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ON EXPONENTIAL DICHOTOMY IN BANACH SPACES

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In this paper we study the exponential dichotomy property for linear systems, the evolution of which can be described by a semigroup of class C_0 on a Banach space. We define the class of (p, q) dichotomic semigroups and establish the connections between the dichotomy concepts and admissibility property of the pair (L^p, L^q) for linear control systems. The obtained results are generalizations of well-known results of W.A. Coppel, J.L. Massera and J.J. Schäffer, K.J. Palmer.

1. Introduction

In Perron's classical paper on stability ([8]) a central concern is the relationship, for linear differential equations, between the condition that the nonhomogeneous equation has some bounded solution for every bounded "second member", on the one hand, and a certain form of conditional stability of the solutions of the homogeneous equation on the other. This idea was later extensively developed among others by Massera and Schäffer in [4] and Coppel in [2].

The extension of the bounded input, bounded output criteria of Perron for the case of linear control systems has been studied by several authors [4], [5], [6], [8]. The relationship between the conditional input-output stability and the exponential dichotomy for the case of a finite-dimensional linear control system is considered by Palmer in [7].

The aim of this paper is to study the exponential dichotomy property

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for linear systems, the evolution of which can be described by a semigroup of class C_0 on a Banach space. Using a fundamental inequality established in [4] we define the concept of (p, q) dichotomic semigroup and give a sufficient condition for exponential dichotomy of a large class of such semigroups. We also give a proof for the equivalence between the exponential dichotomy of a C_0 semigroup $T(t)$ and $\left(L^p, L^q\right)$ admissibility property for the case of a linear control system

$$x\left(t, x_0, u\right) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds .$$

The case $T(t) = \exp(At)$, where A is a bounded linear operator on a finite dimensional space has been considered by Palmer in [7].

2. Definitions and terminology

Let $T(t)$ be a C_0 semigroup on a separable Banach space X . Consider the control process described by the following integral model,

$$\left(T, B, U_p\right)x\left(t, x_0, u\right) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds ,$$

under the following standard assumptions: $x(t, \cdot, \cdot)$ belongs to X ; $u \in U_p = L^p(R_+, U)$ where $R_+ = [0, \infty)$ and U is also a Banach space; $B \in L(U, X)$ (the space of bounded linear operators from U to X); finally $x_0 \in X$.

Here U_p is the Banach space of all U-valued, strongly measurable functions u defined almost everywhere on R_+ such that

$$\|u\|_p = \left(\int_0^\infty \|u(s)\|^p ds\right)^{1/p} < \infty , \text{ if } p < \infty ,$$

and

$$\|u\|_\infty = \operatorname*{ess\,sup}_{s\geq 0} \|u(s)\| < \infty , \text{ if } p = \infty .$$

We also denote

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$$X_p = L^p(R_+, X) \quad \text{and} \quad p' = \begin{cases} \infty & , \text{ if } p = 1 , \\ 1 & , \text{ if } p = \quad , \\ p/(p-1) & , \text{ if } 1 < p < \infty . \end{cases}$$

Let X_1, X_2 be two closed complemented subspaces of X such that

$$X = X_1 \oplus X_2 .$$

If we denote by P_1 a projection along X_2 (that is, $\operatorname{Ker} P_1 = X_2$) then $P_1 \in L(X, X_1)$, $P_1^2 = P_1$ and $P_2 = I - P_1$ is a projection along X_1 with analogous properties.

We shall denote $T_1(t) = T(t)P_1$ and $T_2(t) = T(t)P_2$.

DEFINITION 2.1. The subspace X_1 induces

(i) an *exponential dichotomy* for the semigroup $T(t)$ if there exist constants $N > 0$, $\nu > 0$ such that

$$\|T_1(t)x\| \le Ne^{-\nu t}\|P_1x\|$$

and

$$\|T_2(t)x\| \ge Ne^{\nu t}\|P_2x\|$$

for all $t \ge 0$ and $x \in X$;

(ii) a (p, q) dichotomy (where $1 \le p, q \le \infty$) for the semigroup $T(t)$ if there exists $N > 0$ such that

$$\|T_1(\cdot)x\|_{L^q[t+\delta,\infty)} + \|T_2(\cdot)x\|_{L^q[0,t]} \le N\delta^{(1/p)-2}\|T(\cdot)x\|_{L^1[t,t+\delta]}$$

for all $t \in 0$, $\delta > 0$ and $x \in X_k$, $k = 1, 2$.

