SIGNAL METRICS

WILLIAM F. DARSOW

1. Introduction. The purpose of this paper is to introduce a generalization of metric space that arises naturally out of the notion of signal function as it occurs, for example, in **(5)**. In §§ 2–5, the basic definitions and motivation are given. In §§ 6 and 7 several elementary topological properties are proved, and in §§ 8 and 9 an important example from special relativity is developed.

2. A special group. Let Γ be the set of all order-automorphisms of the real line R. For members of Γ we shall find it convenient to use the following function notation: $t\phi$ or $(t)\phi$ is the value of ϕ at t. Thus, Γ is the set of all (necessarily continuous) one-to-one maps ϕ of R onto R such that $s\phi \leqslant t\phi$ whenever $s \leqslant t$. For ϕ and ψ in Γ, composition reads from left to right: the value of $\phi\psi$ at t is $(t\phi)\psi$. With respect to composition, Γ is a group. The group identity e is the functional identity: $te = t$ for all t in R. And the inverse ϕ^{-1} of ϕ in Γ is the functional inverse: $s = t\phi^{-1}$ if and only if $t = s\phi$.

For ϕ, ψ in Γ define $\phi \leqslant \psi$ if and only if $t\phi \leqslant t\psi$ for all t in R. Γ is a distributive lattice with respect to $\leqslant$. The meet $\wedge$ and join $\vee$ operations on Γ satisfy $t(\phi \wedge \psi) = \min[t\phi, t\psi]$ and $t(\phi \vee \psi) = \max[t\phi, t\psi]$ for all t in R. With respect to both $\leqslant$ and composition, Γ is a lattice-ordered group; see **(2**, chap. 14**)**.

3. Signal metrics. A *signal metric* on a set X is any function $f\colon X \times X \to \Gamma$ such that, for all x, y, z in X,

$$f_{xz} \leqslant f_{xy} f_{yz}, \tag{3.1}$$

$$f_{xx} = e, \tag{3.2}$$

$$f_{xy} f_{yx} > e \qquad \text{when } x \neq y. \tag{3.3}$$

Note that f_{xy} is the value of f at (x, y) in $X \times X$. Also the *definiteness* condition (3.3) means that when $x \neq y$ then $tf_{xy} f_{yx} > t$ for at least one t. We shall say that f is *strongly definite* if $tf_{xy} f_{yx} > t$ for all t whenever $x \neq y$. Neither positivity nor symmetry is assumed. $f\colon X \times X \to \Gamma$ is *positive* when $f_{xy} \geqslant e$ for all x, y in X, it is *symmetric* when $f_{xy} = f_{yx}$ for all x, y in X.

$f\colon X \times X \to \Gamma$ is a *signal semi-metric* on X when (3.1) and (3.2), but not necessarily (3.3), hold for all x, y, z in X. In this case (3.1) and (3.2) imply a weak form of (3.3): $f_{xy} f_{yx} \geqslant e$ for all x, y in X.

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A *signal space* is a pair (X, f) where f is a signal metric on X. A subset Y of X is a *subspace* of X in the sense that the restriction of f to $Y \times Y$ is also clearly a signal metric on Y.

For signal spaces (X, f) and (Y, g) a one-to-one map ϕ of X onto Y is a *signal isometry* when $g_{x\phi y\phi} = f_{xy}$ for all x, y in X. Here $x\phi$ is the value of ϕ at x.

A metric d on X may be identified with a signal metric $\tilde{d}$ on X by defining $t\tilde{d}_{xy} = t + d(x, y)$ for all t in R and all x, y in X. As is easily verified, $\tilde{d}$ is positive, strongly definite, and symmetric.

4. Interpretation. For a signal metric or semi-metric f on X we shall generally have the following interpretation in mind. Each member of X is an observer equipped with a clock. The clock of an observer assigns a real number as the time of occurrence to each event occurring at the observer. It is not assumed to assign a time of occurrence to any event occurring elsewhere; see **(9)** for a detailed analysis of the extensive abstraction involved in these notions of event and time. For each ordered pair (x, y) of observers, f_{xy} is the *signal function* from x to y: tf_{xy} is the time by y's clock that he receives a direct light signal emitted by x at x's time t. That f_{xy} is in Γ indicates that light signals are received at y in the same order that they are emitted by x and that at no time from $-\infty$ to $+\infty$ is either observer out of light reach of the other.

Now suppose that y, upon receiving a light signal at his time tf_{xy}, immediately relays the signal on to an observer z. Then z will receive the relayed signal at his time $tf_{xy}f_{yz}$, whereas the direct signal from x is received at z's time tf_{xz}. The *triangle inequality* (3.1) implies that a relayed signal never arrives before the direct signal.

With the idealization involved, each observer would receive his own signal immediately, as is asserted by (3.2).

Next, suppose that y, upon receiving a signal at his time tf_{xy}, immediately reflects the signal back to x. Then x will receive the reflected signal at his time $tf_{xy}f_{xy}$. According to x the time lapse for the round trip of the signal is $tf_{xy}f_{yx} - t$. This is not negative and, in a sense, measures the separation of y from x at x's time t. Condition (3.3) says that when $y \neq x$, then at some time y is separated from x. This condition is imposed to simplify topological considerations. To state that f is strongly definite may be interpreted as saying that no two distinct observers ever collide or intersect.

