Compositio Mathematica **131:** 1–30, 2002.

On the Motive of Moduli Spaces of Rank Two Vector Bundles over a Curve

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(Received: 19 April 1999; accepted in final form: 7 March 2001)

Abstract. We study the motive of the moduli spaces of rank two vector bundles on a curve. In the smooth case we obtain the Hodge numbers, intermediate Jacobians and number of points over a finite field as corollaries. In the singular case our computations yield the Poincaré–Hodge polynomial of Seshadri's smooth model.

Mathematics Subject Classifications (2000). 14H60, 14F42.

Key words. motives, moduli space.

Introduction

The moduli space of stable vector bundles over an algebraic curve is an interesting space related to the curve and has received great attention for the last 20 years. In particular, when the rank and degree are coprime its cohomology has been shown to be torsion free and its Betti numbers are known. However the methods used in studying its cohomology are topological [23], number theoretical [11, 15] or infinite-dimensional [1], and these, at least in principle, do not yield information on the motivic structure of the cohomology of the moduli space.

In this paper we use a recent construction by M. Thaddeus [28] to give a description of the motivic Poincaré polynomial of the moduli space of rank two semistable vector bundles of fixed determinant on an algebraic curve. It is an idea of Grothendieck that one should work in the Grothendieck group K_0 of the category of motives; this is where the motivic Poincaré polynomial lives. We believe that the theory of motives is an effective language to express clearly and precisely how the algebro-geometric properties of the curve influence those of the moduli space of stable vector bundles. As a manifestation of this belief we show how to prove a semisimplicity statement for the action of the Galois group on the étale cohomology of the moduli space.

This work is concerned with the moduli spaces with fixed determinant. The case where only the degree is fixed and the rank is arbitrary is treated, over a field of characteristic zero, in [7]. The connection between these two studies is related to the action of the torsion of the Jacobian on the Chow motive of the moduli spaces.

As far as we know the only known fact in this direction is the result of Harder and Narasimhan [15], stating that this action is trivial on cohomology.

In Section 1 we introduce the moduli spaces and some basic facts of the theory of motives.

The next section contains a study of the motive of the moduli space of rank two stable and odd determinant vector bundles over a curve. This is a smooth projective variety. We use results of Thaddeus for which we have given a short review. We end this section with applications to Hodge theory and ℓ-adic cohomology.

Section 3 is devoted to the study of the singular moduli space, that is the one in which the determinant is even. We obtain an expression of its motivic Poincaré polynomial and that of the canonical smooth model found by Seshadri. As an application we find the Hodge numbers of Seshadri's smooth model. To conclude we study the mixed Hodge structure of the singular moduli space and show that only two weights occur.

1. Preliminaries

Throughout this work k will stand for a field.

1.1. MODULI SPACES

Let C be a smooth projective curve over k, $n > 1$ an integer and $\mathcal{L} \in \mathrm{Pic}\,C(k)$ a line bundle over C. We shall denote by $N_C(n, \mathcal{L})$ the moduli space of semistable vector bundles of rank n and determinant isomorphic to $\mathcal{L}$. $N_C(n, \mathcal{L})^s$ will denote the smooth Zariski open subset of $N_C(n, \mathcal{L})$ whose points parametrize stable bundles. In the case k is algebraically closed it is easy to see, using the divisibility of $\mathrm{Jac}\,C(k)$, that the variety $N_C(n, \mathcal{L})$ only depends on the residue of $\deg(\mathcal{L})$ modulo n. It is also the case that $N_C(n, \mathcal{L})$ is a smooth variety if $(\deg(\mathcal{L}), n) = 1$.

Therefore, in the case $n = 2$, we essentially have two moduli spaces according to whether $\deg(\mathcal{L})$ is even or odd. Except in the case $g = 2$ the even degree moduli space is singular. Seshadri has found a smooth model for $N_C(2, \mathcal{O}_C)$ which we shall note by M.

1.2. MOTIVES

1.2.1. *Definitions*

We shall denote by $\mathcal{V}_k$ the category of smooth projective varieties over k. Let $\mathcal{M}_k^+$ be the category of effective Chow motives over k [18, 25]. This consists of a pseudoabelian tensor $\mathbb{Q}$-linear category $\mathcal{M}_k^+$ together with a functor $h\colon \mathcal{V}_k \to \mathcal{M}_k^+$. By taking classes in K_0 we obtain a map $\chi\colon \mathrm{Ob}\ \mathcal{V}_k \to K_0\mathcal{M}_k^+$ that generalises the Poincaré polynomial, it will be called the *motivic Poincaré polynomial*. For $\mathrm{char}\,k = 0$ the Poincaré–Hodge polynomial extends to a ring morphism

$K_0\mathcal{M}_k^+ \to \mathbb{Z}[x, y]$ which will be denoted by P_{xy}. For $k \simeq \mathbb{F}_q$ a finite field the trace of Frobenius defines a ring morphism v_q: $K_0\mathcal{M}_q^+ \to \mathbb{Q}_\ell$ such that $v_q(\chi(X)) = \#X(k)$.

The motive of Spec(k) is called the trivial motive and denoted by $\mathbb{1}$. If C is a smooth projective curve we have a decomposition $h(C) = \mathbb{1} \oplus h^1 C \oplus \mathbb{L}$ (see [25]). We call $\mathbb{L}$ the Lefschetz motive. It is customary to denote a tensor product like $M \otimes \mathbb{L} \otimes \overset{n}{\cdots} \otimes \mathbb{L}$ by $M(-n)$.

Given a finite subset of Ob $\mathcal{M}_k^+$, we define the pseudo-Abelian tensor category generated by these objects to be the full subcategory of $\mathcal{M}_k^+$ whose objects are subquotients of tensor products of the given objects.

1.2.2. *Motives of Arbitrary Schemes*

In the case k has resolution of singularities, e.g. if char$(k) = 0$, F. Guillén and V. Navarro [13], have extended the functor h to the category of separated schemes of finite type over k, $\mathbf{Sch}_k$. Taking the class in K_0 we obtain a map χ: Ob $\mathbf{Sch}_k \to K_0\mathcal{M}_k^+$. They also obtain an extension corresponding to a theory with compact supports, the corresponding map will be denoted by χ_c. The function χ_c is characterized by the following property

(E) If $X \in$ Ob $\mathbf{Sch}_k$ and $Y \subset X$ is a closed subscheme then $\chi_c(X - Y) = \chi_c(X) - \chi_c(Y)$.

1.2.3. *The Ring $\mathcal{K}$*

Define $\mathcal{K}$ to be the completion of $K_0\mathcal{M}_k^+$ along the ideal generated by the Lefschetz motive $\mathbb{L}$. As $P_{xy}(\mathbb{L}) = xy$ we see that P_{xy} extends to a morphism P_{xy}: $\mathcal{K} \to \mathbb{Z}[[x, y]]$. Over a finite field, $\mathbb{F}_q$, we define a ring morphism v_q^t: $\mathcal{K} \to \mathbb{Q}_\ell[[t]]$ by

$$v_q^t(M) = \sum_i (-1)^i \mathrm{Tr}(\mathrm{Fr}_q\colon H_\ell^i(M) \to H_\ell^i(M)) t^i.$$

We recover the morphism v_q by setting $t = 1$.

1.2.4. *Symmetric Powers*

For a motive M and a nonnegative integer n define $\lambda^n(M)$ to be the image in $M^{\otimes n}$ of the projector $(1/n!) \sum_{\sigma \in \mathfrak{S}_n} \sigma_*$. Let C be a smooth projective curve, let $C^{(n)}$ be the nth symmetric power of C, it is a smooth projective variety. The results of [7] show that the motive of $C^{(n)}$ is given by $hC^{(n)} = \bigoplus_{a+b+c=n} \mathbb{1}^{\otimes a} \otimes \lambda^b h^1(C) \otimes \mathbb{L}^{\otimes c}$ where a, b and c run through nonnegative integers. If $n \geqslant 2g - 1$ and $C(k) \neq \emptyset$ one can view $C^{(n)}$ as a projective bundle over Jac$C$ of rank $n - g$ so that by the results of [18, §7] $hC^{(n)} \simeq h\mathrm{Jac}C \otimes (\mathbb{1} \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^{n-g})$. In fact, this is the case even if $C(k) \neq \emptyset$, as one can easily check from the above description of $hC^{(n)}$.

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Expressions like $\bigoplus_{i=0}^{2g_C} \lambda^i h^1 C(-ni)$, where g_C is the genus of C, will arise in our calculations. We will denote this motive, or its class in $K_0\mathcal{M}_k^+$, by $(\mathbb{1} + \mathbb{L}^n)^{h^1 C}$. Note that $(\mathbb{1} + \mathbb{1})^{h^1 C}$ is just the motive of the Jacobian Jac$C$.

2. The Smooth Case

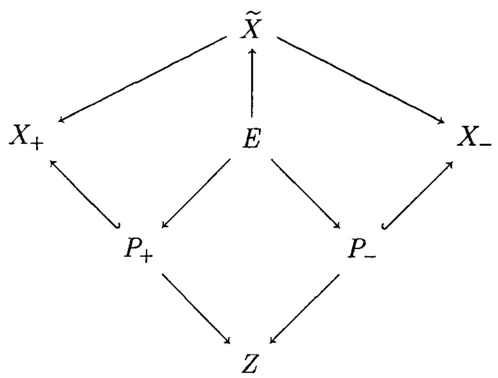
Here we study the moduli space of stable rank two vector bundles with fixed odd determinant $N_C(2, \mathcal{L})$. We will use a construction due to M. Thaddeus which we now recall.

2.1. THADDEUS' CONSTRUCTION

Given a line bundle $\mathcal{L}$ of degree d Thaddeus considers the problem of classifying the isomorphism classes of vector bundles plus a nonzero section. As in the case of ordinary vector bundles, in order to construct a separated moduli space one has to define a concept of stability and restrict to stable objects. However, in contrast with that case, there is not a canonical definition of stability but various. In this way one obtains a list of moduli spaces depending on a real parameter $\sigma \in [0, d/2]$. For σ not in a finite set $S \subset [0, d/2]$ of critical values the moduli space is a smooth projective fine moduli space (fine on the algebraic closure k) which only depends on the connected component of $[0, d/2] - S$ in which σ lies. Let us write $M_0, M_1, \ldots, M_\omega$ for this ordered list of moduli spaces of pairs, then $\omega = [(d-1)/2]$, hence we assume $d \geqslant 3$.

These different moduli spaces are related by a special kind of birational transformation called flips. Let us recall this concept.

DEFINITION 2.1. A birational map $X_+ - - \rightarrow X_-$ is called a smooth flip with centre Z and type (d_+, d_-) if it fits in a diagram of the following type:



in which the two upper diagrams are blow up diagrams, the lower square is a cartesian diagram and P_+ (resp. P_-) is a projective bundle over Z associated to a vector bundle of rank d_- (resp. d_+).

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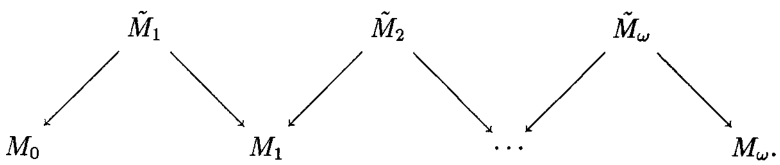
In particular note that the codimension of P_+ (resp. P_-) in X_+ (resp. X_-) is d_- (resp. d_+).

The following summarizes the results of [28] we shall use.

THEOREM 2.2. (M. Thaddeus).

(1) *There is a flip* $M_{i-1} \dashrightarrow M_i$ *with centre the* ith *symmetric power of* C, $C^{(i)}$, *and type* $(i, d+g-2i-1)$.
(2) *The first moduli space of pairs* M_0 *is isomorphic to the* $(d+g-2)$-*dimensional projective space* $\mathbb{P}H^1(C, \mathcal{L}^{-1})$ [28, 3.1].
(3) *There is a natural morphism* $\pi\colon M_\omega \to N_C(2, \mathcal{L})$. *If* d *is odd and greater than* $2g-2$ *it makes* M_ω *into a projective fibration over* $N_C(2, \mathcal{L})$ [28, 3.20].

We can picture this chain of flips:



Note that $\tilde{M}_1 \to M_1$ is the blow up of M_1 along a smooth divisor therefore an isomorphism.

2.2. THE MOTIVE OF $N_C(2, \mathcal{L})$

In this section we use the construction described in the previous section to find the motivic Poincaré polynomial of $N_C(2, \mathcal{L})$.

