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# A NEW SPACE WITH NO LOCALLY UNIFORMLY ROTUND RENORMING

BY

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ABSTRACT. We construct a Banach space $X$ which has no equivalent (wLUR) norm but which has no subspace isomorphic to $l_\infty$.

1. **Introduction.** It was shown by Lindenstrauss [4] that $l_\infty$ admits no locally uniformly rotund (LUR) renorming. Other known spaces for which this is true (such as $l_\infty/c_0$ and $l_\infty(\Gamma)$ with $\Gamma$ uncountable, which actually admit no rotund renorming) contain isomorphic copies of $l_\infty$ and the question has been posed whether $l_\infty$ is in fact the unique obstruction to (LUR) renorming. Similar questions arose in the context of non-reflexive Grothendieck spaces and were answered in [1] and [5]. In this paper, we modify the construction given in [1] to provide an example of a closed sublattice $X$ of $l_\infty$ which has no subspace isomorphic to $l_\infty$ but which allows no (LUR) renorming.

Our notation and terminology for Banach spaces are mostly standard; we write ball $X$ for $\{x \in X : \|x\| \leqq 1\}$ and sph $X$ for $\{x \in X : \|x\| = 1\}$. A Banach space $X$ is said to have a *locally uniformly rotund* (LUR) norm if $\|x - x_n\| \longrightarrow 0$ whenever $x, x_n \in$ sph $X$ are such that $\|(x + x_n)/2\| \longrightarrow 1$. If the above hypothesis on $x$ and $x_n$ implies only that $x_n \longrightarrow x$ weakly then $X$ is said to have a (wLUR) norm. The example we give actually has no (wLUR) renorming.

The plan of the paper is simple. Paragraph 2 introduces the class of "tree-complete" sublattices of $l_\infty$ defined in such a way that argument of [4] may be applied without much modification. In paragraph 3 we follow the methods of [1] to construct a tree-complete sublattice with no subspace ismorphic to $l_\infty$.

2. **Tree complete sublattices of $l_\infty$.** Let $X$ be a closed subspace of $l_\infty$, equipped with a norm $\|\cdot\|$ which satisfies $\|x\|_\infty \leqq \|x\| \leqq M\|x\|_\infty, (x \in N)$. When $x$ is in $X \cap$ sph $l_\infty$ and $A$ is a subset of $\mathbf{N}$, let $X(x, A)$ denote the set

$$\{y \in X : \|y\|_\infty = 1 \quad \text{and} \quad y \mid (\mathbf{N} \backslash A) = x \mid (\mathbf{N} \backslash A)\}$$

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and define

$$\xi(x,A) = \sup\{\|y\| : y \in \mathcal{X}(x,A)\}$$
$$\eta(x,A) = \inf\{\|y\| : y \in \mathcal{X}(x,A)\}.$$

LEMMA 2.1. *If $x$ is in $X \cap$ sph $l_\infty$ and $A$ is an infinite subset of* **N** *then for each $\epsilon > 0$ there exist $x' \in \mathcal{X}(x,A)$ and an infinite subset $A'$ of $A$ such that*

$$\eta(x',A') \geqq \xi(x',A') - \epsilon.$$

PROOF. First choose $x' \in \mathcal{X}(x,A)$ with $\|x'\| \geqq \xi(x,A) - \epsilon/2$ and then $x^* \in X^*$ with $\|x^*\| = 1$ and $\langle x^*, x\rangle \geqq \xi(x,A) - \epsilon/2$. Extend $x^*$ to a function $\mu \in (l_\infty)^*$.

If $A_1, A_2, \ldots$ are disjoint infinite subsets of $A$ then

$$\|u\| \geqq \sum_{n=1}^{\infty} \|\mathbf{1}_{A_n} \cdot \mu\|$$

because $l_\infty^*$ is an (AL)-space. Hence there exists $n$ such that $\|\mathbf{1}_{A_n} \cdot \mu\| < \epsilon/4$. Take $A' = A_n$. Now let $y$ be in $\mathcal{X}(x',A')$. We have

$$\begin{aligned}\langle x^*, y\rangle &= \langle x^*, x'\rangle + \langle x^*, y - x'\rangle \\ &= \langle x^*, x'\rangle + \langle \mathbf{1}_{A_n} \cdot \mu, y - x'\rangle.\end{aligned}$$

Since $\|y - x'\|_\infty \leqq 2$, this leads to

$$\begin{aligned}\langle x^*, y\rangle &\geqq \xi(x,A) - \epsilon/2 - 2 \cdot \epsilon/4 \\ &= \xi(x,A) - \epsilon.\end{aligned}$$

This gives the result since $\xi(x',A') \leqq \xi(x,A)$.

