Can. J. Math., Vol. XXXIII, No. 6, 1981, pp. 1331–1337

SYMBOLIC POWERS OF REGULAR PRIMES

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1. Introduction. In a recent paper [**6**], P. Seibt has obtained the following result: Let k be a field of characteristic 0, $k[T_1, \ldots, T_r]$ the polynomial ring in r indeterminates over k, and let P be a prime ideal of $k[T_1, \ldots, T_r]$. Then a polynomial F belongs to the n-th symbolic power $P^{(n)}$ of P if and only if all higher derivatives of F from the 0-th up to the $(n - 1)$-st order belong to P.

In this work we shall naturally generalize this result so as to be valid for primes of the polynomial ring over a perfect field k. Actually, we shall get a generalization as a corollary to a theorem which asserts: For regular primes P in a k-algebra R of finite type, a certain differential filtration of R associated with P coincides with the symbolic power filtration $(P^{(n)})_{n \geqq 0}$. In order to involve the case in which a ground field has a positive characteristic, we must make an appropriate modification of a differential filtration given in [**6**], which is defined by means of ordinary derivations. This modification is done by making use of higher derivations instead of ordinary derivations.

2. First observations. Throughout this paper, k will denote a field of arbitrary characteristic. Let R be a k-algebra. By a k-higher derivation $\Delta = \{\delta_r\}$ of finite rank n on R, we shall mean a finite sequence of endomorphisms $\delta_0, \delta_1, \ldots, \delta_n$ of R as a k-vector space, which satisfy the following two properties: (a) δ_0 is the identity map of R; and (b) for every r $(0 \leqq r \leqq n)$, and for all $x, y \in R$, we have

$$\delta_r(xy) = \sum_{i+j=r} \delta_i(x)\delta_j(y).$$

The collection of all k-higher derivations of finite rank n on R will be denoted by $H_k^n(R)$. On the other hand we shall let $\mathrm{Der}_k^n(R)$ denote the R-module of all n-th order k-derivations of R to R. Thus $\phi \in \mathrm{Der}_k^n(R)$ if and only if $\phi \in \mathrm{Hom}_k (R, R)$, and for all $x_0, x_1, \ldots, x_n \in R$ we have

$$\phi(x_0x_1 \ldots x_n) = \sum_{s=1}^{n} (-1)^{s-1} \sum_{i_1<\ldots<i_s} x_{i_1} \ldots x_{i_s}\phi(x_0 \ldots \hat{x}_{i_1} \ldots \hat{x}_{i_s} \ldots x_n).$$

For every component δ_r of $\Delta = \{\delta_r\} \in H_k^n(R)$, δ_r is an r-th order derivation

Received May 22, 1980 and in revised form January 29, 1981. This paper is dedicated to Professor Yoshikazu Nakai on his 60th birthday.

of R ([**4**], Proposition 5 in Chapter I). We shall denote by D^n the set of composites $\delta_{\alpha_1}{}^{(1)} \ldots \delta_{\alpha_q}{}^{(q)}$, where each $\delta_{\alpha_i}{}^{(i)}$ is a component of an element of $H_k{}^n(R)$, and $\alpha_1 + \ldots + \alpha_q \leqq n$, q arbitrary. By Corollary 6.1 in Chapter I of [**4**], D^n is a subset of $\mathrm{Der}_k{}^n(R)$. For each $m \leqq n$ we set

$$D_m{}^n = \{\delta_{\alpha_1}{}^{(1)} \ldots \delta_{\alpha_q}{}^{(q)} \in D^n : \alpha_1 + \ldots + \alpha_q \leqq m\}.$$

LEMMA 1. *For $\phi \in D_m{}^n$ there are $\phi_i, \psi_i \in D_{m-1}^n$ such that $\phi_i\psi_i \in D^n$ and*

$$\phi(xy) = \phi(x)y + x\phi(y) + \sum_i \phi_i(x)\psi_i(y)$$

for all $x, y \in R$.

