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WIDTHS OF CROSSINGS IN POISSON BOOLEAN PERCOLATION

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Abstract

We answer the following question: if the occupied (or vacant) set of a planar Poisson Boolean percolation model contains a crossing of an $n \times n$ square, how wide is this crossing? The answer depends on whether we consider the critical, sub-, or super-critical regime, and is different for the occupied and vacant sets.

Keywords: Poisson Boolean percolation; width of crossing; sharp transition; scaling relations

2020 Mathematics Subject Classification: Primary 60K35
Secondary 60G55

1. Introduction

Percolation is the branch of probability theory that investigates the geometry and connectivity properties of random media. Since its introduction in the 1950s to model the diffusion of liquid in porous media [5], percolation theory has attracted great research interest and led to significant discoveries, especially in two dimensions; see, for instance, Kesten's determination of the critical threshold [18] and of the scaling relations [19], Schramm's introduction of the Schramm–Loewner evolution [22, 27], and Smirnov's proof of Cardy's formula [28]. For an introduction to and overview of percolation theory, we recommend [4, 16, 17], among many other texts.

Bernoulli percolation on a symmetric grid lies at the foundation of all percolation models, and embodies the archetypal setting to investigate phase transitions and other phenomena emanating from statistical physics. In the site-percolation version of this model, each node of the grid is independently chosen to be black with probability p and white with probability $1-p$, and a random graph is obtained by removing the white vertices. It is well known that, as the parameter p increases, the model undergoes a sharp phase transition at some critical parameter. At the point of phase transition, percolation is expected to exhibit a universal and conformally invariant scaling limit; unfortunately this was only proved in the particular case of site percolation on the triangular lattice [28].

Boolean (or *continuum*) percolation (Figure 1) first appeared in [15] as an early mathematical model for wireless networks. In recent years, it has been studied in order to determine the

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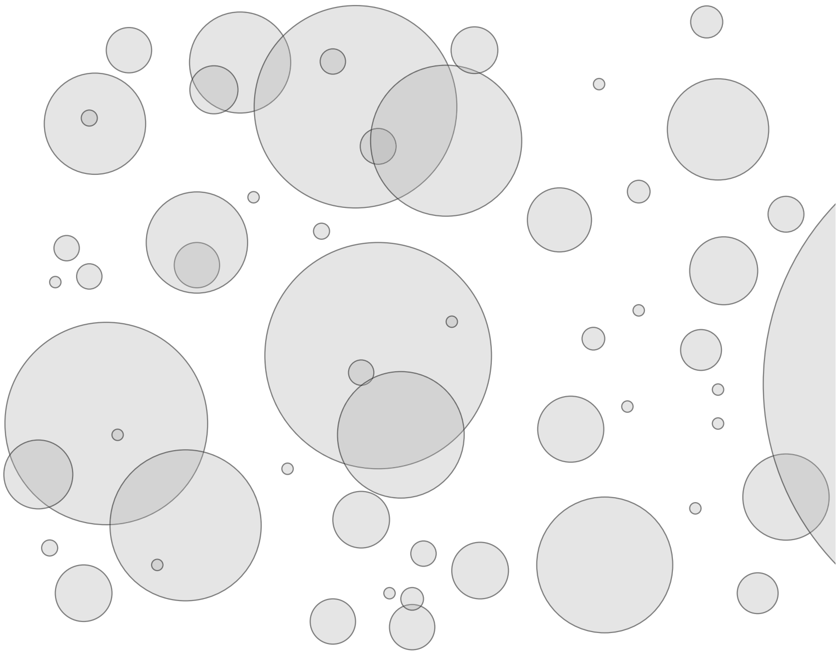

FIGURE 1. Continuum percolation on $\mathbb{R}^2$.

theoretical bounds of information capacity and performance in such networks [10]. See also [14] for a wider perspective on random networks. In addition to this setting, continuum percolation has gained applications in other disciplines, including biology, geology, and physics, such as the study of porous materials and semiconductors. From a mathematical point of view, continuum percolation is particularly interesting as it is expected to exhibit the same features as discrete percolation, but enjoys additional symmetries, such as invariance under rotations. We direct the reader to [23] and the references therein for more background.

1.1. Framework

Let η be a Poisson point process on $\mathbb{R}^2$ with intensity $\lambda \cdot \mathrm{Leb}_{\mathbb{R}^2}$, where $\lambda > 0$ is a parameter of the model. Around each point in the support of η, draw a disk of random radius, sampled independently for each point according to a fixed probability measure μ on $\mathbb{R}_+$; in this paper we only consider the case where the radius is almost surely equal to 1, but describe the general model for future reference. The set $\mathcal{O} = \mathcal{O}_\eta \subset \mathbb{R}^2$ of points which are covered by at least one of the above disks is called the *occupied set*, while its complement $\mathcal{V} = \mathcal{V}_\eta := \mathbb{R}^2 \setminus \mathcal{O}$ is referred to as the *vacant set*. Write $\mathbb{P}_\lambda$ for the measure governing η and the random sets $\mathcal{O}$ and $\mathcal{V}$.