Neither symmetry nor positivity is imposed because of the important example developed in § 9.

The system axiomatized by A. G. Walker **(8)** can be regarded as a signal space of a special type. The interpretation given above essentially agrees with his.

5. Re-graduation. Consider a signal semi-metric f on X. With the interpretation of the preceding section the notion of re-graduation arises. Suppose

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each observer x re-graduates his clock: a new time of occurrence $t\theta_x$ is assigned to any event at x whose old time of occurrence is t. We shall only consider the case where θ_x is in Γ, so that the ordering of events at x is preserved and a lifetime is still from $-\infty$ to $+\infty$. Such a re-graduation induces a new signal function $g_{xy} = \theta_x^{-1} f_{xy} \theta_y$ from x to y. It is easy to verify that g is also a signal semi-metric on X and that g will be definite or strongly definite whenever f is definite or strongly definite, respectively.

Any map $\theta: X \to \Gamma$ will be called a *re-graduation function* on X. And a signal semi-metric g on X will be called a *re-graduation* of f (by way of θ) when $g_{xy} = \theta_x^{-1} f_{xy} \theta_y$ for all x, y in X.

Let $\theta_x = (f_{ax})^{-1}$ for some fixed a in X. Then θ is a re-graduation function, and it is easily verified that the re-graduation of f by way of θ is positive. Thus,

THEOREM 5.1. *A signal metric f on X has a positive re-graduation.*

Let Γ_0 be the set of all ϕ in Γ of the form $t\phi = at + b$ with a, b in R and $a > 0$. A re-graduation function $\theta: X \to \Gamma_0$ is an *affine re-graduation* function.

THEOREM 5.2. *If $f: X \times X \to \Gamma_0$ is a signal metric on X, then there is a re-graduation g of f by way of an affine re-graduation function and there is a non-symmetric metric d on X such that $g = \tilde{d}$.*

Proof. Use the re-graduation function in the proof of Theorem 5.1 and observe that if $at + b \geqslant t$ for all t, then $a = 1$ and $b \geqslant 0$.

Remark. If f is a signal semi-metric on X, then by analogy with the case for semi-metrics members x, y of X may be identified when $f_{xy} f_{yx} = e$. In general the result does not yield a signal metric unless the original f was positive.

6. The induced topology. Consider a signal space (X, f). According to the interpretation of § 4, $tf_{xy} f_{yx} - t$ is the length of time at x for a signal emitted from x at his time t to make the round trip to y and back. From x's point of view: the smaller $tf_{xy} f_{yx} - t$ is, the closer is y to x. This suggests the topology on X which has for a subbase all sets

$$N_x(\epsilon, t) = \{y: tf_{xy} f_{yx} - t < \epsilon\} \quad \text{where } x \in X,\ t \in R, \text{ and } \epsilon > 0.$$

This topology—i.e., the family $\mathbf{T}(X, f)$ of all open sets—is *the topology on X induced by f*. The main result of this section is that the induced topology is metrizable and independent of re-graduation.

For a subset T of R, let $N_x(\epsilon, T) = \bigcap \{N_x(\epsilon, t): t \in T\}$.

LEMMA 6.1. *If $y \in N_x(\epsilon, T)$ where T is a compact subset of R, then there is a finite set S of rationals and a rational $r > 0$ such that*

$$N_y(r, S) \subset N_x(\epsilon, T).$$

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Proof. Consider t in T. Then $tf_{xy}f_{yx} < t + \epsilon$. Hence, $tf_{xy} < (t+\epsilon)(f_{yx})^{-1}$. Since f_{xy} and $(f_{yx})^{-1}$ are continuous, there exist positive $\delta(t)$ and $\epsilon(t)$ such that

$$[t + \epsilon(t)]f_{xy} + \delta(t) < [t - \epsilon(t) + \epsilon](f_{yx})^{-1}.$$

Since T is compact, finitely many of the intervals $(t - \epsilon(t), t + \epsilon(t))$—say, $(t_i - \epsilon(t_i), t_i + \epsilon(t_i))$ for $i = 1, \ldots, n$—cover T. Let

$$\delta_0 = \min\{\delta(t_i): 1 \leqslant i \leqslant n\},\ v_i = [t_i + \epsilon(t_i)]f_{xy}, \text{ and } V = \{v_1, \ldots, v_n\}.$$

Suppose $z \in N_y(\delta_0, V)$. If $t \in T$, then $t_i - \epsilon(t_i) < t < t_i + \epsilon(t_i)$ for some i. Hence,

$$tf_{xz}f_{zx} \leqslant tf_{xy}f_{yz}f_{zy}f_{yx} < [t_i + \epsilon(t_i)]f_{xy}f_{yz}f_{zy}f_{yx} = v_i f_{yz}f_{zy}f_{yx}.$$