Proof. We can write

$$\phi = \delta_{\alpha_1}{}^{(1)} \ldots \delta_{\alpha_q}{}^{(q)}.$$

Here each $\delta_{\alpha_i}{}^{(i)}$ is a component of an element of $H_k{}^n(R)$ and $\alpha_1 + \ldots + \alpha_q \leqq m$. Then we have

$$\phi(xy) = \sum_{r_1+s_1=\alpha_1} \ldots \sum_{r_q+s_q=\alpha_q} \delta_{r_1}{}^{(1)} \ldots \delta_{r_q}{}^{(q)}(x)\delta_{s_1}{}^{(1)} \ldots \delta_{s_q}{}^{(q)}(y)$$

for all $x, y \in R$, and now our assertion is immediate.

For an ideal I of R, define

$$D^n(I) = \{f \in I : \phi(f) \in I \text{ for every } \phi \in D^n\}.$$

PROPOSITION 1. *$D^n(I)$ is an ideal of R, and we have $I^{n+1} \subset D^n(I)$.*

Proof. The first assertion is immediate from Lemma 1. For the second we have $\phi(I^{n+1}) \subset I$ for every $\phi \in D^n$, since $D^n \subset \mathrm{Der}_k{}^n(R)$, and thus $I^{n+1} \subset D^n(I)$.

PROPOSITION 2. *If Q is a primary ideal of R, then so is $D^n(Q)$.*

Proof. Let $f, g \in R$ be such that $fg \in D^n(Q)$. Assume $f \notin D^n(Q)$. Then either $f \notin Q$, whence $g^s \in Q$ for some $s \geqq 1$, and consequently $(g^s)^{n+1} \in D^n(Q)$ by Proposition 1; or $f \in Q$, but $\phi(f) \notin Q$ for some $\phi \in D_m{}^n, m \leqq n$. In the latter case, we choose such an integer m as small as possible. By Lemma 1 there are $\phi_i, \psi_i \in D_{m-1}^n$ such that

$$\phi(fg) = \phi(f)g + f\phi(g) + \sum_i \phi_i(f)\psi_i(g).$$

Then

$$g\phi(f) = \phi(fg) - f\phi(g) - \sum_i \phi_i(f)\psi_i(g) \in Q.$$

Since $\phi(f) \notin Q$, $g^s \in Q$ for some $s \geqq 1$ and hence $(g^s)^{n+1} \in D^n(Q)$.

Consider a localization $\lambda: R \to S^{-1}R$ of R. For every ideal I of R let $S(I) = \lambda^{-1}(S^{-1}I)$ be the S-saturation of I. On the other hand every higher derivation $\Delta = \{\delta_r\}$ on R has a unique extension $\bar{\Delta} = \{\bar{\delta}_r\}$ to $S^{-1}R$. Then we shall denote by $\bar{D}^n$ the set of composites $\bar{\delta}_{\alpha_1}{}^{(1)} \ldots \bar{\delta}_{\alpha_q}{}^{(q)}$, where each $\bar{\delta}_{\alpha_i}{}^{(i)}$ is a component of a unique extension to $S^{-1}R$ of an element in $H_k{}^n(R)$, and $\alpha_1 + \ldots + \alpha_q \leqq n$. We have

$$\bar{D}^n \subset \mathrm{Der}_k{}^n(S^{-1}R),$$

the set of all n-th order k-derivations of $S^{-1}R$ to $S^{-1}R$. For an ideal $\bar{I}$ of $S^{-1}R$, $\bar{D}^n(\bar{I})$ will denote the set of $f \in \bar{I}$ such that $\bar{\phi}(f) \in \bar{I}$ for every $\bar{\phi} \in \bar{D}^n$.

Proposition 3. *$\bar{D}^n(S^{-1}I) = S^{-1}D^n(S(I))$. In particular, $\bar{D}^n(S^{-1}Q) = S^{-1}D^n(Q)$ for a primary ideal Q of R such that $Q \cap S = \emptyset$, the empty set.*

Proof. Let

$$\phi = \delta_{\alpha_1}{}^{(1)} \ldots \delta_{\alpha_q}{}^{(q)} \in D^n,$$

and set

$$\bar{\phi} = \bar{\delta}_{\alpha_1}{}^{(1)} \ldots \bar{\delta}_{\alpha_q}{}^{(q)} \in \bar{D}^n.$$

