{p_0}(\mathbb{R}^n,v,A)}\le C\frac{v(x_j+B_{\ell_j})^{1/p_0}}{\|\chi_{x_j+B_{\ell_j}}\|_{p(\cdot),q(\cdot)}}$$

for every $(p(\cdot),q(\cdot),r/p_0,s)$-atom a associated with $x_j\in\mathbb{R}^n$ and $\ell_j\in\mathbb{Z}$. Here, the constant C can depend on $[v]_{\mathcal{A}_1(\mathbb{R}^n,A)}$ and $[v]_{RH_{(r/p_0)'}(\mathbb{R}^n,A)}$. ■

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We now prove Theorem 1.7 by applying the criteria established in Proposition 6.3.

Proof of Theorem 1.7(i) Assume that a is a $(p(\cdot), q(\cdot), r/p_0, s)$-atom associated with $x_0 \in \mathbb{R}^n$ and $\ell_0 \in \mathbb{Z}$, and $v \in \mathcal{A}_1(\mathbb{R}^n, A) \bigcap RH_{(r/p_0)'}(\mathbb{R}^n, A)$. Here, p_0, r, and s are as in Proposition 6.3. We can write

$$\begin{aligned}\|Ta\|_{L^{p_0}(\mathbb{R}^n,v)}^{p_0} &= \int_{x_0+B_{\ell_0+w}} |T(a)(x)|^{p_0} v(x)dx + \int_{(x_0+B_{\ell_0+w})^c} |T(a)(x)|^{p_0} v(x)dx \\ &= I_1 + I_2.\end{aligned}$$

According to Proposition 6.1 there exits $C > 0$ such that

$$\begin{aligned}I_1 &\le Cv(x_0 + B_{\ell_0+w})^{1-p_0}\Big(\int_{\mathbb{R}^n} |a(x)|v(x)dx\Big)^{p_0} \\ &\le Cv(x_0 + B_{\ell_0})^{1-p_0}|B_{\ell_0}|^{p_0} \\ &\quad\times \Big[\Big(\frac{1}{|B_{\ell_0}|}\int_{x_0+B_{\ell_0}} |a(x)|^{r/p_0}dx\Big)^{p_0/r}\Big(\frac{1}{|B_{\ell_0}|}\int_{x_0+B_{\ell_0}} v(x)^{(r/p_0)'}dx\Big)^{1/(r/p_0)'}\Big]^{p_0}.\end{aligned}$$

We have used that v is a doubling measure.

Taking into account that a is a $(p(\cdot), q(\cdot), r/p_0, s)$-atom and $v \in RH_{(r/p_0)'}(\mathbb{R}^n, A)$, we obtain

$$\begin{aligned}I_1 &\le Cv(x_0 + B_{\ell_0+w})^{1-p_0}|B_{\ell_0}|^{p_0}\Big(\frac{1}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot),q(\cdot)}|B_{\ell_0}|}\int_{x_0+B_{\ell_0}} v(x)dx\Big)^{p_0} \\ &\le C\frac{v(x_0 + B_{\ell_0})}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot),q(\cdot)}^{p_0}}.\end{aligned}$$

Note that $C = C([v]_{A_1(\mathbb{R}^n,A)}, [v]_{RH_{(r/p_0)'}(\mathbb{R}^n,A)})$.

Since a is a $(p(\cdot), q(\cdot), r/p_0, s)$-atom associated with $x_0 \in \mathbb{R}^n$ and $\ell_0 \in \mathbb{Z}$, by using the condition (6.2) and by proceeding as in [4, pp. 64–65] we deduce that for every $x \in (x_0 + B_{\ell_0+w+\ell+1}) \setminus (x_0 + B_{\ell_0+w+\ell})$, with $\ell \in \mathbb{N}$,

$$\begin{aligned}|Ta(x)| &\le Cb^{-\ell_0-\ell} \sup_{z\in B_{-\ell}} |z|^m \int_{x_0+B_{\ell_0}} |a(y)|dy \\ &\le Cb^{-\ell_0-\ell}(\lambda_-^{-\ell})^m |B_{\ell_0}|^{1/(r/p_0)'}\|a\|_{r/p_0} \\ &\le Cb^{-\ell_0}b^{-\ell(\delta+1)}b^{\ell_0/(r/p_0)'}\|a\|_{r/p_0} \\ &\le Cb^{-\ell_0 p_0/r}(\rho(x - x_0)b^{-\ell_0-w})^{-(\delta+1)}\|a\|_{r/p_0},\end{aligned}$$

where $\delta = m \ln \lambda_-/\ln b$.