But $v_i f_{yz} f_{zy} < v_i + \delta_0$ so that

$$tf_{xz}f_{zx} < (v_i + \delta_0)f_{yx} \leqslant [v_i + \delta(t_i)]f_{yx}.$$

Since

$$v_i + \delta(t_i) = [t_i + \epsilon(t_i)]f_{xy} + \delta(t_i) < [t_i - \epsilon(t_i) + \epsilon](f_{yx})^{-1},$$

then

$$tf_{xz}f_{zx} < t_i - \epsilon(t_i) + \epsilon < t + \epsilon.$$

Thus, $z \in N_x(\epsilon, T)$. Consequently, $N_y(\delta_0, V) \subset N_x(\epsilon, T)$. Now, since $v_i < v_i + \delta_0$ for $i = 1, \ldots, n$, there exist rational s_i and a rational $r > 0$ such that $v_i \leqslant s_i < s_i + r \leqslant v_i + \delta_0$. Let $S = \{s_1, \ldots, s_n\}$. Suppose $z \in N_y(r, S)$. Since

$$v_i f_{yz} f_{zy} \leqslant s_i f_{yz} f_{zy} < s_i + r \leqslant v_i + \delta_0 \qquad \text{for all } i,$$

then $z \in N_y(\delta_0, V)$. Thus, $N_y(r, S) \subset N_y(\delta_0, V) \subset N_x(\epsilon, T)$.

THEOREM 6.2. *Each of the following families is an open base at x: $\mathbf{B}_x{}^0$ which consists of all $N_x(\epsilon, T)$ where ϵ is a positive rational and T is a finite set of rationals; $\mathbf{B}_x$ which consists of all $N_x(\epsilon, T)$ where $\epsilon > 0$ and T is a finite subset of R; and $\mathbf{B}_x{}^c$ which consists of all $N_x(\epsilon, T)$ where $\epsilon > 0$ and T is a compact subset of R.*

Proof. Since $\mathbf{B}_x{}^0 \subset \mathbf{B}_x \subset \mathbf{B}_x{}^c$, it is clear from Lemma 6.1 that it suffices to show that $\mathbf{B}_x{}^0$ is an open base at x. This follows from Lemma 6.1.

THEOREM 6.3. *If g is a re-graduation of f, then $\mathbf{T}(X, g) = \mathbf{T}(X, f)$.*

Proof. This is left to the reader.

THEOREM 6.4. *$\mathbf{T}(X, f)$ is Hausdorff.*

Proof. Consider r, t in R and x, y in X such that $r < tf_{xy}f_{yx}$. We shall show that there is an $N_y(\delta, s)$ such that $r < tf_{xz}f_{zx}$ for all z in $N_y(\delta, s)$. Since $r(f_{yx})^{-1} < tf_{xy}$, then there is a $\delta > 0$ such that $r(f_{yx})^{-1} < tf_{xy} - \delta$ and, thus, such that $r < (tf_{xy} - \delta)f_{yx}$.

Let $s = tf_{xy} - \delta$. Suppose $z \in N_y(\delta, s)$. Then, $sf_{yz}f_{zy} < s + \delta$. Since $f_{zy} \geqslant (f_{xz})^{-1}f_{xy}$ and $f_{yz} \geqslant f_{yx}(f_{zx})^{-1}$, then

$$sf_{yx}(f_{xz})^{-1}\,(f_{zx})^{-1}f_{xy} \leqslant sf_{yz}f_{zy} < s + \delta = tf_{xy}.$$

Thus, $sf_{yx} < tf_{xz}f_{zx}$ and $r < (tf_{xy} - \delta)f_{yx} < tf_{xz}f_{zx}$.

Now, consider $x \neq y$ in X. Then there is a t in R and an $\epsilon > 0$ such that $t + \epsilon < tf_{xy}f_{yx}$. By the above, there is an $N_y(\delta, s)$ such that $t + \epsilon < tf_{xz}f_{zx}$ for all z in $N_y(\delta, s)$. Thus $N_y(\delta, s)$ is disjoint from $N_x(\epsilon, t)$. Since these are open by Theorem 6.2, then $\mathbf{T}(X, f)$ is Hausdorff.

THEOREM 6.5. *$\mathbf{T}(X, f)$ is metrizable.*

Proof. By Theorems 5.1 and 6.3, it may be assumed that f is positive. For a rational $\epsilon > 0$ and a finite set T of rationals, let $V(\epsilon, T)$ be the set of all (x, y) in $X \times X$ such that $tf_{xy} < t + \epsilon$ and $tf_{yx} < t + \epsilon$, for all t in T. Let $\mathbf{B}$ be the family of all such $V(\epsilon, T)$. It may be verified that $\mathbf{B}$ is a uniformity on X which induces a topology coincident with $\mathbf{T}(X, f)$. Since $\mathbf{B}$ is countable and the topology is Hausdorff, then the topology is metrizable; cf. (**4**, p. 186).

For this proof I am indebted to J. Cibulskis.

Remark. It should be observed that a metric d on X induces the same topology on X as the corresponding signal metric $\tilde{d}$ as defined in § 3.

7. Continuity of signal metrics. With the topology induced by signal metrics, the appropriate topology to consider on Γ is that of pointwise convergence. Apart from some observations about this topology, the main result of this section is that $f: X \times X \to \Gamma$ is continuous when f is a positive signal metric.

For $\alpha \in \Gamma$, $\epsilon > 0$, and a subset T of R, let $N_\alpha(\epsilon, T)$ be the set of all $\phi \in \Gamma$ such that $|t\alpha - t\phi| < \epsilon$ for all $t \in T$. The following is easily verified.