REMARK 2.1. If X_1 induces an exponential dichotomy for $T(t)$ then

$$\lim_{t\to\infty} \|T(t)x\| = \begin{cases} 0 , \text{ if } x \in X_1 , \\ \infty , \text{ if } P_2x \ne 0 , \end{cases}$$

and hence $X_1 = \{x \in X : \lim_{t\to\infty} T(t)x = 0\}$.

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REMARK 2.2. If X_1 induces an exponential dichotomy for $T(t)$ then

$$X_1 = \{x \in X : T(\cdot)x \in X_q\}$$

where $1 \le q \le \infty$.

DEFINITION 2.2. The C_0 semigroup $T(t)$ is said to be *exponentially dichotomic* ((p, q) *dichotomic*) if there exists a closed complemented subspace X_1 which induces an exponential dichotomy ((p, q) dichotomy) for $T(t)$.

DEFINITION 2.3. Let $1 \le p, q \le \infty$. The pair (U_p, X_q) is *admissible* for (T, B, U_p) if for every $u \in U_p$ there exists $x_u \in X$ such that $x(\cdot, x_u, u) \in X_q$.

Now let us note four assumptions which will be used at various times.

ASSUMPTION 1. We say that the semigroup $T(t)$ satisfies Assumption 1 if for every $q \ge 1$ the set

$$X_1 = \{x \in X : T(\cdot)x \in X_q\}$$

is a closed complemented subspace.

ASSUMPTION 2. The semigroup $T(t)$ satisfies Assumption 2 if for every $t_0 \ge 0$ there exist $t_1 \ge t_0$ and $m_1 > 0$ such that

$$\|T_2(t_1)x_0\| \ge m\|P_2x_0\| ,$$

for all $x_0 \in X$.

ASSUMPTION 3. The system (T, B, U_p) satisfies Assumption 3 if the range of B is of second category in X .

ASSUMPTION 4. The semigroup $T(t)$ satisfies Assumption 4 if

$$T_1(t) \ne 0 \text{ for every } t \ge 0 \text{ and any } x \in X_1 , \ x \ne 0 .$$

3. Preliminary results

We state the following

LEMMA 3.1. *If $T(t)$ is a C_0 semigroup then there exist $M > 1$,*

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$\omega > 0$ *such that*

(i) $\|T(t)\| \le Me^{\omega t}$ *for all* $t \ge 0$;

(ii) $\|T(t)x\| \le Me^{\omega\delta}\|T(s)x\|$ *for all* $\delta > 0$ *and* $0 \le s \le t \le s+\delta$;

(iii) $\delta\|T(t)x\| \le Me^{\omega\delta} \cdot \int_{t-\delta}^{t} \|T(s)x\|ds$ *for any* $\delta > 0$ *and* $t \ge \delta$;

(iv) $\int_{t}^{t+\delta} \|T(s)x\|ds < Me^{\omega\delta}\cdot\|T(t)x\|$ *for all* $t \ge 0$ *and* $\delta > 0$.

Proof. It is well known (see [*1*], pp. 165-166) that if

$$\omega \ge \overline{\lim_{t\to\infty}} \frac{\ln\|T(t)\|}{t} = \inf_{t>0} \frac{\ln\|T(t)\|}{t} = \omega_0 < \infty$$

then there exists $M \ge 1$ such that *(i)* holds.

The inequalities *(ii)*-*(iv)* follow immediately from *(i)* and the semi-group property.

LEMMA 3.2. *Suppose that Assumption* 1 *holds and let* X_2 *be a complementary subspace of* X_1 . *If* (U_p, X_q) *is admissible for* (T, B, U_p) *then there exists* $N > 0$ *such that for every* $u \in U_p$ *there is an unique* $x_2(u) \in X_2$ *with the properties:*

(i) $x(\cdot, x_2(u), u) \in X_q$, *and*

(ii) $\|x(\cdot, x_2(u), u)\|_q \geqq N\|u\|_p$.