Then

$$\begin{aligned}|Ta(x)| &\le C\frac{b^{\ell_0(\delta+1)}}{\rho(x - x_0)^{\delta+1}}b^{-\ell_0 p_0/r}\frac{|B_{\ell_0}|^{p_0/r}}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot),q(\cdot)}} \\ &= C\frac{|B_{\ell_0}|^{\delta+1}}{\rho(x - x_0)^{\delta+1}\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot),q(\cdot)}}, \quad x \notin x_0 + B_{\ell_0+w}.\end{aligned}$$

Thus,

$$I_2 \le C\frac{|B_{\ell_0}|^{p_0(\delta+1)}}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot),q(\cdot)}^{p_0}}\int_{(x_0+B_{\ell_0+w})^c}\frac{v(x)}{\rho(x - x_0)^{p_0(\delta+1)}}dx.$$

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Since

$$(x_0+B_{\ell_0+w})^c=\bigcup_{i=0}^{\infty}(x_0+B_{\ell_0+w+i+1})\smallsetminus(x_0+B_{\ell_0+w+i}),$$
$$b^{\ell_0+w+i}\le\rho(x-x_0)\le b^{\ell_0+w+i+1},$$

for every $x\in(x_0+B_{\ell_0+w+i+1})\smallsetminus(x_0+B_{\ell_0+w+i})$, $i\in\mathbb{N}$, we have that

$$\begin{aligned}
&\int_{(x_0+B_{\ell_0+w})^c}\frac{v(x)}{\rho(x-x_0)^{p_0(\delta+1)}}dx\\
&=\sum_{i=0}^{\infty}\int_{(x_0+B_{\ell_0+w+i+1})\smallsetminus(x_0+B_{\ell_0+w+i})}\frac{v(x)}{\rho(x-x_0)^{p_0(\delta+1)}}dx\\
&\le\sum_{i=0}^{\infty}b^{-(\ell_0+w+i)p_0(\delta+1)}\int_{x_0+B_{\ell_0+w+i+1}}v(x)dx\\
&\le[v]_{A_1(\mathbb{R}^n,A)}\sum_{i=0}^{\infty}b^{-(\ell_0+w+i)p_0(\delta+1)}|B_{\ell_0+w+i+1}|\operatorname*{ess\,inf}_{x\in x_0+B_{\ell_0+w+i+1}}v(x)\\
&\le b[v]_{A_1(\mathbb{R}^n,A)}\sum_{i=0}^{\infty}b^{-(\ell_0+w+i)(p_0(\delta+1)-1)}\operatorname*{ess\,inf}_{x\in x_0+B_{\ell_0}}v(x)\\
&\le b[v]_{A_1(\mathbb{R}^n,A)}\frac{1}{|B_{\ell_0}|}\int_{x_0+B_{\ell_0}}v(z)dz\sum_{i=0}^{\infty}b^{-i(p_0(\delta+1)-1)}b^{-(\ell_0+w)(p_0(\delta+1)-1)}\\
&=b[v]_{A_1(\mathbb{R}^n,A)}\frac{1}{b^{\ell_0}}\frac{b^{-(\ell_0+w)(p_0(\delta+1)-1)}}{1-b^{-p_0(\delta+1)+1}}v(x_0+B_{\ell_0}).
\end{aligned}$$

Note that $p_0>1/(\delta+1)$.

We get

$$I_2\le C[v]_{A_1(\mathbb{R}^n,A)}\frac{v(x_0+B_{\ell_0})}{\|\chi_{x_0+B_{\ell_0}}\|^{p_0}_{p(\cdot),q(\cdot)}},$$

where C does not depend on v.

Hence, for a certain $C=C([v]_{A_1(\mathbb{R}^n,A)},[v]_{RH_{(1/p_0)'}(\mathbb{R}^n,A)})$,

$$\|Ta\|^{p_0}_{L^{p_0}(\mathbb{R}^n,v)}\le C\frac{v(x_0+B_{\ell_0})}{\|\chi_{x_0+B_{\ell_0}}\|^{p_0}_{p(\cdot),q(\cdot)}}.$$

We complete the proof by applying Proposition 6.3(i). ∎

Before proving Theorem 1.7(ii), we establish the following auxiliary result.

