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A GENERALIZATION OF A FIXED POINT THEOREM OF REICH

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The following theorem is the principal result of this paper.

THEOREM 1. *Let (M, d) be a metric space and T a self-mapping of M satisfying the condition for $x, y \in M$*

$$d(Tx, Ty) \leq ad(x, Tx)+bd(y, Ty)+cd(x, Ty)+ed(y, Tx)+fd(x, y), \tag{1}$$

where a, b, c, e, f are nonnegative and we set $\alpha=a+b+c+e+f$. Then

(a) *If M is complete and $\alpha<1$, T has a unique fixed point.*

(b) *If* (1) *is modified to the condition*

$$\text{(1')} \quad \begin{aligned} &x \neq y \text{ implies} \\ &d(Tx, Ty) < ad(x, Tx)+bd(y, Ty)+cd(x, Ty)+ed(y, Tx)+fd(x, y), \end{aligned}$$

and in this case we assume M is compact, T is continuous and $\alpha=1$, then T has a unique fixed point.

Reich in [1] recently obtained a similar conclusion to that in (a) in the case that $\alpha=a+b+f$. Reich's result in turn generalizes the fixed point theorem of Kannan [2] in which $\alpha=a+b$. The conclusion in part (b) is a limiting version of the theorem contained in part (a), and Edelstein obtained this result in [3] for the case $\alpha=f=1$.

Having obtained the basic result which is Theorem 1, we proceed to give a generalization of part (a) of that theorem in Theorem 2, allowing a, b, c, e, f to become monotonically decreasing functions of $d(x, y)$. Theorem 2 generalizes the corresponding result when $c=e=0$ proved in [1] by Reich who improved upon the original theorem of this kind due to Rakotch [4]. For generalizations of a different nature of Rakotch's fixed point theorem, see Meir and Keeler [5].

We continue to generalize the results of Reich in Theorems 3 and 4 which, roughly, give conditions for convergence of operators to imply convergence of their corresponding fixed points. Earlier theorems of this sort are due to Bonsall [6] and Nadler, [7].

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With no loss of generality we may assume $a=b$ and $c=e$, for from condition (1) we can derive

$$(2)\qquad d(Tx, Ty) \leq \left(\frac{a+b}{2}\right)[d(x, Tx)+d(y, Ty)]+\left(\frac{c+e}{2}\right)[d(x, Ty)+d(y, Tx)]+fd(x, y).$$

Before proceeding we require the following lemma.

LEMMA. *Suppose condition* (1) *holds on* (X, d). *Then if* $\alpha<1$ *there exists a* $\beta<1$ *such that*

$$(3)\qquad d(Tx, T^2x) \leq \beta d(x, Tx).$$

Also if condition (1′) *holds with* $\alpha=1$, *then*

$$(3')\qquad x \neq Tx \text{ implies } d(Tx, T^2x) < d(x, Tx).$$

Proof. Suppose $\alpha<1$. Set $y=Tx$ in (1) and simplify to obtain

$$(4)\qquad d(Tx, T^2x) \leq \left(\frac{a+f}{1-b}\right) d(x, Tx)+\frac{c}{1-b} d(x, T^2x).$$

Now by the triangle inequality, $d(Tx, T^2x)\geq d(T^2x, x)-d(Tx, x)$ so from (4) we obtain

$$d(T^2x, x)-d(Tx, x) \leq \left(\frac{a+f}{1-b}\right) d(x, Tx)+\frac{c}{1-b} d(x, T^2x),$$

or simplifying,

$$(5)\qquad d(T^2x, x) \leq \left(\frac{1+a+f-b}{1-b-c}\right) d(x, Tx).$$

Substituting inequality (5) into inequality (4) we obtain

$$(6)\qquad d(Tx, T^2x) \leq \left(\frac{a+c+f}{1-b-c}\right) d(x, Tx),$$

and by symmetry, we may exchange a with b and c with e in (6) to obtain

$$d(Tx, T^2x) \leq \left(\frac{b+e+f}{1-a-e}\right) d(x, Tx).$$

Then

$$\beta = \min[(a+c+f)/(1-b-c), (b+e+f)/(1-a-e)]$$

satisfies the first conclusion of the lemma. The proof for the remaining case is similar, the main difference being that a side argument is needed to show that without loss of generality we may assume the numbers $a+e$ and $b+c$ are less than 1. This follows essentially because of inequality (2) and the comment preceding it.