LEMMA 7.1. *If $\beta \in N_\alpha(\epsilon, T)$ where T is compact, then there is a $\delta > 0$ such that $N_\beta(\delta, T) \subset N_\alpha(\epsilon, T)$.*

From this and the fact that $N_\alpha(\epsilon, T) \subset N_\alpha(\delta, S)$ when $\epsilon \leqslant \delta$ and $T \supset S$, it follows readily that the family of all $N_\alpha(\epsilon, T)$ where $\epsilon > 0$ and T is finite (compact) is an open base at α for the topology of pointwise convergence (uniform convergence on compacta) on Γ.

LEMMA 7.2. *If T is compact and $\epsilon > 0$, there is a rational $r > 0$ and a finite set S of rationals such that $N_e(r, S) \subset N_e(\epsilon, T)$.*

Proof. There exist a positive rational $r \leqslant \epsilon/2$ and a finite set $S = \{s_0, \ldots, s_n\}$ of rationals such that $0 < s_{i+1} - s_i \leqslant r$ and such that T is a subset of the closed interval $[s_0, s_n]$. Consider $\phi \in N_e(r, S)$ and $t \in T$. For some i, $s_i \leqslant t \leqslant s_{i+1}$. Hence,

$$s_i\phi \leqslant t\phi \leqslant s_{i+1}\phi \quad \text{and} \quad s_i\phi - s_{i+1} \leqslant t\phi - t \leqslant s_{i+1}\phi - s_i.$$

Since $s_{i+1} - s_i \leqslant r$, then

$$s_i\phi - s_i - r \leqslant t\phi - t \leqslant s_{i+1}\phi - s_{i+1} + r.$$

But $-r < s_k\phi - s_k < r$ for all k since $\phi \in N_e(r, S)$. Thus,

$$-2r < t\phi - t < 2r$$

and $|t\phi - t| < \epsilon$. Consequently, $\phi \in N_e(\epsilon, T)$ and $N_e(r, S) \subset N_e(\epsilon, T)$.

THEOREM 7.3. *For* Γ *the topologies of pointwise convergence and uniform convergence on compacta coincide.*

The proof of this theorem follows directly from the two preceding lemmas. Moreover, since Γ is a homeomorphism group, it is known that with respect to the compact open topology, it is a topological group whose two-sided uniformity is complete; see **(1)**. But for Γ the compact open topology and the topology of uniform convergence on compacta coincide; c.f. **(4**, p. 230**)**. Also by Lemma 7.2, the topology of pointwise convergence has a countable base at e. Hence this topology is metrizable; cf. **(3**, p. 70**)**. Thus we have

THEOREM 7.4. *The topology of pointwise convergence on* Γ *is metrizable. With respect to it,* Γ *is a topological group whose two-sided uniformity is complete. Moreover, the lattice operations of meet and join are continuous.*

THEOREM 7.5. $f: X \times X \to \Gamma$ *is continuous when* f *is a positive signal metric.*

Proof. To avoid excessive subscripts, we shall write xy instead of f_{xy}. Since the spaces involved are metrizable, it suffices to show that if $x_n \to x$ and $y_n \to y$ in X, then $tx_n y_n \to txy$ for all t in R.

Consider $\epsilon > 0$ and t in R. Since xy is continuous, there is a $\delta > 0$ such that

$$(t + \delta)xy - \epsilon < txy < (t - \delta)xy + \epsilon.$$

Let $\epsilon_1 = \min[\epsilon, \delta]$. Then, $\epsilon_1 > 0$ and

$$(t + \epsilon_1)xy - \epsilon < txy < (t - \epsilon_1)xy + \epsilon.$$

Let ϵ_2 be the smaller of $txy - (t + \epsilon_1)xy + \epsilon$ and $(t - \epsilon_1)xy - txy + \epsilon$. Then $\epsilon_2 > 0$ and

$$(1) \qquad (t + \epsilon_1)xy + \epsilon_2 - \epsilon \leqslant txy \leqslant (t - \epsilon_1)xy - \epsilon_2 + \epsilon.$$

Let $T_1 = \{t, t - \epsilon_1\}$ and $T_2 = \{txy - \epsilon, (t + \epsilon_1)xy\}$. Since $x_n \to x$ and $y_n \to y$, there is an integer n_0 such that $x_n \in N_x(\epsilon_1, T_1)$ and $y_n \in N_x(\epsilon_2, T_2)$ when $n \geqslant n_0$. Thus, when $n \geqslant n_0$,

$$txx_n\, x_n\, x < t + \epsilon_1,$$
$$(t - \epsilon_1)xx_n\, x_n\, x < t,$$
$$(txy - \epsilon)yy_n\, y_n\, y < txy - \epsilon + \epsilon_2,$$
$$(t + \epsilon_1)xyyy_n\, y_n\, y < (t + \epsilon_1)xy + \epsilon_2;$$

and since $xx_n, x_n x, yy_n, y_n y \geqslant e$, we obtain

$$tx_n x < t + \epsilon_1, \tag{2}$$

$$(t - \epsilon_1)xx_n < t, \tag{3}$$

$$(txy - \epsilon)y_n y < txy - \epsilon + \epsilon_2, \tag{4}$$

$$(t + \epsilon_1)xyyy_n < (t + \epsilon_1)xy + \epsilon_2. \tag{5}$$

By (4) and (1), $(txy - \epsilon)y_n y < (t - \epsilon_1)xy$. By (3), $(t - \epsilon_1)xy < t(xx_n)^{-1} xy$. Thus,