Proof. Let $u \in U_p$. Then by admissibility of (U_p, X_q) for (T, B, U_p) there exists $x_0 \in X$ such that

$$x(\cdot, x_0, u) \in X_q .$$

If we denote by $x_k = P_k x_0$ $(k = 1, 2)$ then from the definition of X_1 we have that $x(\cdot, x_1, 0) \in X_q$ and hence

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$$x(\cdot,\, x_2,\, u) = x(\cdot,\, x_0,\, u) - x(\cdot,\, x_1,\, 0) \in X_q \ .$$

It follows that for every $u \in U_p$ there is $x_2(u) = P_2 x_0$, X_2 with the property *(i)*.

If we suppose that there exist $x_2',\, x_2'' \in X_2$ such that $x(\cdot,\, x_2',\, u) \in X_q$ and $x(\cdot,\, x_2'',\, u) \in X_q$ then

$$x(\cdot,\, x_2 - x_2'',\, u) = x(\cdot,\, x_2' - x_2'',\, 0) = T(\cdot)(x_2' - x_2'') \in X_q$$

and hence

$$x_2' - x_2'' \in X_1 \cap X_2 = \{0\} \ ,$$

which shows that $x_2' = x_2''$.

Let $\Lambda : U_p \to X_q$ be the operator defined by

$$\Lambda u = x(\cdot,\, x_2(u),\, u) \ .$$

It is easy to see that Λ is linear (from uniqueness of $x_2(u)$). Property *(ii)* is equivalent with the statement that Λ is a bounded operator. From the closed graph theorem it is sufficient to prove that Λ is closed.

Let $u_n \to u$ in U_p and $\Lambda u_n \to x$ in X_q . Let (u_{n_k}) be a subsequence of (u_n) such that $u_{n_k} \to u$ almost everywhere.

Because we may suppose that $x(\cdot)$ is continuously, we have that

$$\lim_{k\to\infty} x_2(u_{n_k}) = \lim_{k\to\infty} (\Lambda u_{n_k})(0) = x(0) \in X_2$$

and hence

$$x(t) = \lim_{k\to\infty} \left[T(t)x_2(u_{n_k}) + \int_0^t T(t-s)Bu_{n_k}(s)ds \right]$$

$$= T(t)x(0) + \int_0^t T(t-s)Bu(s)ds = x(t,\, x(0),\, u) \ .$$

From $x(\cdot) \in X_q$, we have that $x(0) = x_2(u)$ and hence

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$$x(t) = x\big(t, x_2(u), u\big) = (\Lambda u)(t) \quad \text{for all} \quad t \geq 0 .$$

LEMMA 3.3. *If* $T(t)$ *is* (p, q) *dichotomic with* $(p, q) \neq (1, \infty)$ *then there exists a function* $\eta : R_+ \to R_+$ *with* $\lim_{t\to\infty} \eta(t) = 0$ *and such that for all* $\delta_0 > 0$ *and* $\delta > \delta_0$ *we have*

(i) $$\int_t^{t+\delta} \|T_1(s)x\|ds \leq \eta(\delta_0) \cdot \int_{t_0}^{t_0+\delta} \|T_1(s)x\|ds ,$$ *for all* $t_0 \geq 0$, $t \geq t_0+\delta_0$ *and all* $x \in X$; *and*

(ii) $$\int_{t_0}^{t_0+\delta} \|T_2(s)x\|ds \leq \eta(\delta_0) \cdot \int_t^{t+\delta} \|T_2(s)x\|ds$$ *for all* $t_0 \geq 0$, $t \geq t_0+2\delta_0$ *and* $x \in X$.

Proof. Let $\delta > \delta_0 > 0$ and let n be a positive integer such that $n\delta_0 \leq \delta < (n+1)\delta_0$.

If we denote by $\delta_1 = \delta/n$ then from $t > t_0+\delta_0$ and $s = t_0+k\delta_1$, $k = 0, 1, \ldots, n-1$, by (p, q) dichotomy of $T(t)$ and Hölder's inequality we have

$$\int_{s+t-t_0}^{s+t-t_0+\delta_1} \|T_1(\tau)x\|d\tau \leq \delta_1^{1/q'} \cdot \left(\int_{s+\delta_0}^{\infty} \|T_1(\tau)x\|^q d\tau\right)^{1/q}$$

$$\leq \left(2\delta_0\right)^{1/q} \cdot \int_s^{s+\delta_0} \|T_1(\tau)x\|d\tau < \eta\left(\delta_0\right) \cdot \int_s^{s+\delta_1} \|T_1(\tau)x\|d\tau ,$$

where

$$\eta\left(\delta_0\right) = N\left(2\delta_0\right)^{1/q'}\delta_0^{(1/p)-2} .$$