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Proof of Theorem 1. Existence in part (b) follows easily from the second part of the lemma. For $\inf\{d(x, Tx):x \in M\}=d(y, Ty)$ for some $y \in M$ because T is continuous and M is compact. Condition (3′) implies now that y must be fixed under T. Also it is easy to verify that the conditions in both parts (a) and (b) imply uniqueness. We proceed to prove the existence portion of part (a). By the first part of the lemma, there exists $\beta<1$ with $d(Tx, T^2x)\leq\beta\, d(x, Tx)$. Let $m>n$, then

$$d(T^mx, T^nx) \leq d(T^mx, T^{m-1}x)+\cdots+d(T^{n+1}x, T^nx)$$
$$\leq \beta^n(1+\beta+\cdots+\beta^{m-n})\, d(x, Tx) \leq \frac{\beta^n}{1-\beta}\, d(x, Tx).$$

Thus $\{T^nx\}$ is a Cauchy sequence and so converges to some x_0 in M. Then we claim $x_0=Tx_0$, for from condition (1),

$$d(x_0, Tx_0) \leq d(T^{n+1}x, Tx_0)+d(T^{n+1}x, x_0) \leq ad(T^nx, T^{n+1}x)+bd(x_0, Tx_0)$$
$$+cd(T^nx, Tx_0)+(e+1)\, d(T^{n+1}x, x_0)+fd(T^nx, x_0).$$

Taking the limit in this inequality as $n\to\infty$, we obtain

$$d(x_0, Tx_0) \leq (b+c)d(x_0, Tx_0),$$

which contradicts $b+c<1$, unless $x_0=Tx_0$.

Let $O(x)$ denote the range of the sequence $x, Tx, T^2x, \ldots$. An example of the situation $d(T^2x, Tx)<d(Tx, x)$ on $\overline{O(x)}$ in which T is not a contraction is given by $Tx_n=x_{n+1}$, where $x_0=1$, $x_1=-\frac{7}{8}$, $x_2=\frac{3}{4}$, $x_3=-\frac{1}{4}$, $x_{n+4}=\frac{1}{8}x_n$, $n=0, 1, \ldots$, and $TO=0$. Another example of this sort, but in which fixed points aren't unique, can be based on defining the iterates on a spiral which converges to the circumference of a circle. Let x_{n+1} be distance $1/n$ from x_n where the distance is measured on the curve. Define T by $x_{n+1}=Tx_n$ and extend this function continuously to the circumference of the circle by defining $Tx=x$ for each point on the circumference. Then condition (3′) is satisfied, but T is not a contraction.

We proceed to generalize part (a) of this theorem.

THEOREM 2. *Let (X, d) be a complete metric space, a, b, c, e, f be monotonically decreasing functions from* $[0, \infty)$ *to* $[0, 1)$, *and let the sum of these five functions be less than* 1. *Suppose $T:X\to X$ satisfies condition* (1) *with $a=a(d(x, y)), \ldots, f=f(d(x, y))$ for all $x, y \in X$. Then T has a unique point.*

Proof. Following the proof of the lemma, there exists a monotone decreasing function $\beta(t)=\beta(a(t), b(t), c(t), e(t), f(t))$ such that $0\leq\beta(t)<1$ and $d(T^2x, Tx)\leq \beta(d(x, Tx))\, d(x, Tx)$. Then

$$d(T^nx, T^{n+1}x) \leq \beta(d(T^{n-1}x, T^nx))\, d(T^{n-1}x, T^nx) < d(T^{n-1}x, T^nx)$$

