An illustrative derivation of the sum of fifth powers

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1. *Introduction*

In 1631, Johannes Faulhaber published the result that sums of the form

$$\sum_{i=1}^{n} i^k = 1^k + 2^k + \dots + n^k$$

with $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$ are always polynomial functions of degree $k + 1$ in n [1], i.e.

$$\sum_{i=1}^{n} i^k = a_{k+1}n^{k+1} + a_k n^k + \dots + a_2 n^2 + a_1 n + a_0.$$

This formula is known as Faulhaber's formula. However, when we want to calculate the sum $1^k + 2^k + \dots + n^k$ for fixed k and n, we have to ask for the coefficients in Faulhaber's formula depending on k. Of course, one could calculate the values of $\sum_{i=1}^{n} i^k$ for $n = 0, 1, 2, \dots, k - 1, k, k + 1$ for each fixed k and solve the following system of linear equations; however, as one can easily imagine, that is very demanding for large k.

$$\begin{array}{lcllll}
\sum_{i=1}^{0} i^k & = & a_{k+1} \cdot 0^{k+1} & + a_k \cdot 0^k & \dots & + a_1 \cdot 0 & + a_0 \\
\sum_{i=1}^{1} i^k & = & a_{k+1} \cdot 1^{k+1} & + a_k \cdot 1^k & \dots & + a_1 \cdot 1 & + a_0 \\
\sum_{i=1}^{2} i^k & = & a_{k+1} \cdot 2^{k+1} & + a_k \cdot 2^k & \dots & + a_1 \cdot 2 & + a_0 \\
 & & \vdots & & & & \\
\sum_{i=1}^{k} i^k & = & a_{k+1} \cdot k^{k+1} & + a_k \cdot k^k & \dots & + a_1 \cdot k & + a_0 \\
\sum_{i=1}^{k+1} i^k & = & a_{k+1}(k + 1)^{k+1} & + a_k(k + 1)^k & \dots & + a_1(k + 1) & + a_0
\end{array}$$

Another point to think about is that using systems of linear equations to determine the coefficients indeed provides the desired result but no intuition for it. A famous example is the triangular numbers $\sum_{i=1}^{n} i$, i.e. $k = 1$. The method described returns, with $\sum_{i=1}^{0} i = 0, \sum_{i=1}^{1} i = 1$ and $\sum_{i=1}^{2} i = 1 + 2 = 3$, the following system of linear equations:

$$\begin{array}{lcllll}
0 & = & 0 & + 0 & + a_0 & \\
1 & = & a_2 & + a_1 & + a_0 & . \\
3 & = & 4a_2 & + 2a_1 & + a_0 &
\end{array}$$

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So we get $a_0 = 0$, $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{2}$, i.e. $\sum_{i=1}^{n} i = \frac{1}{2}n^2 + \frac{1}{2}n$.

A simpler derivation for the triangular numbers—found by the young Carl Friedrich Gauss, among others [2]—is the following:

$$\begin{aligned} 2\sum_{i=1}^{n} i &= \quad 1 \quad + 2 \quad \ldots \quad + (n-1) \quad + n \\ &\quad + n \quad + (n-1) \quad \ldots \quad + 2 \quad + 1 \\ &= (n+1) + (n+1) \ldots + (n+1) + (n+1) \\ &= n(n+1). \end{aligned}$$

We guess that Gauss's method is much easier and also more intuitive. The idea of using systems of linear equations works for all sums of the form $\sum_{i=1}^{n} i^k$. If we did not know Faulhaber's formula, we could also derive the coefficients via systems of linear equations; however, we had to prove them by induction. But thanks to Faulhaber's formula and the uniqueness of the solutions of such systems of linear equations, we can skip the induction. By the way, since the empty sum is always zero, $a_0 = 0$ is true for all k in Faulhaber's formula.

In this work, we show a well-known illustrative derivation for the coefficients in Faulhaber's formula for $k = 3$, i.e. the sum of cubes. We will then generalise this idea for illustrative derivation to the case $k = 5$ to obtain finally a closed-form formula for the sum of the fifth powers, which is the aim of this work. At this point, we note that there might be shorter or more elegant derivations of this formula, e.g. using Bernoulli numbers or summation by parts. However, the aim of this paper is not to provide the shortest or the most elegant derivation, it is rather to provide an illustrative derivation giving some insights.