Lemma 6.4 *Let $\phi\in\mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp}\phi\subset B_0$ and $\int\phi(x)dx\ne0$. Assume that $L\in\mathcal{S}'(\mathbb{R}^n)$ and that T_L is a Calderón–Zygmund singular integral of order m. Then for every $\ell\in\mathbb{Z}$, the operator $S_{(\ell)}=T_{\phi_\ell}\circ T_L$ is a Calderón–Zygmund singular integral of order m. Moreover, if $S_{(\ell)}$ is associated with the kernel K_ℓ, there exists $C>0$ such that*

$$\sup_{\ell\in\mathbb{N}}\Big\{\|\widehat{S_{(\ell)}}\|_\infty,C_{K_\ell},C_{K_\ell,m}\Big\}\le C.$$

Proof Let $\ell\in\mathbb{Z}$. For every $\psi\in\mathcal{S}(\mathbb{R}^n)$ we have that

$$S_{(\ell)}(\psi)=T_{\phi_\ell}(T_L(\psi))=\phi_\ell*(L*\psi)=(L*\phi_\ell)*\psi.$$

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Hence, $S_{(\ell)} = T_{L*\phi_\ell}$. Since $|\widehat{\phi_\ell}| \le \|\phi\|_1$, the interchange formula leads to $\|\widehat{S_{(\ell)}}\|_\infty = \|\widehat{L}\widehat{\phi_\ell}\|_\infty \le \|\widehat{L}\|_\infty \|\phi\|_1$. According to [53, p. 248], $L * \phi_\ell$ is a multiplier for $\mathcal{S}(\mathbb{R}^n)$ and, for every $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$S_{(\ell)}(\psi)(x) = \int_{\mathbb{R}^n} (L * \phi_\ell)(x-y)\psi(y)dy, \quad x \in \mathbb{R}^n.$$

Note that this integral is absolutely convergent for every $x \in \mathbb{R}^n$. Then $S_{(\ell)}$ is associated with the kernel $L * \phi_\ell$ which is in $C^\infty(\mathbb{R}^n)$. We define, for every $k \in \mathbb{Z}$, $L_k \in \mathcal{S}'(\mathbb{R}^n)$ as follows:

$$\langle L_k, \psi\rangle = \langle L, \psi(A^k \cdot)\rangle, \quad \psi \in \mathcal{S}(\mathbb{R}^n).$$

It is not hard to see that, for every $k \in \mathbb{Z}$, $(L * \phi_\ell)_k = L_k * \phi_{\ell+k}$. Then $(L * \phi_\ell)_{-\ell} = L_{-\ell} * \phi$.

Suppose that T_L is associated with the kernel K, that is, for every $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$(L * \psi)(x) = \int_{\mathbb{R}^n} K(x-y)\psi(y)dy, \quad x \notin \operatorname{supp}\psi,$$

and K satisfies (ii) and (iii) after (6.1).

Let $k \in \mathbb{Z}$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$. We have that

$$\begin{aligned}(L_k * \psi)(x) &= \langle L_k(y), \psi(x-y)\rangle = \langle L(y), \psi(x - A^k y)\rangle \\ &= \big\langle L(y), \psi(A^k(A^{-k}x - y))\big\rangle \\ &= (L * \psi(A^k \cdot))(A^{-k}x) \\ &= \int_{\mathbb{R}^n} K(A^{-k}x - y)\psi(A^k y)dy, \quad A^{-k}x \notin \operatorname{supp}\psi(A^k \cdot).\end{aligned}$$

Then

$$(L_k * \psi)(x) = b^{-k} \int_{\mathbb{R}^n} K(A^{-k}(x-y))\psi(y)dy, \quad x \notin \operatorname{supp}\psi.$$

We are going to see that there exists $C > 0$ that does not depend on ℓ such that:

(i) $|(L_{-\ell} * \phi)(x)| \le C/\rho(x)$, $x \in \mathbb{R}^n \setminus \{0\}$,

and, if $\delta = \min\{\gamma, \ln\lambda_-/\ln b\}$,

(ii) $|(L_{-\ell} * \phi)(x-y) - (L_{-\ell} * \phi)(x)| \le C\rho(y)^\delta/(\rho(x))^{\delta+1}$, when $b^{2w}\rho(y) \le \rho(x)$.

