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Indecomposable Higher Chow Cycles

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Abstract. Let X be a projective smooth variety over a field k. In the first part we show that an indecomposable element in $CH^2(X,1)$ can be lifted to an indecomposable element in $CH^3(X_K,2)$ where K is the function field of 1 variable over k. We also show that if X is the self-product of an elliptic curve over $\mathbb{Q}$ then the $\mathbb{Q}$-vector space of indecomposable cycles $CH^3_{ind}(X_{\mathbb{C}},2)_{\mathbb{Q}}$ is infinite dimensional.

In the second part we give a new definition of the group of indecomposable cycles of $CH^3(X,2)$ and give an example of non-torsion cycle in this group.

1 Introduction

Let X be a projective smooth variety over a field k of characteristic 0. The higher Chow group $CH^2(X,1)$ of X is given as the cohomology of the complex

$$K_2(k(X)) \to \bigoplus_{x\in X^1} \kappa(x)^* \to \bigoplus_{y\in X^2} \mathbb{Z}$$

where the first map is the tame symbol and the second is divisors of functions. Here is a description of the tame symbol. Let F be a field with discrete valuation ν, and let $\kappa(\nu)$ be the residue field. Then the tame symbol $t_\nu\colon K_2F \to \kappa(\nu)^*$ is described as

$$t_\nu\big(\{f,g\}\big) = (-1)^{\mathrm{ord}_\nu(f)\mathrm{ord}_\nu(g)} f^{\mathrm{ord}_\nu(g)} g^{-\mathrm{ord}_\nu(f)}.$$

Similarly the higher Chow group $CH^3(X,2)$ is the cohomology of the complex

$$K_3(k(X)) \to \bigoplus_{x\in X^1} K_2(\kappa(x)) \to \bigoplus_{y\in X^2} \kappa(y)^*$$

where the first map is given by localization of algebraic K-theory and the second one is the tame symbol.

There is a product $CH^i(X,j) \otimes CH^n(X,m) \to CH^{i+n}(X,j+m)$ on higher Chow groups [Bl2]. We call the image of $\mathrm{Pic}(X) \otimes CH^1(X,1) = \mathrm{Pic}(X) \otimes k^*$ the group of decomposable cycles. This group is denoted as $CH^2_{\mathrm{dec}}(X,1)$.

We define the group of indecomposable cycles $CH^2_{\mathrm{ind}}(X,1)$ as the quotient of $CH^2(X,1)$ by $CH^2_{\mathrm{dec}}(X,1)$. Some examples of non-torsion elements in $CH^2_{\mathrm{ind}}(X,1)$ for $X = C\times C$, a product of curves, are known [AMS, Fl, GL, Ki3, Mi, MS2, MSC, Sp].

Similarly we consider the subgroup of those cycles which become decomposable after finite extension of base field k and denote it as $CH^2_{g-\mathrm{dec}}(X,1)$. The group $CH^2_{g-\mathrm{ind}}(X,1)$ of geometrically indecomposable cycles is the quotient of $CH^2(X,1)$

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by $CH^2_{g-\mathrm{dec}}(X,1)$. In [Sa] it is shown that the cycles in the above references are also non-torsion in $CH^2_{g-\mathrm{ind}}(X,1)$.

In section 2 we define the decomposable part $CH^3(X,2)_{\mathrm{dec}}$ of $CH^3(X,2)$ as the image of $\mathrm{Pic}(X)\otimes CH^1(X,1)\otimes CH^1(X,1)=\mathrm{Pic}(X)\otimes k^*\otimes k^*$ under the product.

Let $K=k(t)$ be the function field of one variable over k and let $X_K:=X\times_k K$. The main result of this paper is that if one is given a non-torsion cycle in $CH^2(X,1)_{\mathrm{ind}}$ (resp., $CH^2_{g-ind}(X,1)$) then one can construct from it a non-torsion cycle in $CH^3(X_K,2)_{\mathrm{ind}}$ (resp., $CH^3_{g-ind}(X_K,2)$) (Theorem 2.4). As a corollary we show that if X is the self-product of an elliptic curve over $\mathbb{Q}$, the vector space $CH^3_{\mathrm{ind}}(X_{\mathbb{C}},2)\otimes\mathbb{Q}$ is infinite dimensional.

In section 3 we give another definition of decomposable cycles in $CH^3(X,2)$. There we define the decomposable part as the image of $CH^2(X,1)\otimes CH^1(X,1)$. Note that this is a larger subgroup than the image of $\mathrm{Pic}(X)\otimes k^*\otimes k^*$. The group of indecomposable cycles $CH^3(X,2)_{\mathrm{ind}}$ is defined to be the quotient

$$CH^3(X,2)/CH^3(X,2)_{\mathrm{dec}}.$$

Then we give an example of non-torsion cycles in this group for $X=C\times C'$ a product of two projective smooth curves.