By Lemma 1 there exist $\phi_i, \psi_i \in D^n$ such that ϕ_i, ψ_i induce $\bar{\phi}_i, \bar{\gamma}_i \in \bar{D}^n$, and

$$\bar{\phi}(xy) = \bar{\phi}(x)y + x\bar{\phi}(y) + \sum_i \bar{\phi}_i(x)\bar{\psi}_i(y)$$

for all $x, y \in S^{-1}R$. Thus we have

$$\bar{\phi}\left(\frac{f}{s}\right) = \phi(f)\frac{1}{s} + f\bar{\phi}\left(\frac{1}{s}\right) + \sum_i \phi_i(f)\bar{\psi}_i\left(\frac{1}{s}\right)$$

and

$$\phi(f) = \bar{\phi}\left(\frac{f}{s}\right)s + \frac{f}{s}\phi(s) + \sum_i \bar{\phi}_i\left(\frac{f}{s}\right)\psi_i(s)$$

for all $f \in R$, $s \in S$. These yield for $f \in S(I)$:

$$f/s \in \bar{D}^n(S^{-1}I) \text{ if and only if } f \in D^n(S(I));$$

$s \in S$ arbitrary. The first assertion is now immediate. The rest follows from the fact that we have $S(Q) = Q$ for a primary ideal Q satisfying $Q \cap S = \emptyset$.

Proposition 4. *Let $\lambda: R \to S^{-1}R$ be a localization of R, and let Q be a primary ideal of R such that $Q \cap S = \emptyset$. Then*

$$D^n(Q) = \lambda^{-1}\bar{D}^n(S^{-1}Q).$$

Proof. This is immediate from Propositions 2 and 3.

3. Main result. Let R be a local integral domain containing a field k and having the quotient field K. Let L be the residue class field of R and assume L is a separable extension of k. Let $\{v_1, \ldots, v_s\}$ be a separating transcendence basis of L/k and let $u_1, \ldots, u_s$ be representatives of $v_1, \ldots, v_s$ in R. The elements $u_1, \ldots, u_s$ are algebraically independent over k and hence $F = k(u_1, \ldots, u_s)$ is a subfield of R. If it is possible to find u_i's and v_i's as above such that K is a separable extension of F, then we say $\{u_1, \ldots, u_s\}$ is a system of separating representatives in R.

From now on let k denote a perfect field, that is, every extension field of k is separable. A k-algebra R is said to be of *finitely generated type* if R is a localization of a k-algebra of finite type.

LEMMA 2. *Let k be a perfect field and let R be a k-algebra of finitely generated type which is a regular local ring. Then R has a system of separating representatives.*

Proof. Let M be the maximal ideal of R. Since k is perfect, the residue class field R/M is separable over k. Hence the sequence

$$(*) \quad 0 \rightarrow M/M^2 \rightarrow R/M \otimes_R \Omega_k^{\,1}(R) \rightarrow \Omega_k^{\,1}(R/M) \rightarrow 0$$

is exact, where for a k-algebra $S(\Omega_k^{\,1}(S), d)$ denotes the universal object for k-derivations of order 1 on S ([**3**], (8.1) p. 187). Let $\{t_1, \ldots, t_r\}$ be a regular system of parameters for R and let $u_1, \ldots, u_s \in R$ be such that $\bar{u}_1, \ldots, \bar{u}_s \in R/M$ form a separating transcendence basis over k. Then the exact sequence $(*)$ shows that $\Omega_k^{\,1}(R)$ is a free R-module, and $dt_1, \ldots, dt_r$, $du_1, \ldots, du_s$ form a free basis of $\Omega_k^{\,1}(R)$. Now $\{t_1, \ldots, t_r, u_1, \ldots, u_s\}$ is a separating transcendence basis for the quotient field of R over k, and consequently $\{u_1, \ldots, u_s\}$ is a system of separating representatives in R.

PROPOSITION 5. *Let k be a perfect field and let R be a k-algebra of finitely generated type which is a regular local ring with the maximal ideal M. Then $D^n(M) = M^{n+1}$ for all $n \geqq 1$.*