We now introduce some notation for the *dyadic tree* $T$. We define $T$ to be $\bigcup_{n\in\mathbf{N}}\{0,1\}^n$; its elements are finite (possible empty) strings of 0's and 1's. The empty string ( ) is the unique string of length 0; more generally, the *length* $|t|$ of a string $t$ is $n$ if $t \in \{0,1\}^n$. The *tree-order* is defined by $s \prec t$ if $|s| < |t|$ and $t(m) = s(m)(m < |s|)$. Each $t \in T$ has exactly two immediate successors, which we shall denote by $t.0$ and $t.1$. For each infinite sequence of 0's and 1's, that is to say, for each $b \in \{0,1\}^{\mathbf{N}}$, there is a unique *branch* of $T$,

$$B(b) = \{b \mid n : n \in \mathbf{N}\}.$$

We shall say that a sub-lattice $X$ of $l_\infty$ is *tree-complete* if, whenever $(y_t)_{t\in T}$ is a bounded, disjoint family in $X$, there exists $b \in \{0,1\}^{\mathbf{N}}$ such that the (pointwise) sum

$$\sum_{n\in\mathbf{N}} y_{b|n}$$

is in $X$.

Notice that if $X$ contains $c_0$ and is tree-complete then, for every infinite subset $B$ of $\mathbf{N}$, there is an infinite subset $C$ of $B$ with $\mathbf{1}_C \in X$. Thus when we apply Lemma 2.1 to such an $X$ we may always arrange that $\mathbf{1}_{A'} \in X$ and $x' \mid A' = 0$. (Replace $A'$ by an infinite $A'' \subset A'$ with $\mathbf{1}_{A''} \in X$ and replace $x'$ by $x'' = (x' \wedge \mathbf{1}_{A''}) \vee (-\mathbf{1}_{A''})$.)

THEOREM 2.2. *If $X$ is a tree-complete sublattice of $l_\infty$ and $X$ contains $c_0$ then $X$ admits no equivalent (wLUR) norm.*

PROOF.. Let $\|\cdot\|$ be an equivalent norm on $X$. We shall give a recursive definition of a family $(x_t)_{t\in T}$ in $X \cap$ sph $l_\infty$, a family $(A_t)_{t \in T}$ of infinite subsets of $\mathbf{N}$ and a family $(m_t)_{t\in T\setminus\{(\ )\}}$ of natural numbers. These will have the following properties:

(i) $A_t \subset A_s$ if $s \prec t$;
(ii) $A_t \cap A_s = \phi$ if $s, t$ are incomparable;
(iii) $m_{t\cdot i} \in A_t$ $(t \in T, i \in \{0, 1\})$;
(iv) $x_t \mid A_t = 0$, $x_{t.i}(m_{t.i}) = 1$;
(v) $\xi(x_t, A_t) - \eta(x_t, A_t) < 2^{-|t|}$;
(vi) $x_t \in \mathcal{X}(x_s, A_s)$ if $s \prec t$.

To start, we apply Lemma 2.1 with $A = \mathbf{N}$, $\epsilon = 1$ and $x$ any element of $X \cap$ sph $l_\infty$. We obtain $x_{(\ )}$ and $A_{(\ )}$ with

$$\xi(x_{(\ )}, A_{(\ )}) - \eta(x_{(\ )}, A_{(\ )}) < 1$$

and may assume that $x_{(\ )} \mid A_{(\ )} = 0$.

If $x_s, A_s$ have been obtained already, we choose distinct $m_{s.0}, m_{s.1}$ in $A_s$ and disjoint infinite subsets $B_{s.0}$, $B_{s.1}$ of $A_s\setminus\{m_{s.0}, m_{s.1}\}$. By inductive hypothesis, $\|x_s\|_\infty = 1$ and $x_s \mid A_s = 0$; so $\|x_s + e_{m_{s.i}}\|_\infty = 1$ for $i \in \{0, 1\}$. Moreover, $x_s + e_{m_{s.i}}$ is in $X$ since $X$ contains $c_0$. We apply Lemma 2.1 with $\epsilon = 2^{-|s|-1}, x = x_s + e_{m_{s.i}}, A = B_{s.i}$ and obtain $x_{x.i}, A_{s.i}$ as required.