2. *The sum of cubes*

Before we turn to the sum of fifth powers, we first have a look at the case $k = 3$. We provide a well-known illustrative derivation of the closed-form formula for $\sum_{i=1}^{n} i^3$, By use of the so-called principle of double counting, we will show that $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$ holds. Since we know the closed-form formula for the triangular numbers, it then follows immediately that

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

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1	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20
5	10	15	20	25

FIGURE 1: Schematic representation of the partition of $(1 + 2 + \ \dots \ + n)^2$ (for $n = 5$)

We rewrite according to the pattern in Figure 1:

$$\left(\sum_{i=1}^{n} i\right)^2 = \begin{array}{cccccc} & 1 & +\ 2 & +\ 3 & +\ 4 & \dots & +\ n \\ +\ 2 & +\ 4 & +\ 6 & +\ 8 & \dots & +\ 2n \\ +\ 3 & +\ 6 & +\ 9 & +\ 12 & \dots & +\ 3n \\ +\ 4 & +\ 8 & +\ 12 & +\ 16 & \dots & +\ 4n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ +\ n & +\ 2n & +\ 3n & +\ 4n & \dots & +\ n^2 \end{array}$$

and sort the summands:

$$\left(\sum_{i=1}^{n} i\right)^2 = \begin{array}{llllllll} & (& 1 & & & & &) \\ + & (& & 2 & & & & \\ & & +\ 2 & +\ 4 & & & &) \\ + & (& & & 3 & & & \\ & & & & +\ 6 & & & \\ & & +\ 3 & +\ 6 & +\ 9 & & &) \\ + & (& & & & 4 & & \\ & & & & & +\ 8 & & \\ & & & & & +\ 12 & & \\ & & +\ 4 & +\ 8 & +\ 12 & +\ 16 & &) \\ \vdots & & & & & & & \\ + & (& & & & & n & \\ & & & & & & +\ 2n & \\ & & & & & & +\ 3n & \\ & & & & & & +\ 4n & \\ & & & & & & \vdots & \\ & & +\ n & +\ 2n & +\ 3n & +\ 4n \ \dots & +\ n^2 &) \end{array}$$

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We denote the sum of the numbers in the jth brackets with L_j and determine their value:

$$L_j = 2j\left(\sum_{i=1}^{j-1} i\right) + j^2 = 2j \cdot \frac{(j-1)j}{2} + j^2 = j^3.$$

Here we twice summed the numbers on the edges of the jth square leading to the corner j^2 without the value of the corner j^2 and added the number of the corner itself—as illustrated in Figure 2.

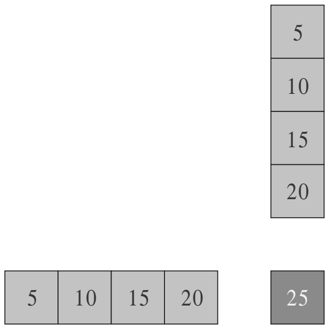

FIGURE 2: Direct decomposition of the difference between $(1 + 2 + \ldots + n)^2$ and $(1 + 2 + \ldots + (n-1))^2$ (for $n = 5$)

Another way to determine the value of L_j is to subtract the sum of the numbers of the $(j-1)$th square from the sum of the numbers of the jth square; see Figure 3.

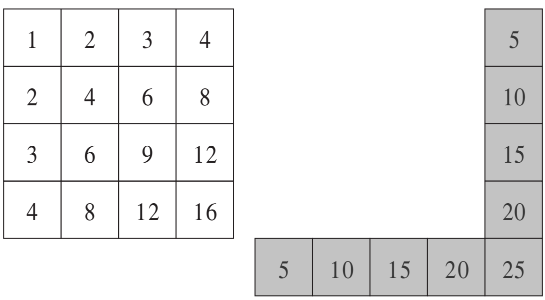

FIGURE 3: Illustration of the difference between $(1 + 2 + \ldots + n)^2$ and $(1 + 2 + \ldots + (n-1))^2$ (for $n = 5$)

$$\begin{aligned} L_j &= \left(\sum_{i=1}^{j} i\right)^2 - \left(\sum_{i=1}^{j-1} i\right)^2 \\ &= \left(\frac{j(j+1)}{2}\right)^2 - \left(\frac{j(j-1)}{2}\right)^2 \\ &= \frac{j^2}{4}(j+1)^2 - \frac{j^2}{4}(j-1)^2 \\ &= j^3. \end{aligned}$$