Proof. By Proposition 1 we have $M^{n+1} \subset D^n(M)$. We shall show the converse inclusion relation. Let $\{t_1, \ldots, t_r\}$ be a regular system of parameters for R. Consider $\hat{R}$, the M-adic completion of R. $\hat{R}$ may be identified with $K[[t_1, \ldots, t_r]]$, the ring of formal power series in $t_1, \ldots, t_r$ over $K = R/M$. Let

$$\Delta^{(i)} = \{\delta_j^{(i)}\}_{j \leqq n} \in H_K^{\,n}(\hat{R}), \, 1 \leqq i \leqq r,$$

be the higher derivation defined by

$$\delta_j^{(i)}(t_1^{m_1} \ldots t_i^{m_i} \ldots t_r^{m_r}) = \binom{m_i}{j} t_1^{m_1} \ldots t_i^{m_i - j} \ldots t_r^{m_r},$$

https://doi.org/10.4153/CJM-1981-102-2 Published online by Cambridge University Press

where we put $\binom{m_i}{j} = 0$ for $j > m_i$. With $\hat{M} = (t_1, \ldots, t_r)\hat{R}$ we have: If $f \in \hat{M}$ and if $\delta_{j_1}{}^{(1)} \ldots \delta_{j_r}{}^{(r)}(f) \in \hat{M}$ for all $j_1, \ldots, j_r$ such that $j_1 + \ldots + j_r \leqq n$, then $f \in \hat{M}^{n+1}$. For, write $f = F_n + g$, where $F_n \in K[t_1, \ldots, t_r]$ is a polynomial of degree n and $g \in \hat{M}^{n+1}$. Then it is easily seen that $F_n = 0$. Assume that we have shown the existence of $\Gamma^{(i)} = \{\gamma_j{}^{(i)}\}_{j\leqq n} \in H_k{}^n(R)$ such that $\gamma_j{}^{(i)} = \delta_j{}^{(i)}|R$ for all i, j. Then we obtain what we want: Let $f \in M$ be such that $\phi(f) \in M$ for every $\phi \in D^n$. In particular, we have

$$\gamma_{j_1}{}^{(1)} \ldots \gamma_{j_r}{}^{(r)}(f) \in M \quad \text{for all } j_1, \ldots, j_r$$

with $j_1 + \ldots + j_r \leqq n$, hence

$$\delta_{j_1}{}^{(1)} \ldots \delta_{j_r}{}^{(r)}(f) \in \hat{M} \quad \text{for all } j_1, \ldots, j_r$$

with $j_1 + \ldots + j_r \leqq n$. This implies $F_n = 0$ and thus

$$f = g \in \hat{M}^{n+1} \cap R = M^{n+1}.$$

It remains to prove that there exist $\Gamma^{(i)} = \{\gamma_j{}^{(i)}\} \in H_k{}^n(R)$, $1 \leqq i \leqq r$, satisfying $\gamma_j{}^{(i)} = \delta_j{}^{(i)}|R$ for every i, j.

Let $\Omega_k(R)$ be the universal algebra of higher differentials on R over k and let

$$\Delta = \{\delta_j\}: R \rightarrow \Omega_k(R)$$

be the canonical k-higher derivation of infinite rank (Cf. [**1**]). By Lemma 2 R has separating representatives $u_1, \ldots, u_s$. Then $\Omega_k(R)$ is a free R-algebra with a free basis

$$\{\delta_j(t_l), \delta_j(u_m): l = 1, \ldots, r, m = 1, \ldots, s, j = 1, 2, \ldots, \infty\}$$

([**1**], Theorem 3). On the other hand it is easily shown that each $\Delta^{(i)} = \{\delta_j{}^{(i)}\}_{j\leqq n} \in H_K{}^n(\hat{R})$ can be imbedded into a higher derivation $\{\delta_j{}^{(i)}\}$ of infinite rank. Hence there are uniquely determined k-higher derivations $\Gamma^{(i)} = \{\gamma_j{}^{(i)}\}$ on R of infinite rank such that

$$\gamma_j{}^{(i)}(t_l) = \delta_j{}^{(i)}(t_l) \quad \text{and} \quad \gamma_j{}^{(i)}(u_m) = 0$$

for all i, j, l, m. Then it is obvious that $\gamma_j{}^{(i)} = \delta_j{}^{(i)}|R$ for all i, j. Thus $\{\gamma_j{}^{(i)}\}_{j\leqq n} \in H_k{}^n(R)$, $1 \leqq i \leqq r$, are the required ones.