$$txy - \epsilon < t(xx_n)^{-1} xy(y_n y)^{-1}. \tag{6}$$

By (2) and (5), $tx_n xxyyy_n < (t + \epsilon_1)xy + \epsilon_2$. By (1),

$$(t + \epsilon_1)xy + \epsilon_2 \leqslant txy + \epsilon.$$

Thus,

$$tx_n xxyyy_n < txy + \epsilon. \tag{7}$$

Since $xy \leqslant xx_n x_n y_n y_n y$ and since $x_n y_n \leqslant x_n xxyyy_n$, then from (6) and (7) we obtain $txy - \epsilon < tx_n y_n < txy + \epsilon$ when $n \geqslant n_0$. Thus, $tx_n y_n \to txy$.

Remark. There are a number of unresolved questions about the completeness of $\mathbf{T}(X,f)$. However, by Theorem 7.5 it is easy to prove the useful fact that $\mathbf{T}(X,f)$ is the smallest topology on X for which all maps $x \to f_{xa} f_{ax}$ are continuous.

8. Lorentz transformations. In this section some notation and properties of Lorentz transformations that will be used in the next section are given.

In Cartesian 3-space R^3 let $x \cdot y$ and $|x|$ be the usual inner product of x, y and norm of x. For v in R^3, let v^* be the linear functional on R^3 defined by $xv^* = x \cdot v$ for all x in R^3, and let v^*v be the linear transformation of R^3 defined by $xv^*v = (x \cdot v)v$ for all x in R^3. For v in R^3 let

$$a = (|v|^2 + 1)^{\frac{1}{2}} \quad \text{and} \quad \alpha = 1/(a + 1). \tag{1}$$

When subscripts are attached to v, the corresponding subscripts will be attached to a and α.

For v in R^3, an orthogonal map $V\colon R^3 \to R^3$, and $\epsilon = \pm 1$, let $[v, V, \epsilon]$ be the linear transformation of $R^3 \times R$ defined by

$$(x, t)[v, V, \epsilon] = (x, t)\begin{bmatrix} I + \alpha v^*v & v^* \\ v & a \end{bmatrix}\begin{bmatrix} V & 0 \\ 0 & \epsilon \end{bmatrix}$$

for all (x, t) in $R^3 \times R$. I is the identity transformation of R^3 and a, α are given by (1). It is not difficult to see that the transformations $[v, V, \epsilon]$ are precisely the Lorentz transformations of $R^3 \times R$, i.e. the linear transformations of $R^3 \times R$ that leave the quadratic form $|x|^2 - t^2$ invariant.

The *affine Lorentz group* L is the group of all transformations $[v, V, \epsilon, w, r]$ defined by

$$(x, t)[v, V, \epsilon, w, r] = (x, t)[v, V, \epsilon] + (w, r)$$

where $[v, V, \epsilon]$ is a Lorentz transformation and (w, r) is in $R^3 \times R$. Let L^* be the subgroup of all $[v, V, \epsilon, w, r]$ with $\epsilon = +1$.

The following lemmas are easily verified.

LEMMA 8.1. $(I + \alpha v^*v)^{-1} = I - (\alpha/a)v^*v$.

LEMMA 8.2. *If* $[v_0, V_0, \epsilon_0, w_0, r_0] = [v, V, \epsilon, w, r]^{-1}$, *then*

$$v_0 = -\epsilon v V, \qquad V_0 = V^{-1}, \qquad \epsilon_0 = \epsilon,$$

$$w_0 = -wV^{-1}(I + \alpha v^*v) + \epsilon r v,$$

$$r_0 = wV^{-1}\cdot v - \epsilon a r.$$

LEMMA 8.3. $|x(I + \alpha v^*v) - |x|\, v| = |x|\, a - x\cdot v$ *for all* x, v *in* R^3.

Remark. The results of this and the next section extend with minor changes to the case where R^3 is replaced by any real Hilbert space and the speed of light is not taken as unity.

9. Relativistically related observers. Consider a system (X, E, ϕ, σ) that consists of non-empty sets X and E, a one-to-one map ϕ_i of E onto $R^3 \times R$ for each i in X, and a map σ_{ij} on R into the two element set $\{-1, +1\}$ for each i, j in X. Assume that $\phi_i^{-1}\,\phi_j \in L$ for all i, j.