We start with a lemma which explains how a flip operates on the motive of a variety.

LEMMA 2.3. *Let* X_+, X_- *and* Z *be smooth projective varieties. If* X_+ *and* X_- *are related by a flip with centre* Z *and type* (d_+, d_-), *then*

$$\begin{aligned} &h(X_+) \oplus ((\mathbb{L} \oplus \cdots \oplus \mathbb{L}^{d_- -1}) \otimes (\mathbb{1} \oplus \cdots \oplus \mathbb{L}^{d_+ -1}) \otimes h(Z)) \\ &\quad \simeq h(X_-) \oplus ((\mathbb{L} \oplus \cdots \oplus \mathbb{L}^{d_+ -1}) \otimes (\mathbb{1} \oplus \cdots \oplus \mathbb{L}^{d_- -1}) \otimes h(Z)) \end{aligned} \tag{1}$$

and

$$\chi(X_+) - \chi(X_-) = \chi(Z) \cdot \frac{\mathbb{L}^{d_+} - \mathbb{L}^{d_-}}{\mathbb{1} - \mathbb{L}}. \tag{2}$$

Proof. This is consequence of Definition 2.1, and of the results of Manin on the structure of the motive of a blow up and a projective bundle ([18, §7 and §9]). □

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COROLLARY 2.4. *Let X_-, X_+ and Z be smooth projective varieties. If $X_+ - - \to X_-$ is a flip with centre Z then $h(X_-)$ lies in the pseudo-Abelian subcategory of $\mathcal{M}_k^+$ generated by $h(X_+)$, $h(Z)$ and $\mathbb{L}$.*

Proof. This follows from (1) which exhibits an isomorphism between $h(X_-)$ and a subobject of

$$h(X_+) \oplus ((\mathbb{L} \oplus \cdots \oplus \mathbb{L}^{d_- -1}) \otimes (\mathbb{1} \oplus \cdots \oplus \mathbb{L}^{d_+ -1}) \otimes h(Z)). \qquad \square$$

THEOREM 2.5. *The Chow motive of $N_C(2, \mathcal{L})$ lies in the pseudo-Abelian tensor category of $\mathcal{M}_g^+$ generated by $h^1(C)$ and $\mathbb{L}$.*

Proof. First we prove by induction on i that this is the case for the moduli spaces of pairs M_i. As M_0 is a projective space the statement is certainly true for $i = 0$. Assume $h(M_{i-1})$ lies in the stipulated category. The motive of $h(C^{(i)})$ also lies in this category by Subsection 1.2.4; the previous corollary and 2.2.1 show the same holds for $h(M_i)$.

By tensoring our vector bundles by the canonical line bundle, K_C, we obtain an isomorphism $N_C(2, \mathcal{L}) \to N(2, K_C^{\otimes 2} \otimes \mathcal{L})$, hence we may assume that $\deg \mathcal{L} > 2g - 2$. By 2.2.3 there is a natural map π: $M_\omega \to N_C(2, \mathcal{L})$ which is a projective bundle. But via π^*, $h(N_C(2, \mathcal{L}))$ is identified with a subobject of $h(M_\omega)$ thus proving the assertion. $\square$

In [23], P. E. Newstead shows that the Poincaré polynomial of the variety $N_C(2, \mathcal{L})$ is $((1 + t^3)^{2g} - (1 + t)^{2g})/((1 - t^2)(1 - t^4))$. The following is our motivic version of this result.

THEOREM 2.6. *Let C be a smooth projective curve over a field k and $\mathcal{L} \in \mathrm{Pic}^1 C(k)$ a line bundle of degree 1. Then*

$$\chi(N_C(2, \mathcal{L})) = \frac{(\mathbb{1} + \mathbb{L})^{h^1(C)} - (\mathbb{1} + \mathbb{1})^{h^1(C)}(-g)}{(\mathbb{1} - \mathbb{L})(\mathbb{1} - \mathbb{L}^2)}.$$

Proof. We may substitute $\mathcal{L}$ by a line bundle of degree $4g - 3$ for there is an isomorphism

$$N_C(2, \mathcal{L}) \xrightarrow{\otimes K_C} N(2, K_C^{\otimes 2} \otimes \mathcal{L}) \quad \text{and} \quad \deg K_C^{\otimes 2} \otimes \mathcal{L} = 4g - 3.$$

For this degree the sequence of moduli spaces of pairs has $\omega = 2g - 2$. Also $M_0 \simeq \mathbb{P}^{5g-5}$, M_ω is a projective bundle over $N_C(2, \mathcal{L})$ of relative dimension $2g - 2$ and $M_{i-1} - - \to M_i$ is a flip of type $(i, 5g - 4 - 2i)$ and centre $C^{(i)}$.

By inductively applying Lemma 2.3 we obtain

$$\begin{aligned}\chi(M_\omega) &= \chi(M_0) + \sum_{i=1}^{\omega} \chi(C^{(i)}) \frac{\mathbb{L}^i - \mathbb{L}^{5g-4-2i}}{\mathbb{1} - \mathbb{L}} \\ &= \sum_{i=1}^{\omega} \chi(C^{(i)}) \frac{\mathbb{L}^i - \mathbb{L}^{5g-4-2i}}{\mathbb{1} - \mathbb{L}}.\end{aligned}$$

In order to avoid negative powers of $\mathbb{L}$, we shall work in the ring $\mathcal{K}[[T]]$, compute

$$\frac{1}{1-T}\left(\sum_{i=0}^{\omega} \chi(C^{(i)}) T^i - \sum_{i=0}^{\omega} \chi(C^{(i)}) T^{5g-4-2i}\right) \tag{3}$$

in this ring and then apply the natural ring morphism R: $\mathcal{K}[[T]] \to \mathcal{K}$ that takes T to $\mathbb{L}$. We write $\equiv$ for congruence modulo ker R in $\mathcal{K}[[T]]$.

We first calculate the first summand in (3)

$$\sum_{i=0}^{2g-2} \chi C^{(i)} T^i = \sum_{i=0}^{\infty} \chi C^{(i)} T^i - \sum_{i=2g-1}^{\infty} \chi C^{(i)} T^i \tag{4}$$

Note that $\chi C^{(i)} = \sum_{a+b+c=i} \lambda^b h^1 C \cdot \mathbb{L}^c$ so the first summand in the right-hand side of (4) equals

$$\sum_{a=0}^{\infty} T^a \sum_{b=0}^{\infty} \lambda^b h^1 C T^b \sum_{c=0}^{\infty} \mathbb{L}^c T^c = \frac{(1+T)^{h^1 C}}{(1-T)(1-\mathbb{L}T)}$$

For the second summand in (4) note that for $i \geqslant 2g-1$ by Subsection 1.2.4

$$\chi C^{(i)} = (1+1)^{h^1 C} \cdot \frac{1 - \mathbb{L}^{i-g+1}}{1-\mathbb{L}},$$

therefore (4) equals

$$\begin{aligned}&\frac{(1+T)^{h^1 C}}{(1-T)(1-\mathbb{L}T)} - \frac{(1+1)^{h^1 C}}{1-\mathbb{L}} \sum_{i=2g-1}^{\infty} (1 - \mathbb{L}^{i-g+1}) \\ &\quad = \frac{(1+T)^{h^1 C}}{(1-T)(1-\mathbb{L}T)} - \frac{(1+1)^{h^1 C}}{1-\mathbb{L}} \left(\frac{T^{2g-1}}{1-T} - \frac{\mathbb{L}^g T^{2g-1}}{1-\mathbb{L}T}\right).\end{aligned}$$

For the second sum in (3), $T^{5g-4} \sum_{i=0}^{\omega} \chi C^{(i)} T^{-2i}$, we work in $\mathcal{K}[[T^{-1}]]$, by arguments similar to the previous

$$\begin{aligned}\sum_{i=0}^{\omega}\chi C^{(i)}T^{-2i} &= \sum_{i=0}^{\omega}\chi C^{(i)}T^{-2i} - \sum_{i=2g-1}^{\omega}\chi C^{(i)}T^{-2i}\\ &= \frac{(1+T^{-2})^{h^1C}}{(1-T^{-2})(1-\mathbb{L}T^{-2})} - \frac{(1+1)^{h^1C}}{1-\mathbb{L}}\sum_{i=2g-1}^{\omega}(1-\mathbb{L}^{i-g+1})T^{-2i}\\ &= \frac{(1+T^{-2})^{h^1C}}{(1-T^{-2})(1-\mathbb{L}T^{-2})} - \frac{(1+1)^{h^1C}}{1-\mathbb{L}}\left(\frac{T^{-4g+2}}{1-T^{-2}} - \frac{\mathbb{L}^g T^{-4g+2}}{1-\mathbb{L}T^{-2}}\right)\end{aligned}$$

now

$$\begin{aligned}T^{5g-4}\sum_{i=0}^{\omega}\chi C^{(i)}T^{-2i} &= \frac{T^{5g-4}(1-T^{-2})^{h^1C}}{(1-T^{-2})(1-\mathbb{L}T^{-2})} - \frac{(1+1)^{h^1C}}{1-\mathbb{L}}\left(\frac{T^{g-2}}{1-T^{-2}} - \frac{\mathbb{L}^g T^{g-2}}{1-\mathbb{L}T^{-2}}\right)\\ &= \frac{T^{5g}(1-T^{-2})^{h^1C}}{(T^{-2}-1)(T^{-2}-\mathbb{L})} - \frac{(1+1)^{h^1C}}{1-\mathbb{L}}\left(\frac{-T^{g}}{1-T^{2}} - \frac{\mathbb{L}^g T^{g}}{T^{2}-\mathbb{L}}\right),\end{aligned}$$

this lives in $\mathcal{K}[[T]]$, and modulo $\ker R$ it is congruent to

$$\frac{T^{5g-1}(1+T^{-2})^{h^1C}}{(1-T^{-2})(1-T)} - \frac{(1+1)^{h^1C}}{1-\mathbb{L}}\left(\frac{T^{2g-1}}{1-T} - \frac{T^g}{1-T^2}\right).$$

By Künnemann's motivic version of hard Lefschetz we have $\lambda^i h^1 C(i-g) \simeq \lambda^{2g-i} h^1 C$. This implies that

$$\begin{aligned}T^{5g-1}(1-T^{-2})^{h^1C} &= \sum_{i=0}^{2g} T^{5g-1-2i}\lambda^i h^1 C\\ &= \sum_{i=0}^{2g} T^{4g-1-i}\mathbb{L}^{g-i}\lambda^i h^1 C = T^{2g-1}\sum_{i=0}^{2g} T^{2g-i}\lambda^{2g-i}h^1 C\\ &= T^{2g-1}(1+T)^{h^1C}.\end{aligned}$$

Therefore $\chi N_C(2,\mathcal{L})$ is the result of applying R to

$$\begin{aligned}&\frac{1}{1-T^{2g-1}}\left(\sum_{i=0}^{\omega}\chi C^{(i)}T^{i} - T^{5g-4}\sum_{i=0}^{\omega}\chi C^{(i)}T^{-2i}\right)\\ &\equiv \frac{1}{1-T^{2g-1}}\left[\frac{(1-T^{2g-1})(1+T)^{h^1C}}{(1-T)(1-T^2)} - \frac{(1+1)^{h^1C}}{1-\mathbb{L}}\times\right.\\ &\qquad\left.\times\left(\frac{T^{2g-1}-T^{2g-1}}{1-T} - \frac{T^{3g-1}-T^g}{1-T^2}\right)\right]\\ &= \frac{(1+T)^{h^1C} - (1+1)^{h^1C}T^g}{(1-T)(1-T^2)}.\end{aligned}$$

This proves the theorem. □

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In the case when we use a semisimple category of motives we obtain an explicit description of $hN_C(2, \mathcal{L})$. In such categories there is a cancellation property stating that $A \oplus C \simeq B \oplus C \Rightarrow A \simeq B$. This shows that two objects have the same class in the Grothendieck ring if and only if they are isomorphic. Examples of such semisimple categories of motives are the category of motives with respect to numerical equivalence [17] and the category of absolute Hodge motives [9].