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so that $\{d(T^nx, T^{n+1}x)\}$ is a decreasing sequence. Let $\lim_{n\to\infty} d(T^nx, T^{n+1}x)=p$. If $p>0$, then $d(T^nx, T^{n+1}x)\geq p$ and since β is monotone decreasing $d(T^nx, T^{n+1}x)< \beta(p)\, d(T^{n-1}x, T^nx)\leq\cdots\leq\beta^n(p)\, d(x, Tx)\to 0$, as $n\to\infty$, a contradiction. Therefore $p=0$.

Set $a=a(d(T^{n-1}x, T^{m-1}x))$, and similarly for b, c, e, f. Without loss, we assume $T^{n-1}x\neq T^{m-1}x$. Applying condition (1) and then the triangle inequality, we have

$$\begin{aligned} &d(T^nx, T^mx)\\ &\leq ad(T^{n-1}x, T^nx)+bd(T^{m-1}x, T^mx)+cd(T^{n-1}x, T^mx)+ed(T^{m-1}x, T^nx)\\ &\quad +fd(T^{n-1}x, T^{m-1}x)\\ &\leq ad(T^nx, T^{n-1}x)+bd(T^mx, T^{m-1}x)+c(d(T^{n-1}x, T^nx)\\ &\quad +d(T^nx, T^mx))+e(d(T^{m-1}x, T^mx)+d(T^mx, T^nx))+f(d(T^{m-1}x, T^mx)\\ &\quad +d(T^mx, T^nx)+d(T^{n-1}x, T^nx)). \end{aligned}$$

Simplifying this expression we obtain

$$\begin{aligned} (7)\qquad d(T^nx, T^mx) &\leq \frac{(a+c+f)}{1-(c+e+f)}\, d(T^{n-1}x, T^nx)+\frac{(b+e+f)}{1-(c+e+f)}\, d(T^mx, T^{m-1}x)\\ &= \beta_1 d(T^{n-1}x, T^nx)+\beta_2 d(T^mx, T^{m-1}x), \end{aligned}$$

where $\beta_1=(a+c+f)/(1-(c+e+f))$ and $\beta_2=(b+e+f)/(1-(c+e+f))$. Choose $\varepsilon>0$. If $d(T^{n-1}x, T^{m-1}x)\geq\varepsilon$ then from inequality (7),

$$(8)\qquad d(T^nx, T^mx)\leq \beta_1(\varepsilon)\, d(T^{n-1}x, T^nx)+\beta_2(\varepsilon)\, d(T^mx, T^{m-1}x),$$

while if $d(T^{n-1}x, T^{m-1}x)\leq\varepsilon$,

$$\begin{aligned} (9)\qquad d(T^nx, T^mx) &\leq d(T^nx, T^{n-1}x)+d(T^{n-1}x, T^{m-1}x)+d(T^{m-1}x, T^mx)\\ &< d(T^nx, T^{n-1}x)+d(T^{m-1}x, T^mx)+\varepsilon. \end{aligned}$$

Since $d(T^nx, T^{n-1}x)\to 0$, it is clear from inequalities (8) and (9) that $\{T^nx\}$ is Cauchy sequence. Let $T^nx\to z$.

We may assume that $T^{n-1}x\neq z$ and that $b+c<\frac{1}{2}$. Put $a=a(d(T^{n-1}x, z))$, etc. We have

$$\begin{aligned} d(z, Tz) &\leq d(z, T^nx)+d(T^nx, Tz)\\ &\leq d(z, T^nx)+ad(T^{n-1}x, T^nx)+bd(z, Tz)+cd(T^{n-1}x, z)\\ &\quad +cd(z, Tz)+ed(z, T^nx)+fd(T^{n-1}x, z)\\ &\leq 2d(z, T^nx)+2d(z, T^{n-1}x)+d(T^{n-1}x, T^nx)+\tfrac{1}{2}d(z, Tz)\to\tfrac{1}{2}d(z, Tz). \end{aligned}$$

Consequently, z must be a fixed point of f. Uniqueness follows easily from condition (1).