Our interpretation is as follows (cf. the system of relativistically related frames of P. Suppes **(7)**). E is space-time. X is a collection of observers. Each observer i in X has a coordinate frame $\phi_i\colon E \to R^3 \times R$ with respect to which i is stationary at his spatial origin. That is, the *world line* of i is the set $E_i = \{(0, t)\phi_i^{-1}\colon t \in R\}$. Suppose i emits a light signal at his time t and that the signal is received by j at j's time $\bar{t}$. According to j's coordinate frame the events of emission and reception have coordinates $(y, s) = (0, t)\phi_i^{-1}\phi_j$ and $(0, \bar{t})$, respectively. If the speed of light is unity, then in special relativity $|y| = |\bar{t} - s|$. Hence, $\bar{t} = s \pm |y|$, that is, the *signal* function f_{ij} from i to j has the form

$$(1) \qquad tf_{ij} = s + t\sigma_{ij}\,|y|, \qquad (y, s) = (0, t)\phi_i^{-1}\,\phi_j.$$

Now, suppose that i emits a signal at a time t when E_i and E_j do not intersect. That is, suppose $(0, t)\phi_i^{-1} \notin E_j$. Since the event of emission does not occur at j, it is plausible to assume that the return time $tf_{ij}f_{ji}$ of the signal reflected back from j will be later than t. Thus,

$$(\mathrm{I}) \qquad tf_{ij}f_{ji} > t, \qquad \text{when} \quad (0, t)\phi_i^{-1} \notin E_j.$$

It will also be desirable to identify observers with the same world line; see **(6; 8)**. So

$$\text{(II)} \qquad E_i \neq E_j \quad \text{when } i \neq j.$$

When (I) and (II) hold, the system (X, E, ϕ, σ) will be called *admissible.* We shall show that in this case f is a signal metric on X. Also, up to signal isometry and re-graduation, we shall obtain an explicit realization of the signal space (X, f).

Since $\phi_i^{-1}\phi_j$ has the form $[v_{ij}, V_{ij}, \epsilon_{ij}, w_{ij}, r_{ij}]$, it is directly verifiable that y and s in (1) are given by

$$(2) \qquad y = tv_{ij} V_{ij} + w_{ij}, \qquad s = t\epsilon_{ij} a_{ij} + r_{ij}.$$

Thus,

$$(3) \qquad tf_{ij} = t\epsilon_{ij} a_{ij} + r_{ij} + t\sigma_{ij} |tv_{ij} V_{ij} + w_{ij}|.$$

LEMMA 9.1. *Suppose that f_{ij} is continuous. Then f_{ij} is an order automorphism or an order anti-automorphism of R depending on whether $\epsilon_{ij} = +1$ or -1, respectively. Moreover, σ_{ij} is continuous at every t for which*

$$tv_{ij} V_{ij} + w_{ij} \neq 0.$$

Proof. Upon considering (3), the last sentence of the lemma is clearly true. The rest of the proof is clear when $tv_{ij} V_{ij} + w_{ij} = 0$ for all t. Otherwise there is at most one t for which $tv_{ij} V_{ij} + w_{ij} = 0$ and, consequently, there is a t_0 such that σ_{ij} is constant to the left of t_0 and constant to the right of t_0. By (3), then,

$$(tf_{ij})/t \to \epsilon_{ij} a_{ij} \pm (t_0 \pm 1)\sigma_{ij} |v_{ij}| \qquad \text{as } t \to \pm\infty.$$

This shows that f_{ij} is neither bounded above nor below; for if such a bound existed, it would follow that $a_{ij} \leqslant |v_{ij}|$, which is impossible. The proof will be complete if $(t_1 f_{ij} - t_2 f_{ij})\epsilon_{ij} < 0$ whenever $t_1 < t_2$. If $t_1\sigma_{ij} = t_2\sigma_{ij}$, then

$$\begin{aligned}(t_1 f_{ij} - t_2 f_{ij})\epsilon_{ij} &= (t_1 - t_2)a_{ij} \pm (|t_1 v_{ij} V_{ij} + w_{ij}| - |t_2 v_{ij} V_{ij} + w_{ij}|)\\ &\leqslant (t_1 - t_2)a_{ij} + |t_1 v_{ij} V_{ij} - t_2 v_{ij} V_{ij}|\\ &= (t_1 - t_2)(a_{ij} - |v_{ij}|) < 0.\end{aligned}$$

If $t_1\sigma_{ij} \neq t_2\sigma_{ij}$, then t_0 is a point of discontinuity of σ_{ij}, so that

$$t_0 v_{ij} V_{ij} + w_{ij} = 0,$$

$t_1 \leqslant t_0 \leqslant t_2$, and $t_1\sigma_{ij} = -(t_2\sigma_{ij})$. Hence,

$$\begin{aligned}(t_1 f_{ij} - t_2 f_{ij})\epsilon_{ij} &= (t_1 - t_2)a_{ij} \pm (|t_1 v_{ij} V_{ij} + w_{ij}| + |t_2 v_{ij} V_{ij} + w_{ij}|)\\ &= (t_1 - t_2)a_{ij} \pm (|(t_1 - t_0)v_{ij} V_{ij}| + |(t_2 - t_0)v_{ij} V_{ij}|)\\ &= (t_1 - t_2)a_{ij} \pm [(t_0 - t_1) + (t_2 - t_0)]\,|v_{ij}|\\ &= (t_1 - t_2)(a_{ij} \mp |v_{ij}|) < 0.\end{aligned}$$

LEMMA 9.2. *For i, j, k, in X*

$$tf_{ij}f_{jk} - tf_{ik} = t\sigma_{ij}\,|A|\,\epsilon_{jk} + tf_{ij}\,\sigma_{jk}\,|B| - t\sigma_{ik}\,|A + B|, \tag{4}$$

$$tf_{ij}f_{ji} - t = (t\sigma_{ij}\,\epsilon_{ij} + tf_{ij}\,\sigma_{ji})\,|\tau v_{ij}\,V_{ij} + w_{ij}| \tag{5}$$

where

$$A = y(I + \alpha_{jk}\,v_{jk}{}^{*}v_{jk}) - t\sigma_{ij}\,|y|\,v_{jk},$$

$$B = sv_{jk} + w_{jk}\,V_{jk}{}^{-1} + t\sigma_{ij}\,|y|\,v_{jk},$$

$$\tau = ta_{ij} + t\sigma_{ij}\,|y|\,\epsilon_{ij} + \alpha_{ij}(w_{ij}\cdot v_{ij}\,V_{ij}),$$

and where y, s are given by (2).