First, we prove (i). We have that $L_{-\ell} * \phi \in L^2(\mathbb{R}^n)$ and $\widehat{L_{-\ell} * \phi} = \widehat{L_{-\ell}}\widehat{\phi} \in L^1(\mathbb{R}^n)$. Then we can write

$$(L_{-\ell} * \phi)(x) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot y}\widehat{L_{-\ell}}(y)\widehat{\phi}(y)dy, \quad x \in \mathbb{R}^n.$$

Note that the two sides in the last equalities define smooth functions in $\mathbb{R}^n$. Since $\|\widehat{L_{-\ell}}\|_\infty = \|\widehat{L}\|_\infty$, we deduce that

$$|(L_{-\ell} * \phi)(x)| \le \|\widehat{L_{-\ell}}\|_\infty \int_{\mathbb{R}^n} |\widehat{\phi}(y)|dy, \quad x \in \mathbb{R}^n.$$

We obtain

$$|(L_{-\ell} * \phi)(x)| \le \frac{b^{1+w}\|\widehat{L_{-\ell}}\|_\infty}{\rho(x)} \int_{\mathbb{R}^n} |\widehat{\phi}(y)|dy, \quad x \in B_{1+w} \setminus \{0\}.$$

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On the other hand, since $\rho(x+y) \le b^w(\rho(x)+\rho(y))$, $x,y \in \mathbb{R}^n$, we have that $\rho(x-y) \ge b^{-w}\rho(x) - \rho(y)$, $x,y \in \mathbb{R}^n$. Then if $x \notin B_{1+w}$ and $y \in B_0$, it follows that $\rho(x-y) \ge b^{-w}\rho(x) - b^{-w-1}\rho(x) = b^{-w}(1-b^{-1})\rho(x)$. We can write

$$|(L_{-\ell} * \phi)(x)| \le b^\ell \int_{B_0} |K(A^\ell(x-y))||\phi(y)|dy \le C_K b^\ell \int_{B_0} \frac{|\phi(y)|}{\rho(A^\ell(x-y))} dy$$
$$\le C_K \int_{B_0} \frac{|\phi(y)|}{\rho(x-y)} dy \le \frac{C_K b^w}{(1-b^{-1})\rho(x)} \int_{B_0} |\phi(y)|dy, \quad x \notin B_{w+1}.$$

We conclude that $|(L_{-\ell} * \phi)(x)| \le C/\rho(x)$, $x \in \mathbb{R}^n \setminus \{0\}$, where $C>0$ is independent of ℓ, and (i) is proved.

We now establish (ii). We can write

$$(L_{-\ell} * \phi)(x-y) - (L_{-\ell} * \phi)(x)$$
$$= \int_{\mathbb{R}^n} \left(e^{-2\pi i(x-y)\cdot z} - e^{-2\pi i x\cdot z}\right) \widehat{L_{-\ell}}(z)\widehat{\phi}(z)dz, \quad x,y \in \mathbb{R}^n.$$

The mean value theorem leads to

$$|(L_{-\ell} * \phi)(x-y) - (L_{-\ell} * \phi)(x)| \le C|y|\|\widehat{L}\|_\infty \int_{\mathbb{R}^n} |z||\widehat{\phi}(z)|dz, \quad x,y \in \mathbb{R}^n.$$

According to [4, (3.3) p. 11] , $|y| \le C\rho(y)^{\ln\lambda_-/\ln b}$, when $\rho(y) \le 1$. Also, by [4, (3.2) p. 11], we get

$$|y| \le C\rho(y)^{\ln\lambda_+/\ln b} \le C\rho(y)^{\ln\lambda_-/\ln b}, \quad 1 \le \rho(y) \le b^{4w}.$$

Hence,

$$|(L_{-\ell} * \phi)(x-y) - (L_{-\ell} * \phi)(x)| \le C\rho(y)^{\ln\lambda_-/\ln b}$$
$$\le C\frac{\rho(y)^{\ln\lambda_-/\ln b}}{\rho(x)^{\ln\lambda_-/\ln b+1}}, \quad b^{2w}\rho(y) \le \rho(x) \le b^{4w}.$$