THEOREM 3. *Let (X, d) be a complete metric space and $T_n: X\to X$, $n=1, 2, \ldots$ satisfy the conditions of Theorem 2 with the same coefficients a, b, c, e, f. Let*

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$T_nx_n=x_n$ and suppose $T_n \to T$ pointwise on X. Then $\lim_{n\to\infty} x_n=x$ *is the unique fixed point of T.*

Proof. By continuity of the metric, and condition (1), the limit map T also satisfies condition (1) and so has a unique fixed point, call it x. Setting $a=a(d(x_n, x))$ and similarly with b, c, e, f we have

$$\begin{aligned} d(x_n, x) = d(T_nx_n, Tx) &\le d(T_nx_n, T_nx)+d(T_nx, Tx) \le ad(x_n, T_nx_n) \\ &\quad +bd(x, T_nx)+cd(x_n, T_nx)+ed(T_nx_n, x)+fd(x_n, x)+d(T_nx, Tx) \\ &\le bd(x, T_nx)+c(d(x_n,x)+d(Tx, T_nx))+ed(T_nx_n, x)+fd(x_n, x) \\ &\quad +d(T_nx, Tx). \end{aligned}$$

Simplification yields

$$d(x_n, x) \le \frac{(1+b+c)}{1-(c+e+f)} d(T_nx, x) \le \frac{2}{1-(c+e+f)} d(T_nx, x).$$

Choose $\varepsilon>0$. There exists N such that $n \ge N$ implies

$$d(T_nx, Tx) < \frac{\varepsilon}{2}(1-c(\varepsilon)+e(\varepsilon)+f(\varepsilon)).$$

Take $n\ge N$, then if $d(x, x_n)\ge\varepsilon$, it follows that

$$d(x_n, x) \le \frac{2}{1-(c(\varepsilon)+e(\varepsilon)+f(\varepsilon))} d(T_nx, x) < \varepsilon.$$

This contradiction implies $d(x, x_n)<\varepsilon$. Hence $x_n\to x$, and the proof is complete.

THEOREM 4. *Let (X, d) be a complete metric space and $T_n:X\to X$, $n=1, 2, \ldots$ be functions with at least one fixed point x_n, $n=1, 2, \ldots$. Let T satisfy the hypothesis of Theorem 2 and suppose $T_n\to T$ uniformly on X. Then x_n converges to x, the unique fixed point of T.*

Proof. Let $a=a(d(x_n, x))$ and similarly for the other coefficients. We have

$$\begin{aligned} d(x_n, x) = d(T_nx_n, Tx) &\le d(T_nx_n, Tx_n)+d(Tx_n, Tx) \le d(T_nx_n, Tx_n) \\ &\quad +ad(x_n, Tx_n)+bd(x, Tx)+cd(x_n,Tx)+ed(Tx_n, x)+fd(x_n, x) \\ &\le d(T_nx_n, Tx_n)+ad(T_nx_n, Tx_n)+cd(x_n, x)+e(d(Tx_n, T_nx_n)+d(x_n, x)) \\ &\quad +fd(x_n,x). \end{aligned}$$

Therefore,

$$d(x_n, x) \le \frac{1+a+e}{1-(c+e+f)} d(T_nx_n, Tx_n) \le \frac{2}{1-(c+e+f)} d(T_nx_n, Tx_n).$$

Choose $\varepsilon > 0$ and N such that $n \geq N$ implies

$$d(T_n x_n, T x_n) < \frac{\varepsilon}{2}[1-(c(\varepsilon)+e(\varepsilon)+f(\varepsilon))].$$

It is now clear that we must have $d(x_n, x) < \varepsilon$ and the proof is complete.

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