We present two methods of construction. One uses Deligne cohomology and the other uses continuous etale cohomology. We make use of Kato's element in K_2 of a CM elliptic curve given in [BlKa] in the latter case.

2 Indecomposable Cycles in $CH^3(X,2)$

Let $j\colon X_K\hookrightarrow X\times\mathbb{P}^1_k$ be the map given by identifying K with the function field of $\mathbb{P}^1_k$ and let $i\colon X\hookrightarrow X\times\mathbb{P}^1_k$ be the map given by identifying X with $X\times\{0\}$.

The image of $\mathrm{Pic}(X_K)\otimes K^*\otimes K^*$ in $CH^3(X_K,2)$ is called the group of decomposable cycles and denoted as $CH^3_{\mathrm{dec}}(X_K,2)$. We define the group of indecomposable cycles $CH^3_{\mathrm{ind}}(X_K,2)$ and that of geometrically indecomposable cycles $CH^3_{g-ind}(X_K,2)$ in the similar way as the case of $CH^2(X,1)$.

There is a boundary map

$$\partial_0\colon CH^3(X_K,2)\to CH^2(X,1)$$

which is described as follows: Let $\sum_l(D_l,\alpha_l)$ be an element of $CH^3(X_K,2)$. Here D_l is an irreducible curve on X_K and α_l is an element of $K_2(\kappa(D_l))$ for each l. Let $C_{l,j}$ for $j=1,\dots,n_l$ be the irreducible components of $\overline{j(D_l)}\cap i(X)$. Here $\overline{j(D_l)}$ is the closure of $j(D_l)$ in $X\times\mathbb{P}^1$. Then

$$\partial_0\Big(\sum_i(D_l,\alpha_l)\Big)=\sum_l\sum_{j=1}^{n_l}(C_{l,j},t_{C_{l,j}}(\alpha_l)).$$

Here $t_{C_{l,j}}$ is the tame symbol at the generic point of $C_{l,j}$. When $\overline{j(D_l)}$ is not normal at the generic point of $C_{l,j}$ then $t_{C_{l,j}}(\alpha_l)$ should be understood as follows. Let

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$\tilde{D}_l$ be the normalization of $\overline{j(D_l)}$ and π be the projection from $\tilde{D}_l$ to $\overline{j(D_l)}$. Let C_i for $i = 1, \dots, n$ be the irreducible components of $\pi^{-1}(C_{l,j})$. Then

$$t_{C_{l,i}}(\alpha_l) = \prod_{i=1}^{n} N_{\kappa(C_i)/\kappa(C_{l,j})} t_{C_i}(\pi^* \alpha_l).$$

Let $\alpha = \sum_l (D_l, f_l)$ be an element of $CH^2(X, 1)$. Consider the element

$$A = pr_1^* \alpha \cdot pr_2^* t = \sum_l \big(pr_1^*(D_l), \{pr_1^* f_l, t\} \big) \in CH^3(X_K, 2).$$

Here pr_i are the projections on $X_K = X \times K$ and t is the parameter of $K = k(t)$.

Lemma 2.1

$$\partial_0(A) = \alpha.$$

Proof Since $\overline{j(p^* D_l)} = D_l \times \mathbb{P}^1$, we see that $\overline{j(p^* D_l)} \cap i(X) = D_l \times \{0\}$. The functions $p^* f_l$ have no zero or pole along $D_l \times \{0\}$. So it follows that $t_{D_l \times \{0\}}\{f_l, t\} = f_l^{\operatorname{ord}_{D_l \times \{0\}}(t)} = f_l$. This finishes the proof. ■

Lemma 2.2 *The group $CH^3_{\mathrm{dec}}(X_K, 2)_{\mathbb{Q}}$ (resp., $CH^3_{g-\mathrm{dec}}(X_K, 2)_{\mathbb{Q}}$) is mapped under ∂_0 to $CH^2_{\mathrm{dec}}(X, 1)_{\mathbb{Q}}$ (resp., $CH^2_{g-\mathrm{dec}}(X, 1)_{\mathbb{Q}}$). Here the subscript $\mathbb{Q}$ means tensor with $\mathbb{Q}$.*

Proof Let D be an irreducible curve on X_K and $\beta = \{f, g\}$ be an element of $K_2(K)$. Assume we have the equality

$$\overline{j(D)} \cap X \times \{0\} = \sum_i m_i D_i.$$

Then we have

$$\partial_0(D, \{f, g\}) = \sum_i \big(D_i, (t_0\{f, g\})^{m_i} \big)$$

where $t_0\{f, g\} \in k^*$ is the tame symbol of $\{f, g\}$ at $t = 0$.