COROLLARY 2.7. *In any semisimple category of motives $h(N_C(2, \mathcal{L}))$ is isomorphic to*

$$\bigoplus_{k=0}^{g} \lambda^k h^1 C \otimes (\mathbb{1} \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^{g-k-1}) \otimes (\mathbb{1} \oplus \mathbb{L}^2 \oplus \cdots \oplus \mathbb{L}^{2g-2k-2}) \otimes \mathbb{L}^k. \qquad (5)$$

Proof. By the previous theorem

$$\chi N_C(2, \mathcal{L}) = \frac{(\mathbb{1} + \mathbb{L})^{h^1 C} - (\mathbb{1} + \mathbb{1})^{h^1 C}(-g)}{(\mathbb{1} - \mathbb{L})(\mathbb{1} - \mathbb{L}^2)}$$

this is equal to

$$\bigoplus_{k=0}^{2g} \lambda^k h^1 C \frac{\mathbb{L}^k - \mathbb{L}^g}{(\mathbb{1} - \mathbb{L})(\mathbb{1} - \mathbb{L}^2)},$$

use the duality isomorphism $\lambda^{2g-i} h^1 C \simeq \lambda^i h^1 C \otimes \mathbb{L}^{g-i}$ of Künnemann to get

$$\bigoplus_{k=0}^{g} \lambda^k h^1 C \frac{\mathbb{L}^k - \mathbb{L}^g}{(\mathbb{1} - \mathbb{L})(\mathbb{1} - \mathbb{L}^2)} \oplus \bigoplus_{k=0}^{g} \lambda^k h^1 C(-g+k) \frac{\mathbb{L}^{2g-k} - \mathbb{L}^g}{(\mathbb{1} - \mathbb{L})(\mathbb{1} - \mathbb{L}^2)}$$

adding this we obtain the following expression for $\chi N_C(2, \mathcal{L})$

$$\bigoplus_{k=0}^{g} \lambda^k h^1 C \frac{\mathbb{1} - \mathbb{L}^{g-k}}{\mathbb{1} - \mathbb{L}} \cdot \frac{\mathbb{1} - \mathbb{L}^{2k-2g}}{\mathbb{1} - \mathbb{L}^2}(-k).$$

Note that this is the class in K_0 of an actual motive so by the aforementioned cancellation property we have $hN_C(2, \mathcal{L})$. □

Remark 2.8. The previous corollary allows one to guess what the Chow groups of $N_C(2, \mathcal{L})$ should look like. Indeed, if the expression (5) were equal to $h(N_C(2, \mathcal{L}))$ in the category of Chow motives we would have

$$CH^j_{\mathbb{Q}}(N_C(2, \mathcal{L})) = \bigoplus_{a,b} CH^a_{\mathbb{Q}}(\lambda^b h^1(C))^{d_{a,b}(j)}$$

for certain computable integers $d_{a,b}(j)$. Of course, the category of Chow motives is not semisimple so the above is only a guess.

2.3. REALIZATIONS

Next we show how the theorems in the previous section give results for concrete cohomology theories.

The algebraic cohomology of the moduli space has been studied in [3]. In [6] (see also [7]) we show how this is connected with the motivic Poincaré polynomial.

2.3.1. *Poincaré–Hodge Polynomials*

COROLLARY 2.9. *The Poincaré–Hodge polynomial of* $N_C(2, \mathcal{L})$ *is*

$$\frac{(1+xy^2)^g(1+x^2y)^g - x^g y^g (1+x)^g (1+y)^g}{(1-xy)(1-x^2y^2)}.$$

Proof. We will apply the morphism $P_{xy}\colon \mathcal{K} \to \mathbb{Z}[[x, y]]$ to the expression in Theorem 2.6. Taking into account that P_{xy} is a ring morphism we just need to evaluate it on

$$\mathbb{L}, \quad (\mathbb{1}+\mathbb{1})^{h^1 C}, \quad \text{and} \quad (\mathbb{1}+\mathbb{L})^{h^1(C)}.$$

It is clear that $P_{xy}(\mathbb{L}) = P_{xy}(H^2(\mathbb{P}^1_k)) = xy$, also, as $(\mathbb{1}+\mathbb{1})^{h^1(C)} = h(\mathrm{Jac}C)$, we have

$$P_{xy}((\mathbb{1}+\mathbb{1})^{h^1(C)}) = P_{xy}(\mathrm{Jac}C) = (1+x)^g(1+y)^g.$$

It remains to show that

$$P_{xy}(\mathbb{1}+\mathbb{L})^{h^1(C)} = (1+xy^2)^g(1+x^2y)^g.$$

We shall give a general expression for $P_{xy}(\mathbb{1}+\mathbb{L}^n)^{h^1(C)}$. From the definition, as P_{xy} is a ring morphism, we have

$$P_{xy}(\mathbb{1}+\mathbb{L}^n)^{h^1 C} = \sum_i P_{xy}(\lambda^i h^1 C)\cdot (x^n y^n)^i.$$

If we replace the $x^n y^n$ coming from $\mathbb{L}^n$ by an indeterminate T we get

$$\begin{aligned}\sum_i P_{xy}(\lambda^i h^1 C)\cdot T^i &= \sum_i \sum_{p+q=i} \dim H^{p,q}(\mathrm{Jac}C, \mathbb{C}) x^p y^q \cdot T^i \\ &= P_{Tx,Ty}(\mathrm{Jac}C) = (1+Tx)^g(1+Ty)^g.\end{aligned}$$

Therefore we have shown that

$$P_{xy}(\mathbb{1}-\mathbb{L}^n)^{h^1 C} = (1+x^{n+1}y^n)^g(1+x^n y^{n+1})^g.$$

This proves the corollary. □

Recall that by [24] the varieties $N_C(2, \mathcal{L})$ are rational. As a consequence the Hodge numbers $h^{0,p} N_C(2, \mathcal{L})$ are zero for all $p > 0$. That is, the border of the Hodge diamond

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contains zeroes. In fact the Hodge diamond is quite thin, for this recall the definition of the level of a Hodge structure: $\mathrm{Max}_{h^{p,q}\neq 0}|p-q|$. Then one can prove that the level of the Hodge structure $H^i(N_C(2,\mathcal{L}),\mathbb{Q})$ is less than or equal to $[i/3]$. This can be proven by working out the Poincaré–Hodge polynomial in the following way: Put $A=(1+xy^2)(1+x^2y)$ and $B=xy(1+x)(1+y)$ then

$$P_{xy}N_C(2,\mathcal{L})=\frac{A^g-B^g}{A-B}=A^{g-1}+A^{g-2}B+\cdots+B^{g-1},$$

and as the only monomials in A and B are x^iy^j with $i=j$, $i=2j$ or $2i=j$ one can now see that the level of H^i is less than or equal to $[i/3]$. Pictorially, the nonzero Hodge numbers of $N_C(2,\mathcal{L})$ lie in the shadowed rhombus in Figure 1.

2.3.2. *Intermediate Jacobians*

Let X be a smooth complex projective variety. Hodge theory shows that the natural map $H^1(X,\mathbb{Z})\to H^1(X,\mathbb{C})\to H^{1,0}(X,\mathbb{C})$ is an injection of $H^1(X,\mathbb{Z})$ as a lattice in $H^{1,0}(X,\mathbb{C})$. The quotient, $J(X)$, is thus a complex torus called the Albanese or Jacobian variety. By using Hodge theory one can show that this complex torus is actually an Abelian variety.

The higher intermediate Jacobians, $J^i(X)$, are analogues of this using higher cohomology groups H^{2i-1}.

Weil defines $J^i(X)$ to be the quotient

$$\frac{H^{2i-1}(X,\mathbb{C})}{H^{0,2i-1}\oplus H^{2,2i-3}\oplus\cdots\oplus H^{2i-2,1}+H^{2i-1}(X,\mathbb{Z})}$$

and shows this to be an Abelian variety. However, as X varies in moduli $J^i(X)$ do not.

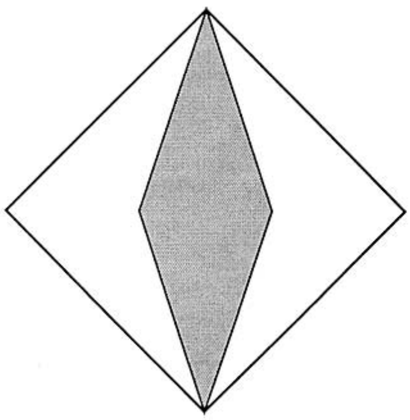

Figure 1. The Hodge diamond of $N_C(2,\mathcal{L})$.

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Griffiths defines $J^i(X)$ to be the quotient

$$\frac{H^{2i-1}(X,\mathbb{C})}{F^iH^{2i-1}(X,\mathbb{C})+H^{2i-1}(X,\mathbb{Z})},$$

where F is the Hodge filtration. This is not in general an Abelian variety but it varies in moduli with X.

An ℓ-adic analogue of Griffiths definition is inspired by the following result of Carlson [5]: Let $\mathbf{MHS}_{\mathbb{Z}}$ be the category of integral mixed Hodge structures, then there is a natural isomorphism of Abelian groups

$$J^i(X)\simeq \mathrm{Ext}^1_{\mathbf{MHS}_{\mathbb{Z}}}(\mathbb{Z}(-i),H^{2i-1}(X,\mathbb{Z})).$$

This motivates the following definition of intermediate Jacobian of a variety X defined over a field k,

$$J^i(X)=\mathrm{Ext}^1_{\mathbf{Rep}_{\mathbb{Z}_\ell}\mathrm{Gal}(\bar{k}|k)}(\mathbb{Z}_\ell(-i),H^{2i-1}(X\otimes\bar{k},\mathbb{Z}_\ell)),$$

where $G_k=\mathrm{Gal}(k|\bar{k})$ is the Galois group of k. Define the ℓ-adic intermediate Jacobians *up to isogeny* to be the same groups replacing $\mathbb{Z}_\ell$ by $\mathbb{Q}_\ell$.

In the following corollary, we assume either (i) k is finitely generated over $\mathbb{Q}$ and a prime number ℓ has been chosen, or (ii) an embedding of k in $\mathbb{C}$ has been chosen together with a choice between Griffiths's and Weil's definition.

COROLLARY 2.10. *Let $\mathcal{L}$ be a line bundle of degree 1. The* ith *intermediate Jacobian of $N_C(2,\mathcal{L})$ is isogenous to*

$$\prod_{k=1}^{[\frac{g+1}{2}]}(J^k\mathrm{Jac}C)^{c_{i,k,g}},\tag{6}$$

where

$$c_{i,k,g}=\mathrm{coeff}_{t^{i-3k+1}}(1+t+t^2+\cdots+t^{g-2k})(1+t^2+t^4+\cdots+t^{2g-4k}).$$

Proof. Taking the piece of weight $2i-1$ in (5), we see that for $k=\mathbb{C}$ the rational pure Hodge structure $H^{2i-1}(X,\mathbb{Q})$ is isomorphic to

$$\bigoplus_{j=1}^{g}\wedge^jH^1(C,\mathbb{Q})(-j)\otimes\left(\substack{\text{weight } 2i-1-3j \text{ part of}\\ (\mathbb{Q}\oplus\mathbb{Q}(-1)\oplus\cdots\oplus\mathbb{Q}(-g+j+1))\otimes(\mathbb{Q}\oplus\mathbb{Q}(-2)\oplus\cdots\oplus\mathbb{Q}(-2g+2j+2))}\right).$$

Given that the Lefschetz–Hodge structure, $\mathbb{Q}(-1)$, has weight 2, in the sum above the only nonzero summands arise when $2i-1-3j$ is even, so we can assume that j is odd. Put $j=2k-1$, then k runs from 1 to $[(g+1)/2]$. As $2i-1-3j=$

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$2i-6k+2$, the Hodge structure $H^{2i-1}(N_C(2,\mathcal{L}),\mathbb{Q})$ is isomorphic to

$$\bigoplus_{k=1}^{[\frac{g+1}{2}]} \wedge^{2k-1} H^1(C,\mathbb{Q})(-2k+1)\otimes\mathbb{Q}(-i+3k-1)^{c_{i,k,g}}.$$

Given Hodge structure M and N and integers k, i we have natural isomorphisms $J^i(M\oplus N)\simeq J^i(M)\times J^i(N)$ and $J^i(N(k))\simeq J^{i+k}(M)$. By using these properties we see that the ith intermediate Jacobian of $N_C(2,\mathcal{L})$ is isomorphic to

$$\prod_{k=1}^{[\frac{g+1}{2}]} J^i(\wedge^{2k-1}H^1(C,\mathbb{Q})(k-i)^{c_{i,k,g}})$$
$$\simeq \prod_{k=1}^{[\frac{g+1}{2}]} J^k(\wedge^{2k-1}H^1(C,\mathbb{Q}))^{c_{i,k,g}} \simeq \prod_{k=1}^{[\frac{g+1}{2}]} J^k(\mathrm{Jac}(C))^{c_{i,k,g}}$$

as claimed.