It is easy to check that this construction does produce families satisfying all of (i) to (vi). Notice that for each $b \in \{0, 1\}^{\mathbf{N}}$ there is a positive real number $\rho(b)$ such that $\xi(x_{b|n}, A_{b|n})$ decreases to $\rho(b)$ and $\eta(x_{b|n}, A_{b|n})$ increases to $\rho(b)$ as $n \longrightarrow \infty$. Thus, if $z_n \in \mathcal{X}(x_{b|n}, A_{b|n})$ for all $n \in \mathbf{N}$, we have $\|z_n\| \longrightarrow \rho(b)$ as $n \longrightarrow \infty$.

We now define a bounded, disjoint family $(y_t)_{t\in T}$ in $X$ by putting

$$y_{(\ )} = x_{(\ )}$$
$$y_{t.i} = x_{t.i} - x_t = \mathbf{1}_{A_t} \cdot x_{t.i}.$$

By tree-completeness, there exists $b \in \{0, 1\}^{\mathbf{N}}$ such that the pointwise sum

$$x = \sum_{n \in \mathbf{N}} y_{b|n}$$

is in $X$. We note that $x$ is in $\mathcal{X}(x_{b|n}, A_{b|n})$ for all $n$ so that $\|x\|$ must equal $\rho(b)$. Moreover, for each $n, x_{b|n}$ and $(x + x_{b|n})/2$ are in $\mathcal{X}(x_{b|n}, A_{b|n})$ so that $\|x_{b|n}\| \longrightarrow \rho(b)$ and $\|x + x_{b|n})/2\| \longrightarrow \rho(b)$.

We can now see immediately that $(X, \|\cdot\|)$ is not (LUR) since $\|x - x_{b|n_\infty}\| \geqq 1$ for all $n$. (We have $x_{b|n}(m_{b|(n+1)}) = 0$, $x(m_{b|(n+1)}) = 1$.)

To see that $X$ is not (wLUR) we need to find $x^* \in X^*$ such that $\langle x^*, x_{b|n}\rangle \longrightarrow \langle x^*, x\rangle$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbf{N}$ and define $\mu \in l_\infty^*$ by

$$\langle \mu, z\rangle = \lim_{n\to\mathcal{U}} z(m_{b|n}).$$

We have $\langle \mu, x_{b|n}\rangle = 0$ for all $n$ while $\langle \mu, x\rangle = 1$, so that $x^* = \mu bigm|X$ will do.

3. **The construction.** Our aim now is to construct a closed, tree-complete sublattice $X$ of $l_\infty$ which contains $c_0$ but which has no subspace isomorphic to $l_\infty$. Our sublattice $X$ will be the closed linear span of the indicator functions $\mathbf{1}_A$ of sets $A$ in a certain subalgebra $\mathfrak{A}$ of the power set $\mathfrak{P}\mathbf{N}$ of the natural numbers. In order to exclude $l_\infty$ as a subspace of $X$, we ensure that for every infinite subset $N$ of $\mathbf{N}$ there is a subset $M$ of $N$ which is not in the trace $\{A \cap N : A \in \mathfrak{A}\}$ of $\mathfrak{A}$ on $N$. In the lemma that follows, which shows how to carry out the inductive step in a construction by transfinite recursion, we suppose that each of a certain family of subsets $N_\gamma$ of $\mathbf{N}$ has already been assigned a "forbidden" subset $M_\gamma$. The lemma shows how to extend a given subalgebra, in a way that will eventually lead to tree-completeness of $X$, while not going against any of the existing assignments of forbidden subsets.