Taking $s = t_0 + k\delta_1$, $k = 0, 1, 2, \ldots, n-1$ and adding we obtain

$$\int_t^{t+\delta} \|T_1(\tau)x\|d\tau = \int_t^{t+n\delta_1} \|T_1(\tau)x\|d\tau \leq \eta\left(\delta_0\right) \cdot \int_{t_0}^{t_0} \|T_1(\tau)x\|d\tau$$

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and hence *(i)* is proved.

Let $t \geq t_0+2\delta_0$ and $s = t_0 + k\delta_1$ with $k = 0, 1, \ldots, n-1$. Then as before we have

$$\int_s^{s+\delta_1} \|T_2(\tau)x\|d\tau \leq \delta_1^{1/q'} \cdot \left(\int_s^{s+\delta_1} \|T_2(\tau)x\|^q d\tau\right)^{1/q}$$

$$\leq 1^{1/q'} \cdot \left(\int_0^{s+t-t_0} \|T_2(\tau)x\|^q d\tau\right)^{1/q} \leq \eta(\delta_0) \cdot \int_{s+t-t_0}^{s+t-t_0+\delta_0} \|T_2(\tau)x\|d\tau$$

$$\leq \eta(\delta_0) \cdot \int_{s+t-t_0}^{s+t-t_0+\delta_1} \|T_2(\tau)x\|d\tau$$

and adding, we obtain the inequality *(ii)*.

LEMMA 3.4 ([4]). *Let $f : R_+ \to R_+$ be a function with the property that there is $\delta > 0$ such that $f(t+\delta) \geq 2f(t)$ for every $t > 0$ and $2f(t) \geq f(t_0)$ for all $t_0 \geq 0$ and $t \in [t_0, t_0+\delta]$. Then there exists $\nu > 0$ such that*

$$4f(t) \geq e^{\nu(t-t_0)} f(t_0) \quad \textit{for all} \quad t \geq t_0 \geq 0 .$$

The proof is immediate. Indeed, if $\nu = (\ln 2)/\delta$ and n is the positive integer with

$$n\delta \leq t - t_0 < (n+1)\delta$$

then

$$4f(t) \geq 2f(t_0+n\delta) \geq 2^{n+1}f(t_0) = e^{\nu(n+1)\delta}f(t_0) \geq e^{\nu(t-t_0)}f(t_0) .$$

LEMMA 3.5. *If $T(t)$ is (p, q) dichotomic with $(p, q) \neq (1, \infty)$ then there exists $\nu > 0$ such that for every $\delta > 0$ there is $N > 0$ with*

$$\textit{(i)} \quad \int_t^{t+\delta} \|T_1(s)x\|ds \leq Ne^{-\nu(t-t_0)}\|T_1(t_0)x\| , \textit{ and}$$

$$\textit{(ii)} \quad \int_{t_0}^{t_0+\delta} \|T_2(s)x\|ds \leq Ne^{-\nu(t-t_0)}\|T_2(t_0)x\| \quad \textit{for all} \quad t \geq t_0 \geq 0$$

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and $x \in X$.

Proof. Let $\delta > 0$, $x \in X$ and let δ_0 be sufficiently large such that

$$\eta(\delta_0) < \tfrac{1}{2} .$$

Let n be a positive integer such that $\eta\delta > 4\delta_0$ and let us consider the function $f : R_+ \to R_+$ defined by

$$f(t) = \left(\int_t^{t+n\delta} \|T_1(s)x\| ds\right)^{-1} .$$

By Lemma 3.3 we obtain

$$\int_{t_0+\delta_0}^{t_0+\delta_0+n\delta} \|T_1(s)x\| ds \le \eta(\delta_0) \int_{t_0}^{t_0+n\delta} \|T_1(s)x\| ds \le \tfrac{1}{2} \int_{t_0}^{t_0+n\delta} \|T_1(s)x\| ds$$