The analogous expression for ℓ-adic cohomology is obtained in the same way. The result follows. □

2.3.3. *The Siegel Formula*

In the case k is the finite field with q elements, $\mathbb{F}_q$, Theorem 2.6 allows us to recover the following formula due to G. Harder which is essentially equivalent to the fact that the Tamagawa number of SL_2 over the function field of C is one, $\tau_{\mathrm{SL}_2}=1$.

COROLLARY 2.11 ([14, §2]). *The number of $\mathbb{F}_q$-rational points of $N_C(2,\mathcal{L})$ is*

$$\#N_C(2,\mathcal{L})(\mathbb{F}_q)=q^{3g-3}\zeta_C(2)-\frac{q^g}{(1-q)(1-q^2)}\#\mathrm{Jac}C(\mathbb{F}_q).$$

Proof. Let ℓ be a prime number not dividing q. Call $\omega_1,\dots,\omega_{2g}$ the eigenvalues of the geometric Frobenius morphism acting on $H^1(C\otimes_{\mathbb{F}_q}\overline{\mathbb{F}}_q,\mathbb{Q}_\ell)$ and set $P_1(t)=\prod_{i=1}^{2g}(1-\omega_i t)$.

By Subsection 1.2.1 $\#N_C(2,\mathcal{L})(\mathbb{F}_q)=v_q\chi N_C(2,\mathcal{L})$. However, our expression for $\chi N_C(2,\mathcal{L})$ lies in $\mathcal{K}$ and the morphism $v_q\colon K_0\mathcal{M}_k^+\to\mathbb{Q}_\ell$ does not extend to this ring. We shall use the ring morphism $v_q^t\colon\mathcal{K}\to\mathbb{Q}_\ell[[t]]$ defined in Subsection 1.2.3. Then $v_q^t\chi N_C(2,\mathcal{L})$ is a polynomial in t and its value at $t=1$ gives $\#N_C(2,\mathcal{L})(\mathbb{F}_q)=v_q\chi N_C(2,\mathcal{L})$.

Thus we need to evaluate

$$v_q^t\left(\frac{(\mathbb{1}+\mathbb{L})^{h^1C}-(\mathbb{1}+\mathbb{1})^{h^1C}(-g)}{(\mathbb{1}-\mathbb{L})(\mathbb{1}-\mathbb{L}^2)}\right).$$

Given that v_q^t is a ring morphism it suffices to evaluate

$$v_q^t\mathbb{L},\quad v_q^t(\mathbb{1}+\mathbb{L})^{h^1C}\quad\text{and}\quad v_q^t(\mathbb{1}+\mathbb{1})^{h^1C}.$$

Let n be a nonnegative integer, then

$$v_q^t(\mathbb{L}) = \mathrm{Tr}\Big(\mathrm{Fr}_q\Big|_{\mathbb{Q}_\ell(-1)}\Big)t^2 = qt^2,$$

$$\begin{aligned} v_q^t(\mathbb{1}+\mathbb{L}^n)^{h^1C} &= \sum_i \mathrm{Tr}\Big(\mathrm{Fr}_q\Big|_{\wedge^i H^1(\overline{C},\mathbb{Q}_\ell)(-ni)}\Big)t^{i+2ni} \\ &= \sum_i \sum_{j_1<\cdots<j_i} \omega_{j_1}\ldots\omega_{j_i} q^{ni}t^{(2n+1)i} = P_1(q^nt^{2n+1}). \end{aligned}$$

Therefore,

$$v_q^t\chi N_C(2,\mathcal{L}) = \frac{P_1(qt^3)-P_1(t)q^gt^{2g}}{(1-qt^2)(1-q^2t^4)}.$$

The substitution $t=1$ gives

$$\#N_C(2,\mathcal{L}) = \frac{P_1(q)-P_1(1)q^g}{(1-q)(1-q^2)}.$$

Note that by the trace formula $P_1(1)=\#\mathrm{Jac}C(\mathbb{F}_q)$. The functional equation for the zeta function $\zeta_C(s)$ gives

$$q^{3g-3}\zeta_C(2)=\zeta_C(-1)\quad\text{and}\quad \zeta_C(-1)=\frac{P_1(q)}{(1-q)(1-q^2)}.$$

The result follows. □

Another consequence of the previous is the following conjectural statement in [14].

PROPOSITION 2.12. *Let k be a number field or a finite field and $\bar{k}$ an algebraic closure of k. The action of* $\mathrm{Gal}(\bar{k}|k)$ *on* $H_\ell^*(N_C(2,\mathcal{L}))$ *is semisimple.*

Proof. Theorem 2.5 shows that $h(N_C(2,\mathcal{L}))$ lies in the tensor category generated by $h^1(C)$ and $\mathbb{L}$. This implies that the $\mathrm{Gal}(\bar{k}|k)$-module $H_\ell^*(N_C(2,\mathcal{L}))$ is a subobject of a sum of $\mathrm{Gal}(\bar{k}|k)$-module of the type $H_\ell^1(C)^{\otimes n}\mathbb{Q}_\ell(-1)^{\otimes m}$ with $n, m\in\mathbb{N}$.

A well known result due to Faltings [12] for number fields and to Tate for finite fields states that $H_\ell^1(C)$ is a semisimple $\mathrm{Gal}(\bar{k}|k)$-module. The result follows. □

3. The Singular Case

In this section we study the motive of the moduli space $N_C(2,\mathcal{O}_C)$ and that of its canonical smooth model, M.

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We require two restrictions for this to work. The first is that our curve has a k-rational point; this is needed to construct the Hecke correspondence and to guarantee the existence of a universal bundle. The second is that the characteristic of the field is zero. The reason for this is that the calculations involve motivic Poincaré polynomials of some varieties which are not smooth and projective (see Subsection 1.2.2); this is only currently available if the characteristic is zero. In the case of a finite field we can replace the motivic Poincaré polynomial by the pure Poincaré polynomials defined in [22].

Therefore in this section we fix a point $x \in C(k)$ and assume that k is either a field of characteristic zero or a finite field.

If the genus of the curve C is 2 then $N_C(2, \mathcal{O}_C) \simeq \mathbb{P}^3$ [20, §6] and the motive of projective space is well known [18, §6]. Therefore, we take g to be greater than 2.

3.1. PRELIMINARIES: THE HECKE CORRESPONDENCE

We shall now describe a construction due to Narasimhan and Ramanan [21], that relates the moduli spaces $N_C(2, \mathcal{O}_C(x))$ and $N_C(2, \mathcal{O}_C)$. We describe it by using the notion of parabolic bundle over a curve ([19], see also Part 3 in [27]).

3.1.1. *Rank Two Parabolic Bundles*

DEFINITION 3.1 ([27]). A rank two parabolic bundle with nontrivial parabolic structure concentrated on x consists on a pair (E, ℓ) where E is a rank two vector bundle over C and $\ell \subset E_x$ is a subspace of dimension one.

DEFINITION 3.2. Fix parabolic weights $1 > \alpha_2 > \alpha_1 \geqslant 0$. Then a parabolic bundle (E, ℓ) is called stable if for every line bundle $L \subset E$

$$\deg L < \mu E + \frac{\alpha_1 - \alpha_2}{2}, \quad \text{if } L_x = \ell,$$

$$\deg L < \mu E + \frac{\alpha_2 - \alpha_1}{2}, \quad \text{if } L_x \neq \ell.$$

Parabolic semistability is defined as above but using $\leqslant$ instead of $<$.

Remark 3.3. It is not hard to see that if the parabolic weights are small enough, then:

(1) Parabolic stability is independent of the (α_1, α_2).
(2) (E, ℓ) is parabolic stable $\Leftrightarrow$ (E, ℓ) is parabolic semistable.
(3) (E, ℓ) is parabolic stable $\Rightarrow$ E is a semistable vector bundle.
(4) E is a stable vector bundle $\Rightarrow$ (E, ℓ) is parabolic semistable for any ℓ.

DEFINITION 3.4. A family of rank two parabolic bundles with parabolic structure concentrated on x parametrized by a scheme S is a vector bundle $\mathcal{E}$ over $S \times C$ of rank 2 together with a section, σ, of the projective bundle $\mathbb{P}\mathcal{E}_x^\vee \to S$. Two families $(\mathcal{E}, \sigma)$, $(\mathcal{E}', \sigma')$ parametrized by S are said to be isomorphic if there is an isomorphism $f\colon \mathcal{E} \to \mathcal{E}'$ such that $f_x \circ \sigma = \sigma'$.

Remark 3.5. Note that for each point $s \in S$ we get a vector bundle $\mathcal{E}_s$ over C, the section of $\mathbb{P}\mathcal{E}_{s,x}^\vee$ gives a line $\ell_s \subset \mathcal{E}_{s,x}$, that is a parabolic bundle. A family of parabolic vector bundles is called stable if for each $s \in S$ the parabolic bundle $(\mathcal{E}_s, \ell_s)$ is parabolic stable.

DEFINITION 3.6. Let α_1, α_2 be small enough weights. Define $\mathcal{N}_x(2, \mathcal{L})$ to be the functor $\mathbf{Sch}_k \to$ Sets that takes S to

$$\frac{\left\{\begin{array}{c}\text{Isomorphism classes of families of rank 2 parabolic}\\ \text{stable vector bundles with } \det = \mathcal{L} \text{ parametrized by } S\end{array}\right\}}{\operatorname{Pic} S}.$$

The following theorem seems to be well known. Lacking a precise reference we give a proof that fits nicely in our discussion of the Hecke correspondence.

THEOREM 3.7 (cf. [27] Part 3, Théorème 32). (1) *The functor* $\mathcal{N}_x(2, \mathcal{L})$ *is representable by a smooth projective variety* $N_x(2, \mathcal{L})$.

(2) *There is a canonical isomorphism of functors* $\mathcal{N}_x(2, \mathcal{L}) \to \mathcal{N}_x(2, \mathcal{L}(-x))$.

Proof. (1) In view of 3.7.2 it is enough to prove it in the case deg $\mathcal{L}$ is odd. Let $\mathcal{U}$ be a universal bundle over $N_C(2, \mathcal{L}) \times C$ and $\imath\colon \{x\} \hookrightarrow C$ the natural inclusion. Put $\mathcal{U}_x = (1_N \times \imath)^* \mathcal{U}$ and let $N_x(2, \mathcal{L})$ be the projective bundle over $N_C(2, \mathcal{L})$ associated to $\mathcal{U}_x^\vee$ (which does not depend on the particular choice of universal bundle $\mathcal{U}$), we claim this represents the functor $\mathcal{N}_x(2, \mathcal{L})$.

Let $S \in Ob(\mathbf{Sch}_k)$ and $(\mathcal{E}, \sigma) \in \mathcal{N}_x(2, \mathcal{L})(S)$. As deg $\mathcal{L}$ is odd, $\mathcal{E}$ is a family of stable vector bundles and we obtain $f\colon S \to N_C(2, \mathcal{L})$ such that $(f \times 1_C)^* \mathcal{U} \simeq \mathcal{E} \otimes p_S^* \mathcal{M}_S$ where $\mathcal{M}_S \in \operatorname{Pic} S$. The section of $\mathbb{P}\mathcal{E}_x^\vee \to S$ gives a section of $\mathbb{P} f^* \mathcal{U}_x^\vee \to S$, i.e. a lift of f to $\mathbb{P}\mathcal{U}_x^\vee = N_x(2, \mathcal{L})$. Thus we have a map

$$\mathcal{N}_x(2, \mathcal{L})(S) \xrightarrow{\Phi(S)} N_x(2, \mathcal{L})(S)$$

which is functorial in S, that is we have a morphism of functors

$$\mathcal{N}_x(2, \mathcal{L}) \xrightarrow{\Phi} N_x(2, \mathcal{L})$$

(here we identify the variety $N_x(2, \mathcal{L})$ with the functor it represents).

$\Phi(S)$ is injective for we can clearly recover $(\mathcal{E}, \sigma)$ modulo Pic$S$ from $S \to N_x(2, \mathcal{L})$.

$\Phi(S)$ is also surjective for by Remark 3.3.(4) any family of parabolic bundles with underlying stable vector bundle is parabolic stable.

This proves (1).