While sharing many features with site or bond Bernoulli percolation, the continuum model poses significant additional challenges. Apart from being continuous rather than discrete, these come from its asymmetrical nature (the 'open' and 'closed' sets have different properties) and potential long-range dependencies. Nevertheless, similarly to the classical Bernoulli case, the Boolean model undergoes a sharp phase transition as λ increases. Indeed, set $\lambda_c := \sup\{\lambda \geq 0 \colon \mathbb{P}_\lambda(0 \overset{\mathcal{O}}{\leftrightsquigarrow} \infty) = 0\}$, where $0 \overset{\mathcal{O}}{\leftrightsquigarrow} \infty$ is the event that the origin lies in an unbounded connected component of $\mathcal{O}$. Under the *minimal condition* $\int_0^\infty x^2 \, \mathrm{d}\mu(x) < \infty$, which is in fact necessary

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for the model to present non-trivial behavior, it was recently shown by [1, 2] that $0 < \lambda_c < \infty$ and:

- For $\lambda < \lambda_c$ (the *sub-critical phase*) the vacant set has a unique unbounded connected component, and the probability of observing an occupied path from 0 to distance n decays exponentially fast in n.
- For $\lambda = \lambda_c$ (the *critical phase*), no unbounded connected component exists in either the occupied or the vacant set. Moreover, the probability of observing either a vacant or an occupied path from 0 to distance n decays polynomially fast in n.
- For $\lambda > \lambda_c$ (the *super-critical phase*) the occupied set has a unique unbounded component and the probability of observing a vacant path from 0 to distance n decays exponentially fast in n.

1.2. Results

For simplicity, we limit our study to the particular setting where the radii are all equal to 1 (i.e. when μ is the Dirac measure at 1).

Assumption 1. *We assume that the radii are fixed and equal to 1, i.e.* $\mu = \delta_1$.

The proof extends readily to the case where μ is supported on a compact subset of $(0, +\infty)$, and with additional work may include situations where μ has sufficiently fast decay towards 0 and ∞. We do not investigate the optimal conditions for which the results remain valid.

A horizontal *crossing* of the square $[-n, n]^2$ is a path contained in $[-n, n]^2$, connecting its left and right boundaries, namely $\{-n\} \times [-n, n]$ and $\{n\} \times [-n, n]$, respectively. A crossing is said to be occupied (resp. vacant) if it is entirely contained in $\mathcal{O}$ (resp. $\mathcal{V}$). In the following, we write cross (n) and cross$^*(n)$ for the events that there exists an occupied and a vacant horizontal crossing of $[-n, n]^2$, respectively.

The *width* $w(\gamma)$ of an occupied crossing γ is *twice* the radius of the largest ball that can be transported along γ without intersecting the vacant set. Alternatively, it may be viewed as twice the distance between γ and $\mathcal{V}$:

$$2 \cdot \sup\{r \geq 0\colon B(\gamma(t), r) \subset \mathcal{O} \text{ for all } t \in [0, 1]\} = 2 \cdot \text{dist}(\gamma, \mathcal{V}),$$

where γ is parameterized by $[0, 1]$, and $B(x, r)$ denotes the open Euclidean ball of radius r centered at $x \in \mathbb{R}^2$. A similar definition may be given for a vacant crossing, with the roles of $\mathcal{O}$ and $\mathcal{V}$ reversed. See Figure 2 for a graphical interpretation.

Define the *maximal* occupied and vacant widths as

$$\mathrm{w}_n := 2 \cdot \sup_\gamma \text{dist}(\gamma, \mathcal{V}), \qquad \mathrm{w}_n^* := 2 \cdot \sup_\gamma \text{dist}(\gamma, \mathcal{O}), \tag{1}$$

where both supremums are taken over all horizontal crossings of $[-n, n]^2$. Observe that no occupied or vacant crossing exists if and only if $\mathrm{w}_n = 0$ and $\mathrm{w}_n^* = 0$ almost surely, respectively.

Before stating our results, we define the four-arm probabilities. For $r \leq R$, define the four-arm event $A_4(r, R)$ as the existence of four disjoint paths $\gamma_1, \ldots, \gamma_4$ in $\overline{B(0, R)} \setminus B(0, r)$, each starting on $\partial B(0, r)$ and ending on $\partial B(0, R)$, distributed in counterclockwise order and with $\gamma_1, \gamma_3 \in \mathcal{O}$ and $\gamma_2, \gamma_4 \in \mathcal{V}$. Let

$$\pi_4(r, R) = \mathbb{P}_{\lambda_c}[A_4(r, R)], \qquad \pi_4(R) = \pi_4(1, R). \tag{2}$$