Assume that $\rho(x) \ge b^{4w}$ and $b^{2w}\rho(y) \le \rho(x)$. It is clear that $x \notin \operatorname{supp}\phi$. Also, we have that $\rho(x-y) \ge b^{-w}\rho(x) - \rho(y) \ge b^{-w}\rho(x) - b^{-2w}\rho(x) \ge b^{3w} - b^{2w} \ge b$. Then $x-y \notin \operatorname{supp}\phi$. We can write

$$(L_{-\ell} * \phi)(x-y) - (L_{-\ell} * \phi)(x) = \int_{\mathbb{R}^n} (K_{-\ell}(x-y-z) - K_{-\ell}(x-z))\phi(z)dz,$$

where $K_{-\ell}(z) = b^\ell K(A^\ell z)$, $z \in \mathbb{R}^n$.

Suppose that $\rho(y) \le b^{-6w}\rho(x)$ and $z \in \operatorname{supp}\phi$. Since $\rho(z) \le b^{-4w}\rho(x)$, we have that $\rho(x-z) \ge b^{-w}\rho(x) - \rho(z) \ge b^{-w}\rho(x) - b^{-4w}\rho(x) \ge b^{6w}(b^{-w} - b^{-4w})\rho(y) = b^{2w}(b^{3w}-1)\rho(y) \ge b^{2w}\rho(y)$. We obtain

$$|(L_{-\ell} * \phi)(x-y) - (L_{-\ell} * \phi)(x)| \le \int_{B_0} |K_{-\ell}(x-y-z) - K_{-\ell}(x-z)||\phi(z)|dz$$
$$\le C\int_{B_0} |\phi(z)| \frac{\rho(y)^\gamma}{\rho(x-z)^{\gamma+1}} dz$$
$$\le C\frac{\rho(y)^\gamma}{\rho(x)^{\gamma+1}} \int_{\mathbb{R}^n} |\phi(z)|dz.$$

Suppose now that $b^{2w}\rho(y) \le \rho(x) \le b^{6w}\rho(y)$. It follows that $\rho(x-y) \ge b^{-w}\rho(x) - \rho(y) \ge (b^{-w} - b^{-2w})\rho(x)$. From (i) we deduce

$$|(L_{-\ell} * \phi)(x-y) - (L_{-\ell} * \phi)(x)| \le C\Big(\frac{1}{\rho(x-y)} + \frac{1}{\rho(x)}\Big) \le \frac{C}{\rho(x)} \le C\frac{\rho(y)^\gamma}{\rho(x)^{\gamma+1}}.$$

We conclude that, if $\delta = \min\{\gamma, \ln\lambda_-/\ln b\}$,

$$|(L_{-\ell} * \phi)(x-y) - (L_{-\ell} * \phi)(x)| \le C\frac{\rho(y)^\delta}{\rho(x)^{\delta+1}}, \quad b^{2w}\rho(y) \le \rho(x),$$

where $C > 0$ does not depend on ℓ, and (ii) is proved.

Since $L * \phi_\ell = (L_{-\ell} * \phi)_\ell$, from (i) and (ii) we infer that

(i') $|(L * \phi_\ell)(x)| \le C/\rho(x)$, $x \in \mathbb{R}^n \setminus \{0\}$,

and, if $\delta = \min\{\gamma, \ln\lambda_-/\ln b\}$,

(ii') $|(L * \phi_\ell)(x-y) - (L * \phi_\ell)(x)| \le C\rho(y)^\delta/(\rho(x))^{\delta+1}$, when $b^{2w}\rho(y) \le \rho(x)$,

and $C > 0$ does not depend on ℓ.

We are going to prove the m-regularity property for the kernel

$$H_\ell(x,y) = (L * \phi_\ell)(x-y), \quad x, y \in \mathbb{R}^n.$$

We have to show that if $\alpha \in \mathbb{N}^n$, $|\alpha| \le m$, and $x, y \in \mathbb{R}^n$, $x - y \in B_{k+1} \setminus B_k$, with $k \in \mathbb{Z}$, then

$$|(\partial_y^\alpha \widetilde{H_\ell})(x, A^{-k}y)| \le \frac{C}{\rho(x-y)} = \frac{C}{b^k},$$

where $\widetilde{H_\ell}(x,y) = H_\ell(x, A^k y)$ and $C > 0$ is independent of ℓ. In order to prove this, we proceed as in [4, pp. 66–67].