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It follows that

$$\sum_{i=1}^{n} i^3 = \sum_{i=1}^{n} L_i = \left(\sum_{i=1}^{n} i\right)^2 = \left(\frac{n(n+1)}{2}\right)^2.$$

As explained above, we could also derive this formula using a system of linear equations with five variables and five equations.

In the next section we will extend the idea shown here to the sum of fifth powers.

3. *Sum of fifth powers*

Again we will use the idea of double counting. When we consider

$$\left(\sum_{i=1}^{n} i\right)^3 = \left(\frac{n(n+1)}{2}\right)^3$$

and partially multiply the left side of this equation, we get:

$$\left(\sum_{i=1}^{n} i\right)^3 = \left(\sum_{i=1}^{n} i\right) \left(\begin{array}{cccccc} 1 & +\ 2 & +\ 3 & +\ 4 & \ldots & +\ n \\ +\ 2 & +\ 4 & +\ 6 & +\ 8 & \ldots & +\ 2n \\ +\ 3 & +\ 6 & +\ 9 & +\ 12 & \ldots & +\ 3n \\ +\ 4 & +\ 8 & +\ 12 & +\ 16 & \ldots & +\ 4n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ +\ n & +\ 2n & +\ 3n & +\ 4n & \ldots & +\ n^2 \end{array} \right)$$

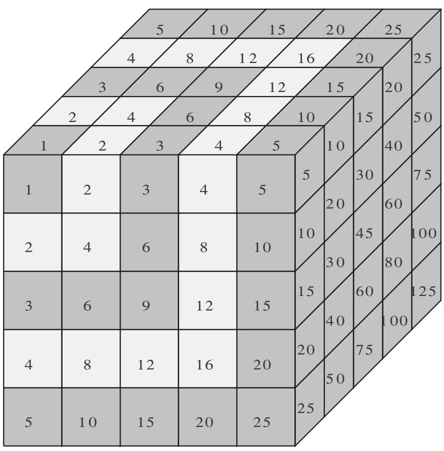

FIGURE 4: Schematic representation of the partition of $(1 + 2 + \ldots + n)^3$ (for $n = 5$)

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Now we partition this product according to the pattern in Figure 4. We obtain:

$$
\begin{array}{lllrrrrcrl}
\left(\sum_{i=1}^{n} i\right)^3 & = [& (1)(& 1 & & & & & &)] \\
 & + [& (1)(& & 2 & & & & & \\
 & & & +2 & +4 & & & & &) \\
 & & +(2)(& 1 & +2 & & & & & \\
 & & & +2 & +4 & & & & &)] \\
 & + [& (1+2)(& & & 3 & & & & \\
 & & & & & +6 & & & & \\
 & & & +3 & +6 & +9 & & & &) \\
 & & +(3)(& 1 & +2 & +3 & & & & \\
 & & & +2 & +4 & +6 & & & & \\
 & & & +3 & +6 & +9 & & & &)] \\
 & + [& (1+2+3)(& & & & 4 & & & \\
 & & & & & & +8 & & & \\
 & & & & & & +12 & & & \\
 & & & +4 & +8 & +12 & +16 & & &) \\
 & & +(4)(& 1 & +2 & +3 & +4 & & & \\
 & & & +2 & +4 & +6 & +8 & & & \\
 & & & +3 & +6 & +9 & +12 & & & \\
 & & & +4 & +8 & +12 & +16 & & &)] \\
 & \vdots & & & & & & & & \\
 & + [& \left(\sum_{i=1}^{n-1} i\right)(& & & & & & n & \\
 & & & & & & & & +2n & \\
 & & & & & & & & +3n & \\
 & & & & & & & & +4n & \\
 & & & & & & & & \vdots & \\
 & & & +n & +2n & +3n & +4n & \ldots & +n^2 &) \\
 & & +(n)(& 1 & +2 & +3 & +4 & \ldots & +n & \\
 & & & +2 & +4 & +6 & +8 & \ldots & +2n & \\
 & & & +3 & +6 & +9 & +12 & \ldots & +3n & \\
 & & & +4 & +8 & +12 & +16 & \ldots & +4n & \\
 & & & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
 & & & +n & +2n & +3n & +4n & \ldots & +n^2 &)]
\end{array}
$$