Take an element $\alpha \in CH^3_{g-dec}(X_K, 2)_{\mathbb{Q}}$. Then there is a finite extension K' of K such that $pr_1^* \alpha \in CH^3_{\mathrm{dec}}(X_{K'}, 2)_{\mathbb{Q}}$. Let C be the smooth projective curve over a finite extension of k whose function field is K' and let f be the map from C to $\mathbb{P}^1$. By the compatibility of the boundary map and the projection, we have the equality

$$\partial_0(pr_{1*} pr_1^* \alpha) = f_* \sum_{p \in f^{-1}(0)} \partial_p pr_1^* \alpha.$$

The left hand side of this equlity is $[K' : K]\partial_0 \alpha$ and the right hand side is geometrically decomposable since it is the image of decomposable element under the projection. ■

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Hence we obtain the following result.

Theorem 2.3 *If α is a non-torsion element in $CH^2_{\text{ind}}(X,1)$ (resp,. $CH^2_{g-\text{ind}}(X,1)$), then A is non-torsion in $CH^3_{\text{ind}}(X_K,2)$ (resp., $CH^3_{g-\text{ind}}(X_K,2)$).*

Corollary 2.4 *For X, the self-product of an elliptic curve over $\mathbb{Q}$, the group*

$$CH^3_{\text{ind}}(X_{\mathbb{C}},2)_{\mathbb{Q}}$$

is infinite dimensional.

Proof We proceed as the proof of Theorem 0.3 in [Sa]. By his theorem we know that $CH^2_{g-\text{ind}}(X,1)_{\mathbb{Q}}$ is infinite dimensional. So the group $CH^3_{g-\text{ind}}(X_K,2)_{\mathbb{Q}}$ is also infinite dimensional. We fix an embedding $K = \mathbb{Q}(t) \to \mathbb{C}$. We need to show that the map

$$CH^3_{g-\text{ind}}(X_K,2)_{\mathbb{Q}} \to CH^3_{\text{ind}}(X_{\mathbb{C}},2)_{\mathbb{Q}}$$

is injective. Assume that $\zeta \in CH^3(X_K,2)_{\mathbb{Q}}$ is decomposable in $CH^3(X_{\mathbb{C}},2)_{\mathbb{Q}}$. Then there exists a finitely generated K-subalgebra R of $\mathbb{C}$, divisors Z_i on $X \otimes R$, $\alpha_i, \beta_i \in R(1 \le i \le a)$ and rational functions f_j, g_j, h_j on $X \otimes R(1 \le j \le b)$ such that

$$\zeta - \sum_i \bigl(Z_i, \{\alpha_i, \beta_i\}\bigr) = \sum_j T\bigl(\{f_j, g_j, h_j\}\bigr)$$

where the letter T means tame symbol. Note that by [MeS, Proposition 11.11] we can replace $K_3(\mathbb{C}(X))$ with $K_3^M(\mathbb{C}(X))$ in the complex (1). By shrinking Spec R if necessary, we can assume that every irreducible component of the divisors of the functions f_j, g_j, h_j is flat over Spec R. By specializing to a general closed point of Spec R, we may assume that R is a finite extension of K by replacing Z_i, α_i, β_i, f_j, g_j, h_j with their restriction to the fiber over the point. Compatibility of the specialization and tame symbol follows from the fact that every irreducible component of the divisors of the functions f_j, g_j, h_j (with reduced scheme structure) has a nonempty open subscheme which is smooth over Spec R. This means $\zeta = 0$ in $CH^3_{g-\text{ind}}(X_K,2)_{\mathbb{Q}}$. ∎

3 Another Definition of Indecomposable Cycles

Definition 3.1 The decomposable part $CH^3(X,2)_{\text{dec}}$ is the image of $CH^2(X,1) \otimes CH^1(X,1)$ under the product. The group of indecomposable cycles $CH^3(X,2)_{\text{ind}}$ is the quotient $CH^3(X,2)/CH^3(X,2)_{\text{dec}}$.