THEOREM. *Let k be a perfect field and let R be a k-algebra of finite type. For $P \in$ Spec (R) suppose R_P is a regular local ring. Then we have $D^n(P) = P^{(n+1)}$ for all $n \geqq 1$.*

Proof. Let $\lambda: R \rightarrow R_P$ be the canonical homomorphism and set $M = PR_P$. Then by Proposition 4

$$D^n(P) = \lambda^{-1}\bar{D}^n(M).$$

Let $\{\delta_j\}_{j\leqq n}$ be a k-higher derivation of R_P of rank n. Then there exist elements $s_i \in R - P$, $i = 1, \ldots, n$, such that $\{\delta_0, s_1\delta_1, \ldots, s_n\delta_n\}$ is a k-higher derivation of rank n on R ([**2**], Lemma 2). Set

$$\gamma_i = s_i\delta_i,\ i = 0, 1, \ldots, n,\ s_0 = 1.$$

We denote by $\{\bar{\gamma}_i\}_{i \leqq n}$ the unique extension of $\{\gamma_i\}_{i \leqq n}$ to R_P. Thus $\delta_i = 1/s_i\bar{\gamma}_i$ on R_P, $i = 0, 1, \ldots, n$. Let

$$\phi = \delta_{\alpha_1}{}^{(1)} \ldots \delta_{\alpha_q}{}^{(q)}$$

be a composite of components of higher derivations on R_P. Then there are elements $s_i \in R - P$, $i = 1, \ldots, q$, and a family of higher derivations $\{\gamma_j{}^{(i)}\}$, $i = 1, \ldots, q$, on R such that

$$\phi = \left(\frac{1}{s_1}\bar{\gamma}_{\alpha_1}{}^{(1)}\right) \ldots \left(\frac{1}{s_q}\bar{\gamma}_{\alpha_q}{}^{(q)}\right).$$

Here $\bar{\gamma}_{\alpha_i}{}^{(i)}$ denotes the unique extension of $\gamma_{\alpha_i}{}^{(i)}$ to R_P. Now we see easily that ϕ is an R_P-linear combination of elements of $\bar{D}^n$, and consequently $\bar{D}^n(M) = M^{n+1}$ by Proposition 5. Thus

$$D^n(P) = \lambda^{-1}(M^{n+1}) = P^{(n+1)} \quad \text{for all } n \geqq 1.$$

Let $R = k[T_1, \ldots, T_r]$ be the polynomial ring in r indeterminates over k. Let

$$\Delta^{(i)} = \{\delta_j{}^{(i)}\}_{j \leqq n} \in H_k{}^n(R), \quad 1 \leqq i \leqq r,$$

be the higher derivation defined by

$$\delta_j{}^{(i)}(T_1{}^{m_1} \ldots T_i{}^{m_i} \ldots T_r{}^{m_r}) = \binom{m_i}{j} T_1{}^{m_1} \ldots T_i{}^{m_i - j} \ldots T_r{}^{m_r}$$

where $\binom{m_i}{j} = 0$ for $j > m_i$. Symbolically we shall write

$$\delta_j{}^{(i)} = \frac{1}{j!}\frac{\partial^j}{\partial T_i{}^j} \quad \text{and} \quad \delta_{j_1}{}^{(1)} \ldots \delta_{j_r}{}^{(r)} = \frac{1}{j_1! \ldots j_r!}\frac{\partial^q}{\partial T_1{}^{j_1} \ldots \partial T_r{}^{j_r}}$$

with $j_1 + \ldots + j_r = q$. The proof of Proposition 18 in [**5**] shows that

$$\frac{1}{j_1! \ldots j_r!}\frac{\partial^q}{\partial T_1{}^{j_1} \ldots \partial T_r{}^{j_r}} \ (j_1 + \ldots + j_r = q, q = 1, \ldots, n)$$

form an R-free basis of $\mathrm{Der}_k{}^n(R)$. Now the following assertion is an immediate consequence of our theorem.

COROLLARY. *For a prime ideal P of $k[T_1, \ldots, T_r]$ we have*

$$P^{(n+1)} = \left\{F \in P: \frac{1}{j_1! \ldots j_r!}\frac{\partial^q F}{\partial T_1{}^{j_1} \ldots \partial T_r{}^{j_r}} \in P,\ j_1 + \ldots + j_r = q, q = 1, \ldots, n\right\}.$$

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https://doi.org/10.4153/CJM-1981-102-2 Published online by Cambridge University Press