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We denote the value of the term in the jth square bracket with V_j and notice that

$$\begin{aligned} V_j &= 3j\left(\sum_{i=1}^{j-1} i\right)^2 + 3j^2\left(\sum_{i=1}^{j-1} i\right) + j^3 \\ &= 3j\left(\frac{(j-1)j}{2}\right)^2 + 3j^2\left(\frac{(j-1)j}{2}\right) + j^3 \\ &= \frac{3j^3}{4}\left(j^2 - 2j + 1\right) + \frac{3j^3}{2}\left(j - 1\right) + j^3 \\ &= \frac{3}{4}j^5 + \frac{1}{4}j^3. \end{aligned}$$

This calculation corresponds to the illustration in Figure 5. Three times we summed the numbers of the surfaces of the jth cube which have j^3 as a corner, without the edges which go to the corner j^3, we added three times the sum of the numbers of the edges of the jth cube which have j^3 as a corner, without the corner j^3, and we added the value of the corner itself.

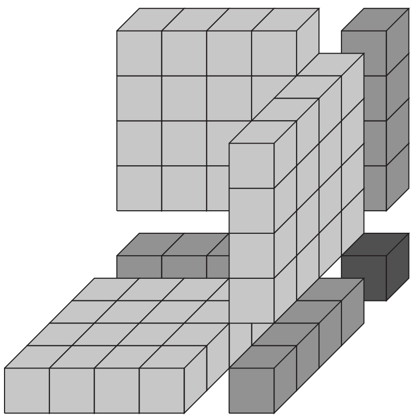

FIGURE 5: Direct decomposition of the difference between $(1 + 2 + \ldots + n)^3$ and $(1 + 2 + \ldots + (n - 1))^3$ (for $n = 5$)

Another way to get the value V_j is to subtract from the sum of the numbers of the jth cube the value of the $(j - 1)$th cube (see Figure 6):

$$\begin{aligned} V_j &= \left(\sum_{i=1}^{j} i\right)^3 - \left(\sum_{i=1}^{j-1} i\right)^3 \\ &= \frac{j^3(j+1)^3}{8} - \frac{(j-1)^3 j^3}{8} \\ &= \frac{j^6 + 3j^5 + 3j^4 + j^3}{8} - \frac{j^6 - 3j^5 + 3j^4 - j^3}{8} \\ &= \frac{3}{4}j^5 + \frac{1}{4}j^3. \end{aligned}$$

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It follows that

$$\left(\frac{n(n+1)}{2}\right)^3 = \left(\sum_{i=1}^{n} i\right)^3$$
$$= \sum_{i=1}^{n} V_i$$
$$= \frac{3}{4}\sum_{i=1}^{n} i^5 + \frac{1}{4}\sum_{i=1}^{n} i^3.$$

whence

$$4\left(\frac{n(n+1)}{2}\right)^3 = 3\sum_{i=1}^{n} i^5 + \left(\frac{n(n+1)}{2}\right)^2$$

and finally

$$\sum_{i=1}^{n} i^5 = \frac{4}{3}\left(\frac{n(n+1)}{2}\right)^3 - \frac{1}{3}\left(\frac{n(n+1)}{2}\right)^2$$
$$= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2.$$

This is exactly the formula we intended to derive. Note that there is another important result derived by Faulhaber, called Faulhaber polynomials, which is sometimes confused with Faulhaber's formula. Faulhaber polynomials state that every sum $\sum_{i=1}^{n} i^k$ with k odd can be represented as a polynomial in $\frac{1}{2}n(n+1)$, i.e. in $\sum_{i=1}^{n} i$, which fits our formula exactly. Since in Figures 1 and 4 the edges which have 1 as a corner contain the numbers $1, 2, \ldots, n$ not only our results but also their derivations fit to Faulhaber polynomials.