Let l be a prime. There are Chern class maps

$$C_{3,2} \colon CH^3(X,2) \to H^4\bigl(X, \mathbb{Q}_l(3)\bigr)$$

and

$$C_{2,1} \colon CH^2(X,1) \to H^3\bigl(X, \mathbb{Q}_l(2)\bigr)$$

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where the cohomology groups on the right are the continuous etale cohomology defined by Dywer and Friedlander [DwFr]. We refer the reader to [So1] for the definition of this map. Jannsen [Ja] shows that there is the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p_{cont}(G_k,\ H^q\big(X_{\bar{k}}, \mathbb{Q}_l(j)\big)\big) \Rightarrow H^{p+q}(X, \mathbb{Q}_l(j)).$$

Here G_k is the absolute Galois group of k. Since Gr^0 of the filtration on $H^4(X, \mathbb{Q}_l(3))$ and on $H^3(X, \mathbb{Q}_l(2))$ given by this spectral sequence vanish for weight reason, we obtain maps to Gr^1:

$$C_{3,2} \colon CH^3(X, 2) \to H^1\big(G_k, H^3(X_{\bar{k}}, \mathbb{Q}_l(3))\big)$$

and

$$C_{2,1} \colon CH^2(X, 1) \to H^1\big(G_k, H^2(X_{\bar{k}}, \mathbb{Q}_l(2))\big).$$

Since the product on higher Chow groups is compatible with the cup product on the cohomology groups under the Chern class map, it maps the decomposable part $CH^3(X, 2)_{\text{dec}}$ into $F^2H^4(X, \mathbb{Q}_l(3))$ of the filtration given by the spectral sequence. Hence we obtain the following map.

$$C_{3,2} \colon CH^3(X, 2)_{\text{ind}} \to H^1\big(G_k, H^3(X_{\bar{k}}, \mathbb{Q}_l(3))\big).$$

If the base field $k = \mathbb{C}$, there are also Chern class maps

$$C_{3,2} \colon CH^3(X, 2) \to H^4_D\big(X, \mathbb{R}(3)\big)$$

and

$$C_{2,1} \colon CH^2(X, 1) \to H^3_D\big(X, \mathbb{R}(2)\big)$$

where the cohomology groups on the right are Deligne cohomology. By a similar argument as above we obtain the following map.

$$C_{3,2} \colon CH^3(X, 2)_{\text{ind}} \to H^4_D\big(X, \mathbb{R}(3)\big).$$

Now let C be a projective smooth curve. Take another projective smooth curve C' with a closed point 0 and let $X = C' \times C$. We denote by i the closed immersion which maps C to $0 \times C$. By Theorem 3.1 and Corollary 3.7 in [Gi] we have the following commutative diagram

$$\begin{array}{ccc} CH^2(C, 2) & \xrightarrow{i_*} & CH^3(X, 2) \\ {\scriptstyle C_{2,2}}\downarrow & & \downarrow{\scriptstyle C_{3,2}} \\ H^1\big(G_{\mathbb{Q}},\ H^1(C_{\bar{\mathbb{Q}}}, \mathbb{Q}_l(2))\big) & \xrightarrow{i_*} & H^1\big(G_{\mathbb{Q}},\ H^3(X_{\bar{\mathbb{Q}}},\ \mathbb{Q}_l(3))\big) \end{array}$$

and in the case $k = \mathbb{C}$ we also have

$$\begin{array}{ccc} CH^2(C, 2) & \xrightarrow{i_*} & CH^3(C, 2) \\ {\scriptstyle C_{2,2}}\downarrow & & \downarrow{\scriptstyle C_{3,2}} \\ H^2_D\big(C, \mathbb{R}(2)\big) & \xrightarrow{i_*} & H^4_D\big(X, \mathbb{R}(3)\big). \end{array}$$

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Here the lower horizontal arrows are Gysin map. Note that the normal bundle $N_{C/C'\times C}$ is trivial. Since the trace map

$$H^2(C_{\bar{\mathbb{Q}}}, \mathbb{Q}_l)(1) \otimes H^1(C'_{\bar{\mathbb{Q}}}, \mathbb{Q}_l)(2) \to H^1(C_{\bar{\mathbb{Q}}}, \mathbb{Q}_l(2))$$

resp., $H^4_D(X, \mathbb{R}(3)) \to H^2_D(C, \mathbb{R}(2))$, gives a splitting for i_*, i_* is injective. Thus we obtain the following result.

Proposition 3.2 *If there is an element $\alpha \in CH^2(C, 2)$ which has a non-vanishing image in $H^1(G_{\mathbb{Q}},\ H^1(C_{\bar{\mathbb{Q}}},\ \mathbb{Q}_l(2)))$ or $H^2_D(C, \mathbb{R}(2))$ then $i_*(\alpha) \in CH^3(X, 2)_{\text{ind}}$ is non-torsion.*

There are several examples of α which satisfy this condition [Be, Be2, Bl, Ki, BlKa].

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