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(2) Let $S \in Ob(\mathbf{Sch}_k)$ and let $(\mathcal{E}, \sigma) \in \mathcal{N}_x(2, \mathcal{L})(S)$. The section σ is equivalent to giving a surjection of $\mathcal{E}_x^\vee$ to a line bundle on S, $\mathcal{M}_x^\vee$ ([16, II.7.12]). By taking duals we get an exact sequence of vector bundles over S

$$0 \to \mathcal{M}_x \to \mathcal{E}_x \to \mathcal{T}_x \to 0.$$

Define $\mathcal{E}' = \ker(\mathcal{E} \to \mathcal{E}_x \to \mathcal{T}_x)$ then we have an exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{T}_x \to 0,$$

in which $\mathcal{E}'$ is locally free for $\mathcal{T}_x$ has a projective resolution of length 2. Tensoring the previous exact sequence of $\mathcal{O}_x$, we get

$$0 \to \mathcal{M}'_x \to \mathcal{E}'_x \to \mathcal{E}_x \to \mathcal{T}_x \to 0$$

where $\mathcal{M}'_x = \mathrm{Tor}_1(\mathcal{O}_x, \mathcal{T}_x)$.

The surjection $\mathcal{E}'^\vee_x \to \mathcal{M}'^\vee_x$ defines a section of $\mathbb{P}\mathcal{E}'^\vee_x$ giving a family of parabolic bundles $(\mathcal{E}', \sigma')$ of rank two and determinant $\mathcal{L}(-x)$.

We shall see that $(\mathcal{E}', \sigma')$ is a family of parabolic stable bundles. It is clearly enough to check this for S the spectrum of a field. Then we have a parabolic stable bundle (E, ℓ) and if $t = E_x/\ell$ the above construction defines a parabolic vector bundle (E', ℓ') by the exact sequences

$$0 \to E' \to E \to t \to 0,$$

$$0 \to \ell' \to E'_x \to E_x \to t \to 0.$$

We want to see that (E', ℓ') is parabolic stable. For this let $L \subset E'$ be a line subbundle of E'.

If $L_x = \ell'$ then in the inclusion $L \subset E$ $L_x \to E_x$ is zero, therefore $L \subset E(-x)$ and as E is a semistable vector bundle

$$\mu L \leqslant \mu E(-x) = \mu E - 1 = \mu E' - \tfrac{1}{2},$$

hence $\mu L \leqslant \mu E' + (\alpha_1 - \alpha_2)/2$.

If $L_x \neq \ell'$ then the map $L_x \to E'_x/\ell' = \ell$ is nonzero, therefore in $L \subset E$ we have $L_x = \ell$. As (E, ℓ) is parabolic stable

$$\mu L < \mu E + \frac{\alpha_1 - \alpha_2}{2} = \mu E' + \frac{1 + \alpha_1 - \alpha_2}{2},$$

hence $\mu L < \mu E' + (\alpha_2 - \alpha_1)/2$.

This way we get a morphism of functors

$$E\colon \mathcal{N}_x(2, \mathcal{L}) \to \mathcal{N}_x(2, \mathcal{L}(-x)).$$

We claim that

$$\mathcal{N}_x(2, \mathcal{L}(-x)) \xrightarrow{E} \mathcal{N}_x(2, \mathcal{L}(-2x)) \xrightarrow{\otimes \mathcal{O}_C(x)} \mathcal{N}_x(2, \mathcal{L})$$

is its inverse.

To prove this let $S \in Ob(\mathbf{Sch}_k)$ and $(\mathcal{E}, \sigma) \in \mathcal{N}_x(2, \mathcal{L})(S)$. Let $(\mathcal{E}', \sigma') = E(\mathcal{E}, \sigma)$ and $(\mathcal{E}'', \sigma'') = E(\mathcal{E}', \sigma')$, we want to show that $(\mathcal{E}'', \sigma'')$ is naturally isomorphic to $(\mathcal{E}, \sigma) \otimes \mathcal{O}_C(-x)$.

The isomorphism $\mathcal{E}'' \simeq \mathcal{E}(-x)$ follows by applying the snake lemma to the morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}' & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{T}_x & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{M}_x = \mathcal{T}'_x & \longrightarrow & \mathcal{E}_x & \longrightarrow & \mathcal{T}_x & \longrightarrow & 0 \end{array}$$

Next recall that $\mathcal{M}''_x = Tor_1(\mathcal{O}_x, \mathcal{T}'_x) = Tor_1(\mathcal{O}_x, \mathcal{M}_x) \simeq \mathcal{M}_x(-x)$.

This concludes the proof. □

3.1.2. *The Hecke Correspondence*

Note that as $N_x(2, \mathcal{L})$ represents a functor, it parametrizes a universal family of parabolic vector bundles over C. By the universal property of $N(2, \mathcal{L})$ and Remark 3.3.3 we get a morphism $N_x(2, \mathcal{L}) \to N(2, \mathcal{L})$. By the isomorphism in 3.7.2 there is a diagram.

$$\begin{array}{ccccc} & & N_x(2, \mathcal{O})_C(x)) \simeq N_x(2, \mathcal{O}_C) & & \\ & \swarrow^{p_1} & & \searrow^{p_0} & \\ N_C(2, \mathcal{O}_C(x)) & & & & N_C(2, \mathcal{O}_C). \end{array} \tag{7}$$

This is the Hecke correspondence.

In the proof of Theorem 3.7 we have seen that p_1 is the projective bundle associated to a vector bundle of rank two. Our next task is to analyze the morphism p_0. Recall that $N_C(2, \mathcal{O}_C)$ is a singular variety so it is natural to consider the stratification given by the singular loci. The singular locus of $N_C(2, \mathcal{O}_C)$ is isomorphic to the Kummer variety KC. The Kummer variety is singular, its singular locus, K_0C, is isomorphic to the 2-torsion of the Jacobian, $\mathrm{Jac}_2 C$, via the quotient map $\mathrm{Jac}C \to KC$. In next proposition will describe the structure of the morphism p_0 over each of these strata.

First we set some notations. Fix $x \in C(k)$. As $C(k) \neq \emptyset$ there is a Poincaré bundle, $\mathcal{P}$, over $\mathrm{Jac}C \times C$. If we normalize it so that $\mathcal{P}_{\mathrm{Jac}C \times \{x\}}$ is trivial, $\mathcal{P}$ is then uniquely defined. The quotient map $\mathrm{Jac}C \to KC$ restricts to a double cover $\mathrm{Jac}C - \mathrm{Jac}_2 C \to KC - K_s C$. The action of $\mathbb{Z}/2\mathbb{Z}$ on $(\mathrm{Jac}C - \mathrm{Jac}_2 C) \times C$ lifts to the vector bundle $\mathcal{P} \oplus \mathcal{P}^{-1}$ and by descent we get a vector bundle on $(KC - K_0 C) \times C$ which we still denote by $\mathcal{P} \oplus \mathcal{P}^{-1}$ (note however that $\mathcal{P}$ is not defined over $(KC - K_0 C) \times C$). The projections of a product, $X \times Y$, on its factors will be written p_X and p_Y.

PROPOSITION 3.8. (1) *The morphism p_0 restricted to $N_C(2, \mathcal{O}_C)^s$ is a conic bundle (cf. [2] Proposition 7).*

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(2) *There is a section of* p_0 *over* $KC - K_0C$, σ, *a locally trivial* $\mathbb{P}^{g-2}$-*bundle* $\pi\colon R \to \mathrm{Jac}C - \mathrm{Jac}_2C$ *and a locally trivial* $\mathbb{P}^1$-*bundle over* R, $Q \to R$, *with a section* σ' *such that there exists an isomorphism,* φ, *making the following diagram commute*

$$\begin{array}{ccc}
Q - \sigma' R & \xrightarrow[\simeq]{\varphi} & p_0^{-1}(KC - K_0C) - \sigma(KC - K_0C) \\
\downarrow & & \\
R & & \downarrow p_0 \\
\pi \downarrow & & \\
\mathrm{Jac}C - \mathrm{Jac}_2C & \longrightarrow & KC - K_0C
\end{array}$$

(3) *The reduced scheme associated to* $p_0^{-1}K_0C$ *is isomorphic to the projective bundle over* K_0C, $\mathbb{P}R^1p_{K_0C*}\mathcal{O}_{C\times K_0C}$.

Proof. (1) Recall that $N_C(2, \mathcal{O}_C)^s$ is the GIT quotient of a subscheme R^s of a *Quot*-scheme by a GL_Nk action (for certain positive integer N). Over $R^s \times C$ there is a universal bundle, $\mathcal{U}$, however the stabilizers of points of R^s in GL_Nk, namely k^*, do not act trivially on the fibres of this vector bundle, therefore the universal bundle does not descend to $C \times N_C(2, \mathcal{O}_C)^s$.

However the stabilizers do act trivially on the projective bundle $\mathbb{P}\mathcal{U}$, therefore it descends to $C \times N_C(2, \mathcal{O}_C)^s$ to a projective bundle. Similarly the projective bundle $\mathbb{P}\mathcal{U}_x^\vee$ over R^s descends to a projective bundle over $N_C(2, \mathcal{O}_C)^s$ which we denote by P_x.

Let p denote the projection $\mathbb{P}\mathcal{U}_x^\vee \to R^s$. If we pull back the universal family by $p \times \mathrm{Id}_C$ we get a family of vector bundles over C parametrized by $\mathbb{P}\mathcal{U}_x^\vee$. There is an natural epimorphism $p^*\mathcal{U}_x^\vee \to \mathcal{O}(1)$ on $\mathbb{P}\mathcal{U}_x^\vee$, this data defines a morphism from $\mathbb{P}\mathcal{U}_x^\vee$ to $N_x(2, \mathcal{O}_C)$ which can be seen to be GL_Nk-equivariant thus yielding a morphism

$$P_x \to p_0^{-1}N_C(2, \mathcal{O}_C)^s \tag{8}$$

that makes the following diagram commute

$$\begin{array}{ccc}
P_x & \longrightarrow & p_0^{-1}N_C(2, \mathcal{O}_C)^s \\
& \searrow \quad \swarrow & \\
& N_C(2, \mathcal{O}_C)^s. &
\end{array}$$

Put $N_x(2, \mathcal{O}_C)^s = p_0^{-1}N_C(2, \mathcal{O}_C)^s$, to see that (8) is an isomorphism we shall construct its inverse. To this effect cover $N_x(2, \mathcal{O}_C)^s$ by affine open sets $\{U_i\}_i$. By the universal property of $N_x(2, \mathcal{O}_C)^s$ we have bundles $\mathcal{E}_i$ over $U_i \times C$ together with sections of $\mathbb{P}\mathcal{E}_{i,x}^\vee$ over U_i. The first data defines (assume, as usual, that the degree is big enough) morphisms $U_i \to R^s$, the second a lift of this morphism to $\mathbb{P}\mathcal{U}_x^\vee$. This way we get for each i a morphism $U_i \to \mathbb{P}\mathcal{U}_x^\vee$. However, these are not uniquely defined because they depend on a trivialization of the trivial bundle $p_{U_i*}\mathcal{E}_i \simeq \mathcal{O}_{U_i}^N$, unique up to an element of $\mathrm{GL}_N k$; in particular they may not patch to give a morphism $N_x(2,\mathcal{O})^s \to \mathbb{P}\mathcal{U}_x^\vee$. However, if we compose with the quotient map $\mathbb{P}\mathcal{U}_x^\vee \to P_x$ we get uniquely defined $U_i \to P_x$ which patch together to give the inverse of (8).

(2) We first define the section $\sigma\colon KC - K_0C \to p_0^{-1}(KC - K_0C) \subset N_x(2,\mathcal{O}_C)$. By the universal property of $N_x(2,\mathcal{O}_C)$ this is equivalent to constructing a family of stable parabolic bundles parametrized by $KC - K_0C$. The family of underlying vector bundles will be $\mathcal{P}\oplus\mathcal{P}^{-1} = \mathcal{F}$ defined in the notations. Let I_x be the Poincaré isomorphism $I_x\colon \mathcal{O}_{\mathrm{Jac}C} \to \mathcal{P}_x$. Then we can construct a section of $\mathcal{P}_x\oplus\mathcal{P}_x^{-1}$ over $\mathrm{Jac}C$ which is invariant under the involution described at the beginning of Subsection 3.1.2, namely $I_x \oplus I_x^{-1}$. This yields a section of $\mathbb{P}\mathcal{F}_x^\vee = \mathbb{P}(\mathcal{P}_x\oplus\mathcal{P}_x^{-1})$ over $KC - K_0C$ different from the ones corresponding to the lines $\mathcal{P}_x \subset \mathcal{F}_x$ and $\mathcal{P}_x^{-1} \subset \mathcal{F}_x$. By Lemma 3.9, this gives a family of stable parabolic bundles independent of the section, i.e. a section $\sigma\colon KC - K_0C \to N_x(2,\mathcal{O}_C)$.