Proof. Let $(y_0, s_0) = (0, t)\phi_i{}^{-1}\phi_k$. And let $(\bar{y}, \bar{s}) = (0, \bar{t})\phi_j{}^{-1}\phi_k$ where $\bar{t} = tf_{ij}$. Since $tf_{ij}f_{jk} = \bar{t}f_{jk} = \bar{s} + \bar{t}\sigma_{jk}\,|\bar{y}|$ and since $tf_{ik} = s_0 + t\sigma_{ik}\,|y_0|$, then

$$tf_{ij}f_{jk} - tf_{ik} = \bar{s} - s_0 + \bar{t}\sigma_{jk}\,|\bar{y}| - t\sigma_{ik}\,|y_0|.$$

Since $\phi_i{}^{-1}\phi_k = (\phi_i{}^{-1}\phi_j)(\phi_j{}^{-1}\phi_j)$, then

$$(y_0, s_0) = (0, t)(\phi_i{}^{-1}\phi_j)(\phi_j{}^{-1}\phi_k) = (y, s)\phi_j{}^{-1}\phi_k$$

and we obtain

$$y_0 = [y(I + \alpha_{jk}\,v_{jk}{}^{*}v_{jk}) + sv_{jk}]V_{jk} + w_{jk},$$

$$s_0 = [y\cdot v_{jk} + sa_{jk}]\epsilon_{jk} + r_{jk}.$$

Since $\bar{t} = tf_{ij} = s + t\sigma_{ij}\,|y|$, it follows upon computing $\bar{s}$ that

$$\bar{s} - s_0 = t\sigma_{ij}[|y|\,a_{jk} - y\cdot(t\sigma_{ij})v_{jk}]\epsilon_{jk}.$$

Applying Lemma 8.3 we obtain $\bar{s} - s_0 = t\sigma_{ij}\,|A|\,\epsilon_{jk}$. Now it is easy to see that $|\bar{y}| = |B|$ and that $|y_0| = |A + B|$. Whereupon, (4) is obtained.

To obtain (5), let $k = i$ in (4). Since $i = k$, then $y_0 = 0$. Hence $B = -A$ and

$$tf_{ij}f_{ji} - t = [(t\sigma_{ij})\epsilon_{ji} + (tf_{ij})\sigma_{ji}]\,|A|.$$

Since $(\phi_i{}^{-1}\phi_j)(\phi_j{}^{-1}\phi_i)$ is the identity transformation, Lemma 8.2 shows that $v_{ji} = -\epsilon_{ij}\,v_{ij}\,V_{ij}$ so that $\alpha_{ji} = \alpha_{ij}$. Whereupon it follows that $A = \tau v_{ij}\,V_{ij} + w_{ij}$. This yields (5).

The following theorem answers, in one way, the question of temporal parity posed by P. Suppes in **(7)**; see, also, **(6)**.

THEOREM 9.3. *Suppose (X, E, ϕ, σ) is admissible. Then f is a signal metric on X. Moreover, $\phi_i{}^{-1}\phi_j$ is in L^* for all i, j in X. Also $t\sigma_{ij} = 1$ whenever $tv_{ij}\,V_{ij} + w_{ij} \neq 0$.*

Proof. If $\tau v_{ij}\,V_{ij} + w_{ij} = 0$ where τ is given by Lemma 9.2, it follows that $\tau = t$. If $tv_{ij}\,V_{ij} + w_{ij} = 0$ for more than one t, then $v_{ij} = w_{ij} = 0$ and $tv_{ij}\,V_{ij} + w_{ij} = 0$ for all t. In this case $E_i \subset E_j$. Moreover, this implies, by Lemma 8.2, that $v_{ji} = w_{ji} = 0$. Hence $tv_{ji}\, V_{ji} + w_{ji} = 0$ for all t and $E_j \subset E_i$. Thus, by (II), $i = j$. We have, then, $tv_{ij}\, V_{ij} + w_{ij} = 0$ for at most one t when $i \neq j$. Moreover, $\tau v_{ij}\, V_{ij} + w_{ij} \neq 0$ whenever

$$tv_{ij}\, V_{ij} + w_{ij} \neq 0.$$

Suppose that $tv_{ij}\, V_{ij} + w_{ij} \neq 0$. By (I) $tf_{ij}f_{ji} > 0$ so that by Lemma 9.2 (5), we have

$$(t\sigma_{ij})\epsilon_{ij} + (tf_{ij})\sigma_{ji} > 0$$

and, thus,

$$(t\sigma_{ij})\epsilon_{ij} = (tf_{ij})\sigma_{ji} = 1.$$

It follows now that f_{ij} is continuous. By Lemma 9.1, $f_{ij}\colon R \to R$ is onto. Hence, $t\sigma_{ji} = 1$ for all except possibly one t. By symmetry, this holds for σ_{ij}; thus $\epsilon_{ij} = 1$ and $f_{ij} \in \Gamma$. When $i = j$, clearly $f_{ij} = e$.

