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ON A CENTRE-LIKE SUBSET OF A RING WITHOUT NIL IDEALS

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We give a new proof of the hypercentre theorem of Herstein.

In [1], Herstein has defined the *hypercentre* of a ring R as follows:

$$T(R) = \{a \in R \mid ax^n = x^n a,\ n = n(x,a) \geq 1 ,\quad \text{all} \quad x \in R\} .$$

Herstein has proved:

THEOREM. *If R is a ring without non-zero nil ideals, then* $T(R) = Z(R)$.

We show that the theorem can be proved by making use of the method which has been given by Herstein in [2] to circumvent the Köthe conjecture. As it has been shown in [1], it suffices to prove the theorem for prime rings without non-zero nil ideals. First we prove the following:

LEMMA. *Let R be a prime ring without non-zero nil right ideals. Then* $T(R) = Z(R)$.

Proof. We prove that $T(R)$ has no non-zero nilpotent elements. If $a \in T(R)$, $a^2 = 0$, then given $x \in R$ there exists $n \geq 1$ such that $0 = a(ax)^n = (ax)^n a$, so $(ax)^{n+1} = 0$. This shows that aR is a nil right ideal, so $a = 0$ by the assumptions on R . Following [1, Lemma 4] we show

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that all the elements of $T(R)$ are regular, so $T(R)$ is a domain. Let $0 \neq a \in T(R)$, and $au = 0$ for some $u \in R$. Then $(uxa)^2 = 0$ for all $x \in R$, and since $T(R)$ is a subring invariant under quasi inner automorphisms on R, we get that

$$a + uxa^2 = (1+uxa)a(1-uxa) = (1+uxa)a(1+uxa)^{-1} \in T(R) .$$

This shows that $uxa^2 \in T(R)$. But $(uxa^2)^2 = 0$, so $uxa^2 = 0$ since $T(R)$ has no non-zero nilpotent elements. We also have $0 \neq a^2$, since $0 \neq a \in T(R)$, so $u = 0$ since R is prime. Now $T(R)$ is a domain, and for all $x, y \in T(R)$ there exists $n = n(x,y) \geq 1$ such that $x^n y = yx^n$, so by [3] $T(R)$ is commutative. By a lemma of Herstein [4, p. 378, $T(R)$ centralizes $J(R)$, so if $J(R) \neq 0$ it follows that $T(R) \subseteq Z(R)$, since R is prime. So we have $T(R) = Z(R)$ if $J(R) \neq 0$, and the same result holds if $J(R) = 0$ by [1, Lemma 2].

Proof of the Theorem. We already know that the result holds if R has no non-zero nil right ideals. Assume R has a non-zero nil right ideal. Since $T(R)$ is a subring invariant under quasi inner automorphisms, it follows by the theorem of Herstein [2], that either $T(R) \subseteq Z(R)$, or $T(R)$ contains a non-zero ideal of R. If $T(R) \subseteq Z(R)$ we are done. If U is a non-zero ideal of R contained in $T(R)$, we prove that $R = Z(R) = T(R)$. For all $x, y \in U$ there exists $n = n(x,y) \geq 1$ such that $x^n y = yx^n$, so by [3] the commutator ideal $C(U)$ of U is nil. Then $UC(U)U$ is a nil ideal of R, so $UC(U)U = 0$ since R has no non-zero nil ideals. This implies $C(U) = 0$ since R is prime, so U is commutative. But a prime ring with a non-zero commutative ideal must be commutative, so $R = Z(R) = T(R)$.

References

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