We have that

$$H_\ell(x,y) = \int_{\mathbb{R}^n} \mathbb{K}(x-z, y)\phi_\ell(z)dz, \quad x - y \in B_\ell,$$

where $\mathbb{K}(x,y) = K(x-y)$, $x, y \in \mathbb{R}^n \setminus \{0\}$.

Suppose that $x_0, y_0 \in \mathbb{R}^n$ and $x_0 - y_0 \in B_{j+2w+1} \setminus B_{j+2w}$, where $j \in \mathbb{N}$, $j \ge \ell$. By [4, (2.11), p. 68] it follows that $x_0 - y_0 - z \notin B_{j+w}$ and $x_0 - y_0 - z \in B_{j+3w+1}$, for every $z \in B_\ell$. By using the regularity of K, we deduce (see [4, (9.29), p. 66]), for every $\alpha \in \mathbb{N}^n$, $|\alpha| \le m$,

$$|(\partial_y^\alpha[\mathbb{K}(\cdot, A^{j+2w}\cdot)])(x_0 - z, A^{-j-2w}y_0)| \le Cb^{-j-2w}, \quad z \in B_\ell.$$

Differentiating under the integral sign we get

$$|(\partial_y^\alpha \widetilde{H_\ell})(x_0, A^{-j-2w}y_0)| \le Cb^{-j-2w}, \quad \alpha \in \mathbb{N}^n,\ |\alpha| \le m,$$

where $C > 0$ does not depend on $\{\ell, j\}$.

Assume that $x_0, y_0 \in \mathbb{R}^n$, and $x_0 - y_0 \in B_{j+1} \setminus B_j$, is $j < \ell + 2w$. Let $\alpha \in \mathbb{N}^n$, $|\alpha| \le m$. We can write

$$\begin{aligned}\widetilde{H_\ell}(x,y) &= \int_{\mathbb{R}^n} e^{-2\pi i z\cdot(x - A^j y)}\widehat{L}(z)\widehat{\phi_\ell}(z)dz \\ &= \int_{\mathbb{R}^n} e^{-2\pi i z\cdot(x - A^j y)}\widehat{L}(z)\widehat{\phi}((A^*)^\ell z)dz, \quad x, y \in \mathbb{R}^n,\end{aligned}$$

where A^* denotes the adjoint matrix of A.

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After making a change of variables, we get

$$\widetilde{H_\ell}(x,y) = b^{-\ell}\int_{\mathbb{R}^n} e^{-2\pi i (A^*)^{-\ell}z\cdot(x-A^j y)}\widehat{L}((A^*)^{-\ell}z)\widehat{\phi}(z)dz$$
$$= b^{-\ell}\int_{\mathbb{R}^n} e^{-2\pi i z\cdot(A^{-\ell}x-A^{-\ell+j}y)}\widehat{L}((A^*)^{-\ell}z)\widehat{\phi}(z)dz, \quad x,y\in\mathbb{R}^n.$$

Then differentiating under the integral sign, we obtain

$$|\partial_y^\alpha \widetilde{H_\ell}(x,y)| \le Cb^{-\ell}\int_{\mathbb{R}^n}|z|^\alpha|\widehat{\phi}(z)|dz, \quad x,y\in\mathbb{R}^n,\ x-y\in B_{j+1}\setminus B_j,$$

because $j-\ell<2w$. Here, $C>0$ does not depend on $\{\ell,j\}$.

Hence,

$$|(\partial_y^\alpha \widetilde{H_\ell})(x_0,A^{-j}y_0)| \le Cb^{-\ell} = Cb^{-\ell+j}b^{-j} \le Cb^{2w}\rho(x_0-y_0)^{-1}.$$