Next we want to analyze $p_0^{-1}(KC - K_0C) - \sigma(KC - K_0C)$, the points of this variety correspond to parabolic vector bundles in which the underlying vector bundle is a nontrivial extension of $\mathcal{L}$ by $\mathcal{L}^{-1}$ where $\mathcal{L}$ is a line bundle of degree zero with $\mathcal{L}\neq\mathcal{L}^{-1}$. These extensions are parametrized by the projective bundle over $\mathrm{Jac}C - \mathrm{Jac}_2C$ associated to the rank $g-1$ vector bundle $(R^1 p_{\mathrm{Jac}C-\mathrm{Jac}_2C*}\mathcal{P}^2)^\vee$. Call R this projective bundle and π the natural projection $R \to \mathrm{Jac}C - \mathrm{Jac}_2C$.

Over $R\times C$ we have a universal extension

$$0 \to (\pi\times 1_C)^*\mathcal{P}^{-1} \to \mathcal{E} \to (\pi\times 1_C)^*\mathcal{P}\otimes p_R^*\mathcal{O}_R(-1) \to 0.$$

If we pull back by i_x we get an exact sequence of vector bundles over R.

$$0 \to \mathcal{O}_R \to \mathcal{E}_x \to \mathcal{O}_R(-1) \to 0. \tag{9}$$

Let Q be the projective bundle over R associated to $\mathcal{E}_x^\vee$. Then Q parametrizes a family of parabolic bundles over C. Lemma 3.10 shows that the stable locus, Q^s, is the complementary of the section of $Q\to R$ defined by the surjection $\mathcal{E}_x^\vee \to \mathcal{O}_R$ obtained by dualizing (9).

The universal property of $N_x(2,\mathcal{O}_C)$ yields a morphism

$$Q^s \to p_0^{-1}(KC - K_0C) - \sigma(KC - K_0C)$$

which is an isomorphism again by Lemma 3.10. This concludes the proof of (2).

(3) See Lemma 7.4.(ii) in [21]. □

LEMMA 3.9 (cf. [2] proof of Proposition 7, case 2). *Let $\mathcal{L}$ be a line bundle over C of degree zero with $\mathcal{L}\neq\mathcal{L}^{-1}$. Then*

(1) *A parabolic bundle $(\mathcal{L}\oplus\mathcal{L}^{-1},\ell)$ is parabolic stable if and only if $\ell\neq\mathcal{L}_x$, $\mathcal{L}_x^{-1}$.*
(2) *Any two stable parabolic bundles with $\mathcal{L}\oplus\mathcal{L}^{-1}$ as underlying vector bundle are isomorphic.*

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Proof. (1) If $\ell = \mathcal{L}_x$ then the subbundle $\mathcal{L} \subset \mathcal{L} \oplus \mathcal{L}^{-1}$ proves that $(\mathcal{L} \oplus \mathcal{L}^{-1}, \ell)$ is not stable for

$$\mu\mathcal{L} = 0 \quad \text{and} \quad \mu(\mathcal{L} \oplus \mathcal{L}^{-1}) + \frac{\alpha_1 - \alpha_2}{2} < 0.$$

The case $\ell = \mathcal{L}_x^{-1}$ is treated in the same way.

Conversely, assume $\ell \neq \mathcal{L}_x, \mathcal{L}_x^{-1}$, take a subbundle $\mathcal{M} \subset \mathcal{L} \oplus \mathcal{L}^{-1}$. If $\mathcal{M} \to \mathcal{L} \oplus \mathcal{L}^{-1} \to \mathcal{L}$ is not zero then either $\mu\mathcal{M} < 0$ in which case we are done or $\mathcal{M} \simeq \mathcal{L}$ and $\mathcal{M} \to \mathcal{L} \oplus \mathcal{L}^{-1} \to \mathcal{L}^{-1}$ will be zero, then $\mathcal{M} = \mathcal{L}$ so $\ell \neq \mathcal{M}_x$ and the stability condition is trivially verified.

(2) This is a consequence of the fact that the action of $\mathrm{Aut}(\mathcal{L} \oplus \mathcal{L}^{-1}) = k^* \times k^*$ on $\mathbb{P}(\mathcal{L}_x \oplus \mathcal{L}_x^{-1})$ is transitive. □

LEMMA 3.10 (cf. [2] proof of Proposition 7, case 1). *Let (E, ℓ) be a parabolic bundle in which E is a nontrivial extension of $\mathcal{L}$ by $\mathcal{L}^{-1}$ where $\mathcal{L}$ is a line bundle of degree zero with $\mathcal{L} \neq \mathcal{L}^{-1}$. Then (E, ℓ) is parabolic stable if and only if $\ell \neq \mathcal{L}_x^{-1}$.*

Proof. If $\ell = \mathcal{L}_x^{-1}$ then the subbundle $\mathcal{L}^{-1} \subset E$ shows that (E, ℓ) is not stable for $\mu\mathcal{L}^{-1} = 0$ and $\mu E + ((\alpha_1 - \alpha_2)/2) < 0$.

Conversely, assume $\ell \neq \mathcal{L}_x^{-1}$. Let $\mathcal{M} \subset E$ be a line subbundle, if $\mathcal{M} \to E \to \mathcal{L}$ is not zero then $\mu\mathcal{M} < 0$ in which case we are done or $\mathcal{M} \simeq \mathcal{L}$ and we have a section of the extension which cannot be the case. If $\mathcal{M} \to E \to \mathcal{L}$ is zero then $\mathcal{M} \subset \mathcal{L}^{-1}$ and $\mu\mathcal{M} \leqslant \mu\mathcal{L}^{-1} = 0$ which is enough to conclude stability thanks to the condition $\ell \neq \mathcal{L}_x^{-1}$. □

3.2. THE MOTIVE OF $N_C(2, \mathcal{O}_C)$

We now use the description of the Hecke correspondence in the previous section to compute the motivic Poincaré polynomial of the singular moduli space $N_C(2, \mathcal{O}_C)$ in terms of the motive of $N_C(2, \mathcal{O}_C(x))$ (see Theorem 2.6), the motive of the Jacobian variety, Jac$C$ and the motive of the Kummer variety, KC.

We need to understand the motivic Poincaré polynomial of a conic bundle. It happens that it behaves as in the locally trivial case.

PROPOSITION 3.11. *Let $C \to X$ be a conic bundle. Then $\chi_c(C) = \chi_c(X).\ (\mathbb{1} + \mathbb{L})$.*

Proof. For a proof of this see [6]. Here we shall content ourselves with proving this fact for any realization of the motive, that is for the image of the motivic Poincaré polynomial via $K_0\mathcal{M}_k^+ \to K_0\mathbf{PHS}_{\mathbb{Q}}$ or $K_0\mathcal{M}_k^+ \to K_0\mathbf{Gr\text{-}Rep}_{\mathbb{Q}_\ell}\mathrm{Gal}(\bar{k}|k)$.

This is then a consequence of the Hirsch theorem that, in our case, yields an isomorphism of mixed Hodge structures $H^*(C, \mathbb{Q}) \simeq H^*(X, \mathbb{Q}) \otimes (\mathbb{Q} \oplus \mathbb{Q}(-1))$. □

THEOREM 3.12. *The motivic Poincaré polynomials of $N_C(2, \mathcal{O}_C)$ and $N_C(2, \mathcal{O}_C)^s$ are*

$$\chi_c N_C(2, \mathcal{O}_C)^s = \chi N_C(2, \mathcal{O}_C(x)) - \frac{\chi \mathrm{Jac}C \cdot \chi\mathbb{P}_k^{g-2} \cdot \mathbb{L} + \chi KC}{\mathbb{1} + \mathbb{L}},$$

$$\chi N_C(2, \mathcal{O}_C) = \chi N_C(2, \mathcal{O}_C(x)) - \frac{\chi \mathrm{Jac} C \cdot \chi \mathbb{P}_k^{g-2} - \chi KC}{\mathbb{1} + \mathbb{L}} \cdot \mathbb{L}\,.$$

Proof. We shall first compute the motivic Poincaré polynomial of $p_0^{-1} N_C(2, \mathcal{O}_C)^s$ with compact supports, $\chi_c p_0^{-1}(2, \mathcal{O}_C)^s$. Using property (E) in 1.2.2 we see that

$$\chi_c p_0^{-1} N_C(2, \mathcal{O}_C)^s = \chi N_x(2, \mathcal{O}_C) - \chi_c p_0^{-1}(KC - K_0 C) - \chi p_0^{-1} K_0 C.$$

The first summand, $\chi N_x(2, \mathcal{O}_C)$ is easy to compute, by Theorem 3.7, $N_x(2, \mathcal{O}_C)$ is isomorphic to $N_x(2, \mathcal{O}_C(x))$ and the proof of the same proposition shows that the former is a projective bundle over $N_C(2, \mathcal{O}_C(x))$ associated to a vector bundle of rank two. Therefore by [18, §7]:

$$\chi N_x(2, \mathcal{O}_C) = (\mathbb{1} + \mathbb{L}) \cdot \chi N_C(2, \mathcal{O}_C(x)).$$

To compute $\chi_x p_0^{-1}(KC - K_0 C)$ we use the geometric description in Proposition 3.8. Property (E) in 1.2.2 together with [18, §7] imply that, in the notation of the mentioned proposition,

$$\begin{aligned}
&\chi_c p_0^{-1}(KC - K_0 C) - \chi_c(KC_K 0 C) \\
&\quad = \chi_c(Q - \sigma' R) = \chi_c Q - \chi_c R \\
&\quad = \chi_c R \cdot \mathbb{L} = \chi_c(\mathrm{Jac} C - \mathrm{Jac}_2 C) \cdot \chi \mathbb{P}_k^{g-2} \cdot \mathbb{L}.
\end{aligned}$$

Again, by Manin's theorem on the structure of the motive of a projective bundle, $\chi p_0^{-1} K_0 C = \chi K_0 C \cdot \chi \mathbb{P}_k^{g-1}$.

In conclusion $\chi_c p_0^{-1} N_C(2, \mathcal{O}_C)^s$ equals

$$\begin{aligned}
&\chi N_C(2, \mathcal{O}) C(x)) \cdot (\mathbb{1} + \mathbb{L}) - \chi_c(\mathrm{Jac} C - \mathrm{Jac}_2 C) \cdot \chi \mathbb{P}_k^{g-2} \cdot \mathbb{L} - \\
&\qquad - \chi K_0 C \cdot \chi \mathbb{P}_k^{g-1} - \chi_c(KC - K_0 C) \\
&\quad = \chi N_C(2, \mathcal{O}_C(x)) \cdot (\mathbb{1} + \mathbb{L}) - \chi \mathrm{Jac} C \cdot \chi \mathbb{P}_k^{g-2} \cdot \mathbb{L} - \chi KC.
\end{aligned}$$

But $p_0\colon p_0^{-1} N_C(2, \mathcal{O}_C)^s \to N_C(2, \mathcal{O}_C)^s$ is a conic bundle, and Proposition 3.11 shows

$$\chi_c N_C(2, \mathcal{O}_C)^s = \chi N_C(2, \mathcal{O}_C(x)) - \frac{\chi \mathrm{Jac} C \cdot \chi \mathbb{P}_k^{g-2} \cdot \mathbb{L} + \chi KC}{\mathbb{1} + \mathbb{L}}$$

The result for $N_C(2, \mathcal{O}_C)$ is a consequence of property (E) in Subsection 1.2.2. □

3.3. THE MOTIVE OF SESHADRI'S SMOOTH MODEL

Next we turn to the study of the motivic Poincaré polynomial of the smooth model of $N_C(2, \mathcal{O}_C)$ constructed by Seshadri in [26]. This consists in a smooth projective variety M together with a birational morphism $\psi\colon M \to N_C(2, \mathcal{O}_C)$. We recall the description of the fibres of ψ.

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(1) Over the stable locus, $N_C(2, \mathcal{O}_C)^s$, ψ is an isomorphism.
(2) The fibre over a point of $KC - K_0C$ is isomorphic to $\mathbb{P}^{g-2} \times \mathbb{P}^{g-2}$.
(3) The fibre over a point of K_0C is the disjoint union of the Grassmannian Gr_3V and a rank $g-2$ vector bundle over Gr_2V where $V = H^1(C, \mathcal{O}_C)$.

