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A NOTE ON STAR COMPACT SPACES WITH POINT-COUNTABLE BASE

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Abstract

In this note we give an example of a Hausdorff, star compact space with point-countable base which is not metrizable.

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1. Introduction

By a space, we mean a topological space. Let A be a subset of a space X and let $\mathcal{U}$ be a family of subsets of X. The star of the set A with respect to $\mathcal{U}$, denoted by $\mathrm{St}(A, \mathcal{U})$, is the set $\bigcup\{U \in \mathcal{U} \mid U \cap A \neq \emptyset\}$.

DEFINITION 1.1 [3]. Let $\mathcal{P}$ be a class (or a property) of a space X. The space X is said to be *star* $\mathcal{P}$ *(or star-determined by* $\mathcal{P}$*)* if, for every open cover $\mathcal{U}$ of X, there exists a subspace Y of X such that $Y \in \mathcal{P}$ and $\mathrm{St}(Y, \mathcal{U}) = X$.

By the above definition, a space X is said to be *star compact* if, for every open cover $\mathcal{U}$ of X, there exists a compact subset K of X such that $\mathrm{St}(K, \mathcal{U}) = X$. In [2], a star compact space is said to be K-starcompact. It is not difficult to see that every countably compact space is star compact (see [2]). Thus it is natural for us to consider the following question.

QUESTION 1.2. Is a star compact space metrizable if it has a point-countable base?

The purpose of this note is to construct an example of a Hausdorff, star compact space with point-countable base which gives a negative answer to this question.

Throughout the paper, the cardinality of a set A is denoted by $|A|$. Let ω be the first infinite cardinal. Other terms and symbols that we do not define will be used as in [1].

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2. The example

EXAMPLE 2.1. There exists a Hausdorff, star compact space with point-countable base which is not metrizable.

PROOF. Let

$$A = \{a_n : n \in \omega\}, \quad B = \{b_m : m \in \omega\},$$
$$Y = \{\langle a_n, b_m \rangle : n \in \omega, m \in \omega\},$$

and

$$X = Y \cup A \cup \{a\} \quad \text{where } a \notin Y \cup A.$$

We topologize X as follows: every point of Y is isolated; a basic neighbourhood of a point $a_n \in A$ for each $n \in \omega$ takes the form

$$U_{a_n}(m) = \{a_n\} \cup \{\langle a_n, b_i \rangle : i > m\} \quad \text{for } m \in \omega$$

and a basic neighbourhood of a takes the form

$$U_a(n) = \{a\} \cup \bigcup \{\langle a_i, b_m \rangle : i > n, m \in \omega\}.$$

Clearly, X is a Hausdorff space by the construction of the topology of X. However, X is not regular, since the point a cannot be separated from the closed subset A by disjoint open subsets of X. Thus X is not metrizable, since it is not regular. By the construction of the topology of X, it is not difficult to see that X is second countable. Thus X has a point-countable base.

We shall now show that X is star compact. Let $\mathcal{U}$ be an open cover of X. For each $n \in \omega$, there exists a $U_n \in \mathcal{U}$ such that $a_n \in U_n$, so there exists an $m_n \in \omega$ such that $\langle a_n, b_{m_n} \rangle \in U_n$. If we put $S_1 = \{\langle a_n, b_{m_n} \rangle : n \in \omega\} \cup \{a\}$, then S_1 is a convergent sequence with the limit point a. Hence S_1 is compact and

$$\{a_n : n \in \omega\} \subseteq \mathrm{St}(S_1, \mathcal{U}).$$

On the other hand, choose $U_a \in \mathcal{U}$ such that $a \in U_a$. Then there exists an $n \in \omega$ such that $U_a(n) \subseteq U_a$, and hence

$$U_a(n) \subseteq \mathrm{St}(S_1, \mathcal{U}),$$

since $U_a \cap S_1 \neq \emptyset$. Finally, for $i \leq n$, $\{a_i\} \cup \{\langle a_i, b_m \rangle : m \in \omega\}$ is compact, so there exists a finite subset $F_i \subseteq \{a_i\} \cup \{\langle a_i, b_m \rangle : m \in \omega\}$ such that

$$\{a_i\} \cup \{\langle a_i, b_m \rangle : m \in \omega\} \subseteq \mathrm{St}(F_i, \mathcal{U}).$$

Put $F = S_1 \cup \bigcup \{F_i : i \leq n\}$. Then F is a compact subset of X and $X = \mathrm{St}(F, \mathcal{U})$, which completes the proof. □

REMARK 2.2. The author does not know if there exists a regular, star compact space with point-countable base which is not metrizable.

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