We conclude that there exists $C>0$ such that for every $\alpha\in\mathbb{N}^n$, $|\alpha|\le m$, and $x,y\in\mathbb{R}^n$, $x-y\in B_{k+1}\setminus B_k$, $k\in\mathbb{Z}$,

$$|(\partial_y^\alpha \widetilde{H_\ell})(x,A^{-k}y)| \le \frac{C}{\rho(x-y)}.$$

Thus, the proof of the property is finished. ■

Proof of Theorem 1.7(ii) Consider r, p_0 and s as in Proposition 6.3. Assume that a is a $(p(\cdot),q(\cdot),r/p_0,s)$-atom associated with $x_0\in\mathbb{R}^n$ and $\ell_0\in\mathbb{Z}$, and $v\in\mathcal{A}_1(\mathbb{R}^n,A)\cap RH_{(r/p_0)'}(\mathbb{R}^n,A)$. We take $\varphi\in\mathcal{S}(\mathbb{R}^n)$ such that $\int\varphi(x)dx\ne 0$ and $\operatorname{supp}\varphi\subset B_0$.

We can write

$$\|Ta\|^{p_0}_{H^{p_0}(\mathbb{R}^n,v,A)}$$
$$\le C\Big(\int_{x_0+B_{\ell_0+w}}(M_\varphi^0(Ta)(x))^{p_0}v(x)dx + \int_{(x_0+B_{\ell_0+w})^c}(M_\varphi^0(Ta)(x))^{p_0}v(x)dx\Big)$$
$$= J_1 + J_2.$$

The Hardy–Littlewood maximal function satisfies Kolmogorov inequality (see [30, p. 91]). Then since $M_\varphi^0(Ta)\le CM_{HL}(Ta)$, we get

$$J_1 \le Cv(x_0+B_{\ell_0+w})^{1-p_0}\Big(\int_{\mathbb{R}^n}|T(a)(x)|v(x)dx\Big)^{p_0}.$$

Here, $C=C([v]_{\mathcal{A}_1(\mathbb{R}^n,A)})>0$.

By splitting the last integral in the same way, we obtain

$$\int_{\mathbb{R}^n}|Ta(x)|v(x)dx = \int_{x_0+B_{\ell_0+w}}|Ta(x)|v(x)dx + \int_{(x_0+B_{\ell_0+w})^c}|Ta(x)|v(x)dx.$$

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Since T is a bounded operator in $L^{r/p_0}(\mathbb{R}^n)$ (see Proposition 6.3) and we have $v \in RH_{(r/p_0)'}(\mathbb{R}^n, A)$, it follows that

$$\begin{aligned}\int_{x_0+B_{\ell_0+w}} &|Ta(x)|v(x)dx\\ &\le \Big(\int_{x_0+B_{\ell_0+w}} |Ta(x)|^{r/p_0}dx\Big)^{p_0/r}\Big(\int_{x_0+B_{\ell_0+w}} v(x)^{(r/p_0)'}dx\Big)^{1/(r/p_0)'}\\ &\le C\|a\|_{r/p_0}|B_{\ell_0+w}|^{1/(r/p_0)'-1}v(x_0+B_{\ell_0+w}) \le C\frac{v(x_0+B_{\ell_0})}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot),q(\cdot)}}.\end{aligned}$$

Here, $C = C([v]_{\mathcal{A}_1(\mathbb{R}^n,A)}, [v]_{RH_{(r/p_0)'}(\mathbb{R}^n,A)})$. We have used that v defines a doubling measure.

Proceeding as in the estimation of I_2 in the proof of Theorem 1.7(i), we get

$$\begin{aligned}\int_{(x_0+B_{\ell_0+w})^c} |Ta(x)|v(x)dx &\le C\frac{|B_{\ell_0}|^{\delta+1}}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot),q(\cdot)}}\int_{(x_0+B_{\ell_0+w})^c}\frac{v(x)}{\rho(x-x_0)^{\delta+1}}dx\\ &\le C\frac{|B_{\ell_0}|^{\delta+1}}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot),q(\cdot)}}v(x_0+B_{\ell_0})|B_{\ell_0}|^{-(\delta+1)}\\ &= C\frac{v(x_0+B_{\ell_0})}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot),q(\cdot)}},\end{aligned}$$

where $C = C([v]_{\mathcal{A}_1(\mathbb{R}^n,A)}) > 0$ and $\delta = m\ln\lambda_-/\ln b$.

We conclude that

$$J_1 \le C\frac{v(x_0+B_{\ell_0})}{\|\chi_{x_0+B_{\ell_0}}\|^{p_0}_{p(\cdot),q(\cdot)}}.$$

Here, $C = C([v]_{\mathcal{A}_1(\mathbb{R}^n,A)}, [v]_{RH_{(r/p_0)'}(\mathbb{R}^n,A)}) > 0$.