and hence

$$f(t_0+\delta_0) \ge 2f(t_0) .$$

If $t \in |t_0, t_0+\delta_0|$ then

$$\int_t^{t+n\delta} \|T_1(s)x\| ds \le \int_{t_0}^{t_0+\delta_0} \|T_1(s)x\| ds + \int_{t_0+\delta_0}^{t_0+\delta_0+n\delta} \|T_1(s)x\| ds$$

$$\le 2 \int_{t_0}^{t_0+n\delta} \|T_1(s)x\| ds ,$$

which implies that

$$2f(t) \ge f(t_0) \quad \text{for every} \quad t \in [t_0, t_0+\delta_0] .$$

From Lemma 3.4 we obtain that there exists $\nu > 0$ such that

$$4f(t) \ge f(t_0)e^{\nu(t-t_0)} \quad \text{for all} \quad t \ge t_0 \ge 0 .$$

By the preceding inequality and Lemma 3.1 we conclude that

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$$\int_t^{t+\delta} \|T_1(s)x\|ds = \frac{1}{f(t)} \le 4e^{-\nu(t-t_0)} \cdot \int_{t_0}^{t_0+n\delta} \|T_1(s)x\|ds$$

$$\le 4Mn\delta e^{n\omega\delta} \cdot e^{-\nu(t-t_0)} \|T_1(t_0)x\|$$

for all $t \ge t_0 \ge 0$. The inequality is proved.

From *(ii)* let g be the function defined by

$$g(t) = \int_t^{t+n\delta} \|T_2(s)x\|ds .$$

Then from inequality *(ii)* of Lemma 3.3 we obtain

$$g(t_0+2\delta_0) = \int_{t_0+2\delta_0}^{t_0+2\delta_0+n\delta} \|T_2(s)x\|ds \ge 2 \int_{t_0}^{t_0+n\delta} \|T_2(s)x\|ds = 2g(t_0)$$

and for $t \in [t_0, t_0+2\delta_0]$ we have

$$g(t_0) = \int_{t_0}^{t_0+n\delta} \|T_2(s)x\|ds \le \int_{t_0}^{t_0+2\delta_0} \|T_2(s)x\|ds + \int_t^{t+n\delta} \|T_2(s)x\|ds$$

$$\le \tfrac{1}{2} \int_{t_0+2\delta_0}^{t_0+4\delta_0} \|T_2(s)x\|ds + g(t) \le 2g(t) .$$

We may now apply Lemma 3.4 to g and on account of Lemma 3.1 it follows that

$$\int_{t_0}^{t_0+\delta} \|T_2(s)x\|ds \le g(t_0) \le 4e^{-\nu(t-t_0)} g(t)$$

$$\le 4Me^{n\omega\delta} e^{-\nu(t-t_0)} \cdot \|T_2(t)x\| = Ne^{-\nu(t-t_0)} \|T_2(t)x\|$$

for all $t \ge t_0 \ge 0$.

4. The main results

The purpose of this section is to establish the connections between the dichotomy concepts and admissibility.

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THEOREM 4.1. *Suppose that Assumption 2 holds. If the subspace* X_1 *indices a* (p, q) *dichotomy with* $(p, q) \neq (1, \infty)$ *then* X_1 *also induces an exponential dichotomy for the semigroup* $T(t)$.

Proof. Let $x \in X$ and $\delta > 0$.

Firstly, we suppose that

$$T_1(t)x \neq 0 \quad \text{for all} \quad t \geq 0 .$$

From Lemmas 3.1 and 3.5 we find that

$$\|T_1(t)x\| \leq Me^{\omega\delta} \cdot \int_{t-\delta}^{t} \|T_1(s)x\|ds \leq MNe^{\omega\delta}e^{-\nu t}\|P_1x\| \quad \text{for all} \quad t > \delta .$$

Let

$$N_1 = \max\left\{MNe^{\omega\delta}, \sup_{t\in[0,\delta]} e^{\nu t}\|T(t)\|\right\} .$$

Then

$$\|T_1(t)x\| \leq N_1e^{-\nu t}\|P_1x\| \quad \text{for all} \quad t \geq 0 .$$

If there exists $t_0 > 0$ such that $T_1\left(t_0\right)x = 0$ then

$$T_1(t)x = 0 \quad \text{for all} \quad t \geq t_0$$

and hence the preceding inequality holds.

Therefore, there exist $N_1, \nu > 0$ such that

$$\|T_1(t)x\| \leq N_1e^{-\nu t}\|P_1x\|$$

for all $t \geq 0$ and $x \in X$.