The morphism ψ over $KC - K_0C$ is not the product of two projective bundles, in fact it is not locally trivial in the Zariski topology. To compute the motive of M we shall need to study in more detail the morphism.

$$\psi\colon \psi^{-1}(KC - K_0C) \to KC - K_0C.$$

As in [2], we use the notation $Y = \psi^{-1}(KC - K_0C)$.

3.3.1. *The Variety Y*

Recall that $\mathcal{P}$ denotes the Poincaré line bundle on $\mathrm{Jac}C \times C$. Note that on the open set $J = \mathrm{Jac}C - \mathrm{Jac}_2C$ the sheaves $R^1p_{J*}\mathcal{P}^2$ and $R^1p_{J*}\mathcal{P}^{-2}$ are locally free, for $\varepsilon \in \{+, -\}$ define $\mathcal{P}_\varepsilon$ to be the projective bundle over J associated to the vector bundle $R^1p_{J*}\mathcal{P}^{\varepsilon 2}$. The involution, τ, of $\mathrm{Jac}C$ is defined by the family of line bundles $\mathcal{P}^{-1}$, so by definition we have $(\tau \times 1_C)^*\mathcal{P} = \mathcal{P}^{-1}$. Flat base change shows $\tau^*R^1p_{J*}\mathcal{P}^2 = R^1p_{J*}\mathcal{P}^{-2}$. Thus the action of τ interchanges the two projective bundles $\mathcal{P}_+$ and $\mathcal{P}_-$. By the proof of corollary 1 in [2] we see that in fact Y is the quotient of $\mathcal{P}_+ \times_J \mathcal{P}_-$ by this involution. Summarizing:

PROPOSITION 3.13. *The pullback of Y via the quotient map $J = \mathrm{Jac}C - \mathrm{Jac}_2C \to KC - K_0C$ is isomorphic to $\mathcal{P}_+ \times_J \mathcal{P}_-$. The induced action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathcal{P}_+ \times_J \mathcal{P}_-$ interchanges the two factors.*

In the proof of the following proposition we use some facts on isotypical decompositions of motives from [10] and [8]. As there, if a finite group G acts on a scheme X and α is a character of G, $\chi_c(X, \alpha)$ stands for the part of $\chi_c(X)$ on which G acts via the character α (see references above for a precise definition).

PROPOSITION 3.14. *The motivic Poincaré polynomial with compact supports of Y, $\chi_c(Y)$, is*

$$\tfrac{1}{2}\chi_c(\mathrm{Jac}C - \mathrm{Jac}_2C)(\chi\mathbb{P}_k^{g-2})^2 + \left(\chi_c(KC - K_0C) - \tfrac{1}{2}\chi_c(\mathrm{Jac}C - \mathrm{Jac}_2C)\right) \times \frac{1 - \mathbb{L}^{2g-2}}{1 - \mathbb{L}^2}.$$

Proof. Denote by 1 and -1 the two characters of $\mathbb{Z}/2\mathbb{Z}$. Let U be an affine $\mathbb{Z}/2\mathbb{Z}$-invariant subset of J where $\mathcal{P}_+$ and $\mathcal{P}_-$ trivialize. Then $\mathcal{P}_+ \times_J \mathcal{P}_-$ restricted to U is isomorphic, as a $\mathbb{Z}/2\mathbb{Z}$-variety, to the product of U and $\mathbb{P}_k^{g-2} \times \mathbb{P}_k^{g-2}$, where the $\mathbb{Z}/2\mathbb{Z}$ acts on the latter variety by permuting the factors.

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By property (3) in Proposition 1.3.3 in [10], we have that

$$\chi_c\left(\mathcal{P}_+ \times_J \mathcal{P}_-\Big|_U, 1\right)$$
$$= \chi_c(U,1)\cdot\chi_c(\mathbb{P}_k^{g-2}\times\mathbb{P}_k^{g-2},1)+\chi_c(U,-1)\cdot\chi_c(\mathbb{P}_k^{g-2}\times\mathbb{P}_k^{g-2},-1).$$

Using property theorem 1.3.1 in [10] we see, applying the previous, that

$$\chi_c(\mathcal{P}_+ \times_J \mathcal{P}_-, 1)$$
$$= \chi_c(J,1)\cdot\chi_c(\mathbb{P}_k^{g-2}\times\mathbb{P}_k^{g-2},1)+\chi_c(J,-1)\cdot\chi_c(\mathbb{P}_k^{g-2}\times\mathbb{P}_k^{g-2},-1).$$

Next note that $h(\mathbb{P}_k^{g-2}\times\mathbb{P}_k^{g-2}) = \bigoplus_{i,j=0}^{g-2}\mathbb{1}(-i-j)$, it is easy to see, by drawing this on a $(g-1)\times(g-1)$ array, that

$$\chi(\mathbb{P}_k^{g-2}\times\mathbb{P}_k^{g-2},1)=\frac{1}{2}\left((\chi\mathbb{P}_k^{g-2})^2+\frac{1-\mathbb{L}^{2g-2}}{1-\mathbb{L}^2}\right),$$

$$\chi(\mathbb{P}_k^{g-2}\times\mathbb{P}_k^{g-2},-1)=\frac{1}{2}\left((\chi\mathbb{P}_k^{g-2})^2-\frac{1-\mathbb{L}^{2g-2}}{1-\mathbb{L}^2}\right).$$

By Corollary 2.1.19 in [8]

$$\chi_c Y = \chi_c(\mathcal{P}_+ \times_J \mathcal{P}_-, 1) \quad \text{and} \quad \chi_c(KC-K_0C)=\chi_c(J,1).$$

The result follows. □

The previous proposition is in full accordance with the results of Balaji and Seshadri as shown in the following corollary.

COROLLARY 3.15 ([4], Proposition 4.1). *If k is the finite field with q elements, $\mathbb{F}_q$, the number of $\mathbb{F}_q$-points of Y is given by*

$$\tfrac{1}{2}\#(\mathrm{Jac}C-\mathrm{Jac}_2C)(\mathbb{F}_q)(\#\mathbb{P}^{g-2}(\mathbb{F}_q))^2+$$
$$-\left(\tfrac{1}{2}\#\mathrm{Jac}C(\mathbb{F}_q)-\#KC(\mathbb{F}_q)+\tfrac{1}{2}\#K_0C(\mathbb{F}_q)\right)\cdot\#\mathbb{P}^{g-2}(\mathbb{F}_{q^2})$$

Proof. By Lemma 1.2.1, $\#Y(\mathbb{F}_q)$ is the result of applying v_q to $\chi_c(Y)$. The result follows from the formula obtained in Proposition 3.14 and the fact that v_q is a ring morphism. □

The statement of this result in Balaji and Seshadri's work reads

$$\#Y(\mathbb{F}_q)=\#A(\mathbb{F}_q)\cdot\#(\mathbb{P}^{g-2}\times\mathbb{P}^{g-2})(\mathbb{F}_q)+\#B(\mathbb{F}_q)\cdot\#\mathbb{P}^{g-2}(\mathbb{F}_{q^2}).$$

The agreement with the previous corollary is clear from the proof of Proposition 3.6 in [4]. There it is shown that the term $\#A(\mathbb{F}_q)$ (noted $N_q(A)$ therein) is equal to

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$\frac{1}{2}\#(\mathrm{Jac}C - \mathrm{Jac}_2 C)(\mathbb{F}_q)$ whereas $\#B(\mathbb{F}_q)$ equals

$$-(\tfrac{1}{2}\#(\mathrm{Jac}C - \mathrm{Jac}_2 C)(\mathbb{F}_q) - \#KC(\mathbb{F}_q) + \tfrac{1}{2}\#K_0C(\mathbb{F}_q)).$$

3.3.2. *The Motive of $\psi^{-1}K_0C$*

PROPOSITION 3.16. *The motive of $\psi^{-1}K_0C$ is given by*

$$\left(\frac{(1-\mathbb{L}^g)(1-\mathbb{L}^{g-1})(1-\mathbb{L}^{g-2})}{(1-\mathbb{L})(1-\mathbb{L}^2)(1-\mathbb{L}^3)} + \frac{(1-\mathbb{L}^g)(1-\mathbb{L}^{g-1})}{(1-\mathbb{L})(1-\mathbb{L}^2)} \cdot \mathbb{L}^{g-2}\right) \cdot \chi K_0 C.$$

Proof. This follows property (E) of the function χ_c and the cell decomposition of the Grassmannian varieties. □

3.3.3. *The Motive of M*

From the structure of the desingularization morphism $\psi\colon M \to N_C(2, \mathcal{O})$ we have just described together with Theorem 3.12 we can deduce the motivic Poincaré polynomial of M.

PROPOSITION 3.17. *The motivic Poincaré polynomial of M is*

$$\chi_c N_C(2, \mathcal{O}_C)^s + \chi_c Y + \chi_c \psi^{-1} K_0 C,$$

where $\chi_c N_C(2, \mathcal{O}_C)^s$ is given in Theorem 3.12, $\chi_C Y$ in Proposition 3.14 and $\chi_c\psi^{-1}K_0C$ by Proposition 3.16.

Upon application of the ring morphism $P_{xy}\colon \mathcal{K} \to \mathbb{Z}[[x, y]]$, we obtain the Poincaré–Hodge polynomial of M.

COROLLARY 3.18. *The Poincaré–Hodge polynomial of M is*

$$\frac{(1+xL)^g(1+yL)^g - L^gA^g}{(1-L)(1-L^2)} - \frac{A^g\dfrac{L^g-L}{1-L} + \frac{1}{2}(A^g+B^g)}{1+L} +$$
$$+ \frac{A^g - 2^{2g}}{2}\left(\frac{1-L^{g-1}}{1-L}\right)^2 + \left(\frac{B^g}{2} - 2^{2g-1}\right)\left(\frac{1-L^{2g-1}}{1-L^2}\right) +$$
$$+ \left(\frac{(1-L^g)(1-L^{g-1})(1-L^{g-2})}{(1-L)(1-L^2)(1-L^3)} + \frac{(1-L^g)(1-L^{g-1})}{(1-L)(1-L^2)} \cdot L^{g-2}\right) \cdot 2^{2g}.$$

where

$$L = xy, \quad A = (1+x)(1+y) \quad \textit{and} \quad B = (1-x)(1-y).$$

The Betti numbers of the variety M were computed by Balaji and Seshadri in [4] using the Weil conjectures. Using the previous corollary we have computed the Hodge diamond of M for small genus.

EXAMPLES 3.19. The Hodge diamond and Betti numbers of Seshadri's smooth model for $g = 3$ and $g = 4$.

1	1
0 0	0
0 2 0	2
0 3 3 0	6
0 3 11 3 0	17
0 0 3 3 0 0	6
0 0 6 84 6 0 0	96
0 0 3 3 0 0	6
0 3 11 3 0	17
0 3 3 0	6
0 2 0	2
0 0	0
1	1

1	1
0 0	0
0 2 0	2
0 4 4 0	8
0 6 19 6 0	31
0 0 8 8 0 0	16
0 1 28 327 28 1 0	385
0 0 4 32 32 4 0 0	72
0 0 1 34 343 34 1 0 0	413
0 0 0 8 56 56 8 0 0 0	128
0 0 1 34 343 34 1 0 0	413
0 0 4 32 32 4 0 0	72
0 1 28 327 28 1 0	385
0 0 8 8 0 0	16
0 6 19 6 0	31
0 4 4 0	8
0 2 0	2
0 0	0
1	1

3.4. THE MIXED HODGE STRUCTURE OF $N_C(2, \mathcal{O}_C)$

If we apply the ring morphism P_{xy} to the expression for $\chi(N_C(2, \mathcal{O}_C))$ in Theorem 3.12 we obtain the pure Poincaré–Hodge polynomial of $N_C(2, \mathcal{O}_C)$ which is defined

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in terms of the weight filtration of the mixed Hodge structure $H^*(N_C(2, \mathcal{O}_C), \mathbb{Q})$ (see [22]). The following result shows how far the pure Poincaré–Hodge polynomial is from the true Poincaré–Hodge polynomial. Given a vector space, V, acted on linearly by $\mathbb{Z}/2\mathbb{Z}$ we shall write V^+ for the subspace of invariants and V^- the subspace on which $\mathbb{Z}/2\mathbb{Z}$ acts via its nontrivial character.

THEOREM 3.20. *The mixed Hodge structure* $H^i(N_C(2, \mathcal{O}_C), \mathbb{Q})$ *has weights* i *and* $i-1$.