To show the triangle inequality for f, it suffices to consider the case where i, j, k are all distinct. Then, except for finitely many t,

$$t\sigma_{ij} = (tf_{ij})\sigma_{jk} = t\sigma_{ik} = 1.$$

Also $\epsilon_{jk} = 1$. Thus, by (4),

$$tf_{ij}f_{jk} - tf_{ik} = |A| + |B| - |A + B| \geqslant 0$$

except for finitely many t. Since $f_{ij}f_{jk} - f_{ik}$ is continuous, the triangle inequality holds everywhere.

We introduce now an explicit system $(\Delta, R^3 \times R, \psi, \rho)$ where Δ is in one-to-one correspondence with $R^3 \times R^3$. For $i \in \Delta$ let (v_i, w_i) be the corresponding member of $R^3 \times R^3$. Let $\psi_i = [v_i, I, 1, w_i, 0] \in L^*$. And for i, j in Δ, let $t\rho_{ij} = 1$ for all $t \in R$. The induced signal function from i to j is given by

$$tF_{ij} = ta_{ij} + r_{ij} + |tv_{ij}\, V_{ij} + w_{ij}| \tag{6}$$

where $[v_{ij}, V_{ij}, 1, w_{ij}, r_{ij}] = \psi_i^{-1}\psi_j$.

We remark that Δ is introduced primarily to avoid complicated subscripts. Δ may be identified with $R^3 \times R^3$.

THEOREM 9.4. *$(\Delta, R^3 \times R^3, \psi, \rho)$ is admissible, and, thus, F is a signal metric on Δ.*

Proof. This is left to the reader.

THEOREM 9.5. *If (X, E, ϕ, σ) is admissible, then within signal isometry and affine re-graduation (X, f) is a subspace of (Δ, F).*

Proof. It suffices to consider a system $(X, R^3 \times R, \phi, \sigma)$ with ϕ_i in L^*. Fix an observer o in X and let $\bar{\phi}_i = \phi_0^{-1}\phi_i$ for all i in X. Then the signal function $\bar{f}_{ij}$ from i to j induced by $(X, R^3 \times R, \bar{\phi}, \sigma)$ is such that $\bar{f} = f$.

Consider such a system. Let $\bar{\phi}_i = [v_i, V_i, 1, w_i, 0]$ and let f_{ij} be the signal functions induced by $(X, R^3 \times R, \bar{\phi}, \sigma)$. Let $t\theta_i = t - r_i$. Then it is easy to see that $\bar{f}_{ij} = \theta_i^{-1} f_{ij}\, \theta_j$ and, thus, that $\bar{f}$ is an affine re-graduation of f.

Consider such a system $(X, R^3 \times R, \phi, \sigma)$ with every ϕ_i of the form $[v_i, V_i, 1, w_i, 0]$. Let $\bar{\phi}_i = [v_i, I, 1, w_i V_i^{-1}, 0]$. By Lemma 8.2, it is straightforward to verify that $\bar{f} = f$ where $\bar{f}$ is the signal metric induced by $(X, R^3 \times R, \bar{\phi}, \sigma)$.

Finally it is clear that X is in one-to-one correspondence with some subset of Δ.

Remark. Consider α, β in L^*. Let $tf_{\alpha\beta} = s + |y|$ where $(y, s) = (0, t)\alpha^{-1}\beta$. The preceding arguments have effectively shown that f is a signal semi-metric on L^*. Let G be the subgroup of L^* that leaves invariant the one-dimensional subspace $\{0\} \times R$ of $R^3 \times R$. The world line E_α, so to speak, of α in L^* is the inverse image of $\{0\} \times R$ under α. Consequently $E_\alpha = E_\beta$ if and only if $\alpha^{-1}\beta \in G$. Hence f is definite on a subset X of L^* if and only if no two distinct members of X belong to the same left coset of G in L^*. Moreover, in this case, there is a translation re-graduation $\bar{f}$ of f such that $(X, \bar{f})$ is signal isometric to a subspace of (Δ, F).

Remark. From the proof of Theorem 9.5 the formula for F_{ij} can be readily obtained:

$$tF_{ij} = ta_{ij} + r_{ij} + |tv_{ij} V_{ij} + w_{ij}|,$$

where

$$a_{ij} = a_i a_j - v_i \cdot v_j,$$

$$v_{ij} V_{ij} = [a_i - \alpha_j(v_i \cdot v_j)]v_j - v_i,$$

$$r_{ij} = w_i \cdot v_{ji} V_{ji},$$

$$w_{ij} = w_j - w_i - \alpha_i(w_i \cdot v_i)v_i + [(w_i \cdot v_i)(1 - \alpha_i \alpha_j v_i \cdot v_j) - \alpha_j(w_i \cdot v_j)]v_j.$$

F is neither symmetric nor positive. To show this, consider $v_i = w_i = 0$ and $v_j = w_j$ $(\neq 0)$.

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Illinois Institute of Technology