Similarly, if $T_2(t) \neq 0$ for every $t \geq 0$ then using Assumption 2 and Lemmas 3.1 and 3.5 one obtains that there exist $\delta, m > 0$ such that

$$m\|P_2x\| \leq \|T_2(\delta)x\| \leq \frac{Me^{\omega\delta}}{\delta} \cdot \int_0^{\delta} \|T_2(s)x\|ds \leq \frac{MNe^{\omega\delta}}{\delta} \cdot e^{-\nu t}\cdot\|T_2(t)x\| \quad \text{for all} \quad t \geq 0 .$$

This yields

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$$\|T_2(t)x\| \geq N_2 e^{\nu t}\|P_2 x\| \quad \text{for all} \quad t \geq 0 ,$$

where

$$N_2 = \frac{m\delta}{MN} e^{-\omega\delta} .$$

If there is $t_0 > 0$ with $T_2(t_0)x_0 = 0$ then $T_2(t)x = 0$ for all $t \geq t_0$ and from Assumption 2 it follows that $P_2 x = 0$. This shows that the inequality

$$\|T_2(t)x\| \geq N_2 e^{\nu t}\|P_2 x\|$$

holds for all $t \geq 0$ and $x \in X$.

THEOREM 4.2. *Assume that Assumptions* 1, 3 *and* 4 *hold. Then if the pair* (U_p, X_q) *is admissible for* (T, B, U_p) *then the semigroup* $T(t)$ *is* (p, q) *dichotomic.*

Proof. According to the more refined version of the open-mapping theorem ([3]) it follows that if (T, B, U_p) satisfies Assumption 3 then there exist an operator $B^+ : X \to U$ and $b > 0$ such that

$$BB^+ = x \quad \text{and} \quad \|B^+ x\| \leq b\|x\| \quad \text{for every} \quad x \in X .$$

Let $t > 0$, $\delta > 0$ and $x \in X$, $x \neq 0$. Let $u_t(\cdot)$ be the input function defined by

$$u_t(s) = \begin{cases} \dfrac{B^+ t(s)x}{\|T(s)x\|} , & \text{if} \quad s \in [t, t+\delta] , \\ 0 , & \text{if} \quad s \notin [t, t+\delta] , \end{cases}$$

and let $x_t^0 = -f(t)P_2 x$ where $f(t) = \int_t^{t+\delta} \frac{ds}{\|T(s)x\|}$ and P_2 is the projection along $X_1 = \{x \in X : T(\cdot)x_1 \in X_q\}$.

The function $u_t \in U_p$, $\|u_t\|_p \leq b\delta^{1/p}$ and

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$$x\left(s, x_t^0, u_t\right) = \begin{cases} f(t)T(s)P_1x & , \text{ if } s > t + \delta , \\ \\ -f(t)T(s)P_2x & , \text{ if } s \le t , \end{cases}$$

where $P_1 = I - P_2$.

Hence $x\left(\cdot, x_t^0, u_t\right) \in X_q$ and by Lemma 3.2 we have that there is $N > 0$ such that

$$\left\|x\left(\cdot, x_t^0, u_t\right)\right\|_q \le N \cdot \|u_t\| < N \cdot b \cdot \delta^{1/p} .$$

This shows that

$$f(t)\|T_2(\cdot)x\|_{L^q[0,t]} + f(t)\|T_1(\cdot)x\|_{L^q[t+\delta,\infty)} \le Nb\delta^{1/p} .$$

By Schwartz's inequality we have

$$\delta^2 = f(t) \cdot \int_t^{t+\delta} \|T(s)x\|ds ,$$

which implies that

$$\|T_1(\cdot)x\|_{L^q[t+\delta,\infty)} + \|T_2(\cdot)x\|_{L^q[0,t]} \le Nb\delta^{(1/p)-2} \cdot \int_t^{t+\delta} \|T(s)x\|ds .$$

The theorem is proved.

COROLLARY 4.3. *Assume that Assumptions* 1-4 *hold. Then if the pair* $\left(U_p, X_q\right)$ *with* $(p, q) \ne (1, \infty)$ *is admissible for* $\left(T, B, U_p\right)$ *then the semigroup* $T(t)$ *is exponentially dichotomic.*

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