According to Lemma 6.4, for every $k \in \mathbb{Z}$, the convolution operator S_k defined by

$$S_{(k)}(\psi) = \varphi_k * (T\psi), \quad \psi \in \mathcal{S}(\mathbb{R}^n),$$

is a Calderón–Zygmund singular integral of order m and this property is uniformly in $k \in \mathbb{Z}$; that is, the characteristic constant does not depend on k.

If $k \in \mathbb{Z}$, by proceeding as in the proof of Theorem 1.7(i), we get

$$|S_{(k)}(a)| \le C\frac{|B_{\ell_0}|^{\delta+1}}{\rho(x-x_0)^{\delta+1}}\frac{1}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot),q(\cdot)}}, \quad x \notin x_0+B_{\ell_0+w},$$

where $C > 0$ does not depend on k.

Then

$$|M^0_\varphi(Ta)(x)| \le C\frac{|B_{\ell_0}|^{\delta+1}}{\rho(x-x_0)^{\delta+1}}\frac{1}{\|\chi_{x_0+B_{\ell_0}}\|_{p(\cdot),q(\cdot)}}, \quad x \notin x_0+B_{\ell_0+w}.$$

We conclude that

$$J_2 \le C([v]_{A_1(\mathbb{R}^n,A)})\frac{v(x_0+B_{\ell_0})}{\|\chi_{x_0+B_{\ell_0}}\|^{p_0}_{p(\cdot),q(\cdot)}}.$$

The proof of this theorem can be completed by putting together the above estimates. ■

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Remark 6.5 In order to prove the boundedness of an operator T defined on Hardy type spaces (or finite atomic Hardy type spaces) and that takes values in a Banach (or quasi-Banach) space, it is usual to add the following condition: T is uniformly bounded on atoms. As it can be seen (for instance, in [6, pp. 3096–3097], the last condition implies the boundedness of T, roughly speaking, proceeding as follows: if $f = \sum_{j=1}^{k} \lambda_j a_j$, then

$$\|Tf\|_X \le \sum_{j=1}^{k} |\lambda_j| \|Ta_j\|_X \le C \sum_{j=1}^{k} |\lambda_j| \le C\|f\|_{H^p}. \tag{6.4}$$

In our case, for the anisotropic Hardy–Lorentz spaces with variable exponents, we do not know if the last inequality in (6.4) holds. In Theorem 1.3 we establish our atomic quasinorm. The condition in Proposition 6.3 is adapted to the quasinorms on the anisotropic Hardy–Lorentz spaces with variable exponents and they replace the uniform boundedness on atoms condition.

Remark 6.6 As is well known, Lorentz and Hardy–Lorentz spaces appear related with interpolation. Fefferman, Rivière, and Sagher ([25]) proved that if $0 < p_0 < 1$, then $(H^{p_0}(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\eta,q} = H^{p,q}(\mathbb{R}^n)$, where $1/p = (1-\eta)/p_0$, $0 < \eta < 1$ and $0 < q \le \infty$. Recently, Liu, Yang, and Yuan ([40, Lemma 6.3])) established an anisotropic version of this result. By using a reiteration argument in [40, Theorem 6.1] the interpolation spaces between anisotropic Hardy spaces are described. Kempka and Vybíral ([36, Theorem 8]) proved that $(L^{p(\cdot)}(\mathbb{R}^n), L^\infty(\mathbb{R}^n))_{\theta,q} = L_{\widetilde{p}(\cdot),q}$, where $0 < \theta < 1$, $0 < q \le \infty$ and $1/\widetilde{p}(\cdot) = (1-\theta)/p(\cdot)$. It is clear that a similar property cannot be expected for the Lorentz space $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, since in the definition of $L^{p(\cdot)}(\mathbb{R}^n)$, p is a measurable function defined in $\mathbb{R}^n$ while in the definition of the Lorentz space $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, p and q are measurable functions defined in $(0,\infty)$. Then the arguments used in [40] to study interpolation in anisotropic Hardy spaces do not work in our variable exponent setting. New arguments must be developed in order to describe interpolation spaces between our anisotropic Hardy–Lorentz spaces with variable exponents.

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