LEMMA 3.1. *Let $\gamma < \mathfrak{c}$ be an ordinal and let $\mathfrak{A}$ be a Boolean subalgebra of $\gamma\mathbf{N}$ with $\#\mathfrak{A} < \mathfrak{c}$. Let $(M_\beta, N_\beta)_{\beta<\gamma}$ be a family of pairs of subsets of $\mathbf{N}$, with $M_\beta \subset N_\beta$, such that $M_\beta \neq A \cap N_\beta$ for all $A \in \mathfrak{A}, \beta < \gamma$. For each $k \in \mathbf{N}$, let $(A_t^k)_{t\in T}$ be a family of elements of $\mathfrak{A}$ and assume that $A_t^k \cap A_s^l = \emptyset$ if $s, t$ are distinct elements of $T$ and $k, l$ are in $\mathbf{N}$. Then there exists $b \in \{0,1\}^{\mathbf{N}}$ such that $M_\beta \neq B \cap N_\beta$ for all $\beta < \gamma$ and all $B$ in the algebra generated by*

$$\mathfrak{A} \cup \left\{ \bigcup_{n\in\mathbf{N}} A_{b|n}^k : k \in \mathbf{N} \right\}.$$

PROOF. For $b \in \{0,1\}^{\mathbf{N}}$ and $k \in \mathbf{N}$ let $B_b^k = \bigcup_{n\in\mathbf{N}} A_{b|n}^k$, let $\mathfrak{B}_b$ be the algebra generated by $\{B_b^k : k \in \mathbf{N}\}$ and let $\mathfrak{A}_b$ be the algebra generated by $\mathfrak{A} \cup \mathfrak{B}_b$. Note that any element of $\mathfrak{A}_b$ may be written in the form $(A_1 \cap B^1) \cup \ldots \cup (A_r \cap B_r)$ with $B_1, \ldots, B_r \in \mathfrak{B}_b$ and $A_1, \ldots, A_r$ *disjoint* elements of $\mathfrak{A}$.

If the assertion of the lemma is false, then by a cardinality argument, there exist disjoint $A_1, \ldots, A_r \in \mathfrak{A}$, an ordinal $\beta < \gamma$ and distinct $b, c, d \in \{0,1\}^{\mathbf{N}}$ such that

$$\begin{aligned} M_\beta &= N_\beta \cap [(A_1 \cap B_1) \cup \ldots \cup (A_r \cap B_r)] \\ &= N_\beta \cap [(A_1 \cap C_1) \cup \ldots \cup (A_r \cap C_r)] \\ &= N_\beta \cap [(A_1 \cap D_1) \cup \ldots \cup (A_r \cap D_r)] \end{aligned}$$

for appropriately chosen $B_j \in \mathfrak{B}_b, C_j \in \mathfrak{B}_c, D_j \in \mathfrak{B}_d$. For some natural number $l$ we have

$$B_j \in \text{alg}\{B_b^k : k < l\}$$
$$C_j \in \text{alg}\{B_c^k : k < l\}$$
$$D_j \in \text{alg}\{B_d^k : k < l\}, \quad (1 \leqq j \leqq r).$$

Let $m$ be the smallest natural number such that $b|m$, $c|m$, $d|m$ are distinct and define

$$E = \bigcup_{\substack{k<l \\ |t|<m}} A_t^k.$$

Notice that $E \in \mathfrak{A}$ and that $E \cap F \in \mathfrak{A}$ whenever $F \in \mathfrak{B}_b$ (or $\mathfrak{B}_c$ or $\mathfrak{B}_d$). It follows from this observation that there exists $A' \in \mathfrak{A}$ such that $M_\beta \cap E = N_\beta \cap A'$.

We now have to consider $M_\beta \backslash E$. Notice that $B_b^i \backslash E, B_c^j \backslash E, B_d^k \backslash E$ are disjoint whenever $i, j, k < l$. For any fixed $j \leqq r$ we have $A_j \cap B_j \cap N_\beta = A_j \cap C_j \cap N_\beta = A_j \cap D_j \cap N_\beta = A_j \cap N_\beta$ (recall that the $A_j$ are disjoint).

We claim that, for each $j$, either

$$(A_j \cap M_\beta)\backslash E = (A_j \cap N_\beta)\backslash E \text{ or } (A_j \cap M_\beta)\backslash E = \phi.$$

If this is not the case, there exist $p \in (A_j \cap M_\beta)\backslash E$ and $q \in (A_j \cap (N_\beta \backslash M_\beta))\backslash E$. Consequently, $p \in B_j \backslash E, q \in (\mathbf{N} \backslash B_j)\backslash E$ which means that, for some $i < l$, one of $p, q$ is in $B_b^i$ and the other not. Similarly, for some $j, k < l, B_c^j \cap \{p, q\} \neq \phi$ and $B_d^k \cap \{p, q\} \neq \phi$. This contradicts the disjointness of $B_b^i \backslash E, B_c^j \backslash E, B_d^k \backslash E$.