Proof. We first compute the mixed Hodge structure $H^i(\mathrm{Jac}C - \mathrm{Jac}_2 C, \mathbb{Q})$. The group H_c^0 is zero because $\mathrm{Jac}C - \mathrm{Jac}_2 C$ is not complete. The exact sequence

$$\cdots \to H^{i-1}(\mathrm{Jac}_2 C, \mathbb{Q}) \to H_c^i(\mathrm{Jac}C - \mathrm{Jac}_2 C, \mathbb{Q}) \to H^i(\mathrm{Jac}C, \mathbb{Q}) \to \\ \to H^i(\mathrm{Jac}_2 C, \mathbb{Q}) \to \cdots$$

shows that $H_c^i(\mathrm{Jac}C - \mathrm{Jac}_2 C, \mathbb{Q}) \simeq H^i(\mathrm{Jac}C, \mathbb{Q})$ for $i > 1$ and $H_c^1(\mathrm{Jac}C - \mathrm{Jac}_2 C, \mathbb{Q})$ is an extension of $H^1(\mathrm{Jac}C, \mathbb{Q})$ by $\mathbb{Q}^{2^{2g}-1}$:

$$0 \to H^0(\mathrm{Jac}_2 C, \mathbb{Q})/H^0(\mathrm{Jac}C, \mathbb{Q}) \to H_c^1(\mathrm{Jac}C - \mathrm{Jac}_2 C, \mathbb{Q}) \\ \to H^1(\mathrm{Jac}C, \mathbb{Q}) \to 0.$$

In fact this mixed Hodge structure is split. This can be seen by using the $\mathbb{Z}/2\mathbb{Z}$-action on $H^*(\mathrm{Jac}C - \mathrm{Jac}_2 C, \mathbb{Q})$ and the fact that the category of mixed Hodge structures is an Abelian category; $\mathbb{Z}/2\mathbb{Z}$ acts trivially on $H^0(\mathrm{Jac}_2 C, \mathbb{Q})/H^0(\mathrm{Jac}C, \mathbb{Q})$ and as multiplication by -1 on a sub-Hodge structure mapping isomorphically on $H^1(\mathrm{Jac}C, \mathbb{Q})$.

To sum up:

$$H_c^i(\mathrm{Jac}C - \mathrm{Jac}_2 C, \mathbb{Q}) = \begin{cases} H^1(\mathrm{Jac}C, \mathbb{Q}) \oplus \mathbb{Q}^{2^{2g}-1}, & \text{if } i = 1, \\ H^i(\mathrm{Jac}C, \mathbb{Q}), & \text{if } i > 1, \\ 0, & \text{otherwise,} \end{cases}$$

the decomposition according to the action of $\mathbb{Z}/2\mathbb{Z}$ is

$$H_c^i(\mathrm{Jac}C - \mathrm{Jac}_2 C, \mathbb{Q})^+ = \begin{cases} \mathbb{Q}^{2^{2g}-1}, & \text{if } i = 1, \\ H^i(\mathrm{Jac}C, \mathbb{Q}), & \text{if } i > 1 \text{ even}, \\ 0, & \text{otherwise,} \end{cases}$$

$$H_c^i(\mathrm{Jac}C - \mathrm{Jac}_2 C, \mathbb{Q})^- = \begin{cases} H^i(\mathrm{Jac}C, \mathbb{Q}), & \text{if } i \text{ odd}, \\ 0, & \text{otherwise,} \end{cases}$$

By the description above, as in the proof of Proposition 3.14 we obtain an isomorphism of mixed Hodge structures:

$$H_c^*(Y, \mathbb{Q}) \simeq H_c^*(\mathrm{Jac}C - \mathrm{Jac}_2 C, \mathbb{Q})^+ \otimes \mathbb{Q}[c, c']^+ \oplus H_c^*(\mathrm{Jac}C - \mathrm{Jac}_2 C, \mathbb{Q})^- \\ \otimes \mathbb{Q}[c, c']^-$$

where c and c' are of degree 2 and subject to the relations $c^{g-1} = c'^{g-1} = 0$. We see

https://doi.org/10.1023/A:1014756205008 Published online by Cambridge University Press

that $H_c^i(Y,\mathbb{Q})$ is the sum of a pure Hodge structure of weight i and the weight $i-1$ part of

$$\mathbb{Q}^{2^{2g}-1}\otimes\mathbb{Q}[c,c']^{+}=H_c^1(Y)\cdot\mathbb{Q}[c,c']^{+}.$$

In particular, $H^i(Y,\mathbb{Q})$ is pure for even i.

Call $\overline{Y}$ the closure of Y in M and set $\partial Y=\overline{Y}-Y$ so that $\overline{Y}=\psi^{-1}KC$ and $\partial Y=\psi^{-1}K_0C$. Taking into account that ∂Y has a cell decomposition (i.e. it is the disjoint union of a finite collection of affine spaces) it is easy to see that $H_c^i(\partial Y,\mathbb{Q})$ is a pure Hodge structure of weight i, zero for odd i. The open immersion $j\colon Y\hookrightarrow\overline{Y}$ gives rise to an exact sequence

$$H^{2i-1}(\partial Y,\mathbb{Q})=0\longrightarrow H_c^{2i}(Y,\mathbb{Q})\xrightarrow{j_!}H^{2i}(\overline{Y},\mathbb{Q})\longrightarrow H^{2i}(\partial Y,\mathbb{Q})\longrightarrow$$
$$\longrightarrow H_c^{2i+1}(Y,\mathbb{Q})\xrightarrow{j_!}H^{2i+1}(\overline{Y},\mathbb{Q})\longrightarrow H^{2i+1}(\partial Y,\mathbb{Q})=0$$

From this we deduce that $H^{2i}(\overline{Y},\mathbb{Q})$ is pure of weight $2i$. To see that $H^{2i+1}(\overline{Y},\mathbb{Q})$ is also pure we need to see that $W_{2i}H_c^{2i+1}(Y,\mathbb{Q})\subset\ker j_!$. As $W_{2i}H_c^{2i+1}(Y,\mathbb{Q})\subset H_c^1(Y,\mathbb{Q})\cdot\mathbb{Q}[c,c']^{+}$ and $j_!$ is a ring morphism it is enough to prove that statement for $i=0$. This follows easily by taking into account the morphism of exact sequences induced by ψ:

$$\begin{array}{ccccc}
H^0(\partial Y,\mathbb{Q}) & \longrightarrow & H_c^1(Y,\mathbb{Q}) & \xrightarrow{j_!} & H^1(\overline{Y},\mathbb{Q})\\
\uparrow\psi^* & & \simeq\uparrow\psi^* & & \uparrow\psi^*\\
H^0(K_0C,\mathbb{Q}) & \longrightarrow & H_c^1(KC-K_0C,\mathbb{Q}) & \xrightarrow{j_!} & H^1(KC,\mathbb{Q})=0.
\end{array}$$

From the exact sequence associated to the immersion $Y\hookrightarrow M$ and the fact that M is smooth and projective we immediately see that $H_c^i(M^s,\mathbb{Q})$ has weights i and $i-1$

$$\cdots\longrightarrow H^{i-1}(\overline{Y},\mathbb{Q})\longrightarrow H_c^i(M^s,\mathbb{Q})\longrightarrow H^i(M,\mathbb{Q})\longrightarrow\cdots$$

The same argument for the immersion $KC\hookrightarrow N_C(2,\mathcal{O}_C)$ proves the statement of the theorem

$$\cdots\longrightarrow H_c^i(N_C(2,\mathcal{O}_C)^s,\mathbb{Q})\longrightarrow H^i(N_C(2,\mathcal{O}_C),\mathbb{Q})\longrightarrow H^i(KC,\mathbb{Q})\longrightarrow\cdots.\qquad\square$$

Acknowledgements

I wish to express my gratitude to Vicente Navarro Aznar and Pere Pascual Gainza for many helpful discussions.

References

1. Atiyah, M. F. and Bott, R.: The Yang–Mills equations over Riemann surfaces, *Philos. Trans. Roy. Soc. London Ser. A* **308** (1982), 523–615.
2. Balaji, V.: Cohomology of certain moduli spaces of vector bundles. *Proc. Indian Acad. Sci. (Math. Sci.)* **98** (1988), 1–24.
3. Balaji, V., King, A. D. and Newstead, P. E.: Algebraic cohomology of the moduli space of rank 2 vector bundles on a curve, *Topology* **36**(2) (1997), 567–577.
4. Balaji, V. and Seshadri, C. S.: Poincaré polynomials of some moduli varieties, In: *Algebraic Geometry and Analytic Geometry*, ICM-90 Satellite Conference Proceedings, Springer, New York, 1990, pp. 1–25.
5. Carlson, J.: Extensions of mixed Hodge structures. In: *Journées de geométries algébrique d'Angers 1979*, Sijthoff and Noordhoff, 1980, pp. 107–127.
6. del Baño, S.: On motives and moduli spaces of stable bundles over a curve, PhD thesis, Universitat Politècnica de Catalunya, Barcelona; available from http://www-ma1.upc.es/recerca/1998.html, 1998.
7. del Baño, S.: On the Chow motive of some moduli spaces, *J. Reine Angew. Math.* **532** (2001), 105–132; available from http://www-ma1.upc.es/recerca/1998.html.
8. del Baño, S. and Navarro Aznar, V.: On the motive of a quotient variety, *Coll. Math.* **49** (1998), 203–226; available from http://www-ma1.upc.es/recerca/1997.html.
9. Deligne, P.: Hodge cycles on abelian varieties, In: *Hodge Cycles, Motives and Shimura Varieties*, Lecture Notes in Math. 900, chapter 1. Springer, New York, 1982.
10. Denef, J. and Loeser, F.: Motivic Igusa zeta functions, *J. Algebraic Geom.* **7** (1998), 505–537.
11. Desale, V. and Ramanan, S.: Poincaré polynomials of the variety of stable bundles, *Math. Ann.* **216** (1975), 233–244.
12. Faltings, G.: Endlichkeitsätze für abelsche Varietäten über zahlkörpern, *Invent. Math.* **73** (1983), 349–366.
13. Guillén Santos, F. and Navarro Aznar, V.: Un critère d'extension d'un foncteur défini sur les schémas lisses, alg-geom/9505008, 1995.
14. Harder, G.: Eine Bemerkung zu einer Arbeit von P. E. Newstead, *J. Math.* **242** (1970), 16–25.
15. Harder, G. and Narasimhan, M. S.: On the cohomology groups of moduli spaces of vector bundles over curves, *Math. Ann.* **212** (1975), 215–248.
16. Hartshorne, R.: *Algebraic Geometry*, Grad. Texts in Math. 52, Springer-Verlag, New York, 1977.
17. Jannsen, J.: Motives, numerical equivalence, and semi-simplicity, *Invent. Math.* **107** (1992), 447–452.
18. Manin, Y.: Correspondences, motives and monoidal transformations, *Math. USSR Sb.* **6** (1968), 439–470.
19. Mehta, V. and Seshadri, C. S.: Moduli of vector bundles on curves with parabolic structures, *Math. Ann.* **248** (1980), 205–239.
20. Narasimhan, M. S. and Ramanan, S.: Moduli of vector bundles on a compact Riemann surface, *Ann. of Math.* **89** (1969), 15–51.
21. Narasimhan, M. S. and Ramanan, S.: Geometry of Hecke cycles I, In: *C. P. Ramanujan – A tribute*, Bombay, 1978.
22. Navarro Aznar, V.: Stratifications parfaites et théorie des poids, *Publ. Mat. UAB.* **36** (1992), 807–825.
23. Newstead, P. E.: Topological properties of some spaces of stable bundles, *Topology* **6** (1967), 241–262.

24. Newstead, P. E.: Rationality of moduli spaces of vector bundles, *Math. Ann.* **251** (1975), 251–268; Correction in *Math. Ann.* **249** (1980), 281.
25. Scholl, A. J.: Classical motives, In: J. P. Serre, U. Jannsen and S. Kleiman (eds), *Motives*, Proc. Sympos. Pure Math. 55, Amer. Math. Soc., Providence, 1994, pp. 163–187.
26. Seshadri, C. S.: Desingularization of the moduli varieties of vector bundles on curves, In: M. Naata (ed.), *Int. Sympos. on Algebraic Geometry*, Kyoto, 1977, pp. 155–184.
27. Seshadri, C. S.: *Fibrés vectoriels sur les courbes algébriques*, *Asterisque* **96**, Société Mathématique de France, 1982.
28. Thaddeus, M.: Stable pairs, linear systems and the Verlinde formula, *Invent. Math.* **117** (1994), 317–353.