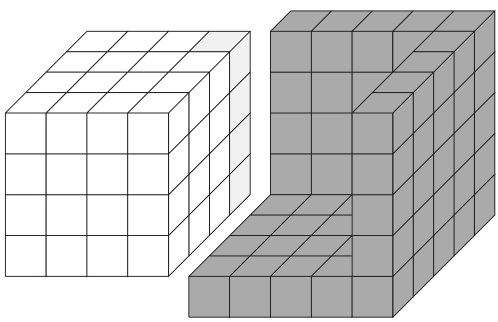

FIGURE 6: Representation of the difference between $(1 + 2 + \ldots + n)^3$ and $(1 + 2 + \ldots + (n-1))^3$ (for $n = 5$)

Note that the consideration of squares as in Figure 1 resulted in a formula for the sum of cubes, and the consideration of cubes as in Figure 4

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resulted in a formula for the sum of fifth powers. The consideration of hypercubes and the usage of the second method (i.e. Figures 3 and 6) for the calculation of the respective summands will lead to further formulae for sums of odd powers. However, this would not be illustrative anymore.

4. *Conclusion*

We derived the coefficients in Faulhaber's formula for $k = 5$:

$$\sum_{i=1}^{n} i^5 = \frac{n^2}{2}\left(\frac{1}{3}n^4 + n^3 + \frac{5}{6}n^2 - \frac{1}{6}\right).$$

We note that the second idea shown in each case for deriving L_j or V_j, respectively, i.e. subtracting the sum of the numbers of the $(j - 1)$th square/cube from the sum of the numbers of the jth square/cube, can be further generalised or systematised. It is up to the reader to decide which way is easier to obtain the coefficients in Faulhaber's formula for $k = 5$: solving a system of linear equations with six equations and six variables or the illustrative idea shown in this paper. The work here may at least contribute to the reader's training in spatial and number imagination.

The presented derivation method does not work for the sum of even powers. However, note that there is a well-known illustrative derivation of the sum of squares which might be generalised to higher even powers:

$$\begin{aligned}
3\sum_{i=1}^{n} i^2 &= 3\left(1^2 + 2^2 + 3^2 + \dots + n^2\right) \\
&= 3\left(1 \times 1 + 2 \times 2 + 3 \times 3 + \dots + n \times n\right) \\
&= 3\left(1 + 2 + 2 + 3 + 3 + 3 + \dots + \dots + n + n + \dots + n\right) \\
&= 1 + 2 + 3 + \dots + n \\
&\quad + 2 + 3 + \dots + n \\
&\quad + 3 + \dots + n \\
&\quad \vdots \\
&\quad + n \\
&\quad + n + n + n + \dots + n \\
&\quad \vdots \\
&\quad + 3 + 3 + 3 \\
&\quad + 2 + 2 \\
&\quad + 1
\end{aligned}$$

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$$
\begin{aligned}
&+ n + \ \ldots \ + 3 + 2 + 1 \\
&+ n + \ \ldots \ + 3 + 2 \\
&+ n + \ \ldots \ + 3 \\
&\ \vdots \\
&+ n \\
= &\ (2n + 1) + (2n + 1) + (2n + 1) + \ \ldots \ + (2n + 1) \\
&+ (2n + 1) + (2n + 1) + \ \ldots \ + (2n + 1) \\
&+ (2n + 1) + \ \ldots \ + (2n + 1) \\
&\ \vdots \\
&+ (2n + 1) \\
= &\ \frac{n(n + 1)}{2} \times (2n + 1).
\end{aligned}
$$

References

1. Johannes Faulhaber, *Academia algebræ, darinnen die miraculosische Inventiones zu den höchsten Cossen weiters continuirt und profitiert werden*, Augspurg (1631).
2. Wolfgang Sartorius Freiherr von Waltershausen, *Gauss zum Gedächtnis*, Leipzig, S. Hirzel (1856).

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