Finally, we see that $M_\beta$ can be written as

$$M_\beta = N_\beta \cap \left[ A' \cup \bigcup_{j \in J} (A_j \backslash E) \right]$$

for a suitable subset $J$ of $l$. This contradicts our original hypothesis.

Proposition 3.2. *There exists a subalgebra $\mathfrak{A}$ of $\mathfrak{P}\mathbf{N}$, containing the finite subsets and satisfying the following two properties:*

*(i) for no infinite $N \subset \mathbf{N}$ do we have $\mathfrak{P}N = \{N \cap A : A \in \mathfrak{A}\}$;*

*(ii) whenever $A_t^k (k \in \mathbf{N}, t \in T)$ are elements of $\mathfrak{A}$ such that*

$$A_t^k \cap A_s^j = \phi \quad (k, j \in \mathbf{N}; s \neq t),$$

*there exists $b \in \{0, 1\}^{\mathbf{N}}$ such that*

$$\bigcup_{n \in \mathbf{N}} a_{b|n}^k \in \mathfrak{A} \quad \text{for all } k \in \mathbf{N}.$$

The proof of this proposition uses the preceding lemma in the same way that 1E was used for 1D in [1].

THEOREM 3.3. *There is a closed sublattice $X$ of $l_\infty$ which admits no equivalent (wLUR) norm but which has no subspace isomorphic to $l_\infty$.*

PROOF. We construct $\mathfrak{A}$ as in 3.2 and take $X$ to be the closed linear span of $\{\mathbf{1}_A : A \in \mathfrak{A}\}$. That $X$ has no subspace isomorphic to $l_\infty$ follows from the argument used in [1]. On the other hand, $X$ contains $c_0$ so that we only need to show that $X$ has the tree-completeness property.

Let $(Y_t)_{t\in T}$ be a a disjointly supported family in $X\cap$ ball $l_\infty$. For each $t \in T$ we can write $y_t$ in the form

$$y_t = \sum_{k=1} 2^{-k}(\mathbf{1}_{A_t^k} - \mathbf{1}_{B_t^k})$$

with $A_t^k, B_t^k \in \mathfrak{A}$ and $A_t^k, B_t^k \subseteq \operatorname{supp} y_t$. If we apply property (ii) of 3.2 we find $b \in \{0,1\}^{\mathbf{N}}$ such that

$$\bigcup_{n\in\mathbf{N}} a_{b|n}^k \in \mathfrak{A} \quad \text{and} \quad \bigcup_{n\in\mathbf{N}} B_{b|n}^k \in \mathfrak{A} \text{ for all } k.$$

But this means that the pointwise sum

$$\sum_{n\in\mathbf{N}} y_{b|n}$$

is in $X$, since we can write it as

$$\sum_{k=1}^{\infty} 2^{-k}(\mathbf{1}_{A_k} - \mathbf{1}_{B_k})$$

with $A_k = \bigcup_{n\in\mathbf{N}} A_{b|n}^k$ and $B_k = \bigcup_{n\in\mathbf{N}} B_{b|n}^k$

4. **Final remarks.** Considerably more is known about the structure of non-reflexive Grothendieck spaces than about that of spaces without (LUR) renormings. The question of whether a non-reflexive Grothendieck space necessarily has $l_\infty$ as a quotient depends upon special set-theoretic axioms ([3]) and [5]); but the dual of such a space always contains $L_1(\{0,1\}^{\omega_1})$ [2]. It is not clear whether the similarity between the example given here and the one in [1] is coincidental or whether results analogous to the above may hold for spaces without (LUR) renormings.

ADDED IN PAGE-PROOFS: G. A. Alexandrov and V. D. Babev [Comptes Rendus de l'Académie Bulgare des Sciences, 41 (1988), 29–32.] have shown that *subsequential completeness* of $\mathfrak{A}$ is enough to guarantee that $X = X_{\mathfrak{A}}$ has no (wLUR)-renorming. Thus the example constructed in [1] fulfills the conditions of Theorem 3.3.

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