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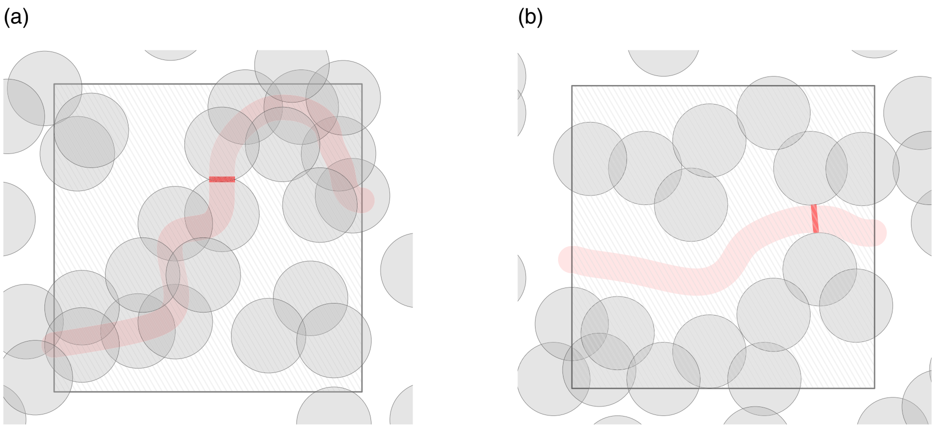

FIGURE 2. The *width* of (a) an occupied and (b) a vacant crossing of an $n \times n$ square.

Our main result concerns the maximal widths of occupied and vacant crossings, when these are conditioned to exist. We formulate two theorems, respectively concerning the vacant and occupied cases.

Theorem 1. *(Widths of vacant crossings.) For any $\delta > 0$ and $\lambda > 0$, there exist constants $0 < c < C$ such that, for large enough n,*

$$\mathbb{P}_\lambda\left[\left|\mathrm{w}_n^* - 2\left(\frac{\lambda_c}{\lambda} - 1\right)\right| \le \frac{C}{\lambda n^2 \pi_4(n)} \mid \mathrm{cross}^*(n)\right] \ge 1 - \delta \qquad \text{if } \lambda < \lambda_c, \tag{3}$$

$$\mathbb{P}_{\lambda_c}\left[\frac{c}{n^2 \pi_4(n)} \le \mathrm{w}_n^* \le \frac{C}{n^2 \pi_4(n)} \mid \mathrm{cross}^*(n)\right] \ge 1 - \delta \qquad \text{if } \lambda = \lambda_c, \tag{4}$$

$$\mathbb{P}_\lambda\left[\frac{c}{n} \le \mathrm{w}_n^* \le \frac{C}{n} \mid \mathrm{cross}^*(n)\right] \ge 1 - \delta \qquad \text{if } \lambda > \lambda_c. \tag{5}$$

Theorem 2. *(Widths of occupied crossings.) For any $\delta > 0$ and $\lambda > 0$, there exist constants $0 < c < C$ such that, for large enough n,*

$$\mathbb{P}_\lambda\left[\frac{c}{\sqrt{n}} \le \mathrm{w}_n \le \frac{C}{\sqrt{n}} \mid \mathrm{cross}(n)\right] \ge 1 - \delta \qquad \text{if } \lambda < \lambda_c, \tag{6}$$

$$\mathbb{P}_{\lambda_c}\left[\frac{c}{n\sqrt{\pi_4(n)}} \le \mathrm{w}_n \le \frac{C}{n\sqrt{\pi_4(n)}} \mid \mathrm{cross}(n)\right] \ge 1 - \delta \qquad \text{if } \lambda = \lambda_c, \tag{7}$$

$$\mathbb{P}_\lambda\left[\mathrm{w}_n \ge 2\sqrt{1 - \left(\frac{\lambda_c}{\lambda} + \frac{C}{\lambda n^2 \pi_4(n)}\right)^2} \mid \mathrm{cross}(n)\right] \ge 1 - \delta \qquad \text{if } \lambda > \lambda_c. \tag{8}$$

Remark 1. The events in the conditioning in Theorems 1 and 2 may be replaced by $\mathrm{w}_n^* > 0$ and $\mathrm{w}_n > 0$, respectively.

Note that the lower bound in (8) is not sharp for λ large, due to the possible existence of crossings of width larger than 2; see also Remark 2. We do expect w_n to be of the order of the lower bound in (8) for $\lambda - \lambda_c > 0$ sufficiently small.

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In (3), (4), (7), and (8), the conditioning has limited effect, as the events have uniformly positive probability (even probability tending to 1 in the first and last cases). However, in (5) and (6), the event in the conditioning is of exponentially small probability, and the resulting measure is highly degenerate. In these cases, the vacant (and respectively occupied) clusters crossing the box have a specific structure, described in detail in [6–9]; these works refer to Bernoulli percolation on the square lattice, but the statements and proofs adapt readily. We will use these results in clearly stated forms, but without re-proving them.

1.3. Organisation of the paper

Section 2 contains certain background on the continuum percolation model. In particular, we state a result concerning the near-critical regime, whose proof we only sketch as it is very similar to existing arguments. Section 3 contains an observation on two distinct increasing couplings of $\mathbb{P}_\lambda$ for $\lambda > 0$ that is the key to our arguments. In Section 4 we prove most of our two main results, Theorems 1 and 2, using the observations of Section 3. Unfortunately, the upper bounds on w_n of Theorem 2 are not accessible with this technique, and in Section 5 we prove these bounds using an alternative approach. Finally, in Section 6, we provide some related open questions.

2. Background and preliminaries

2.1. Positive association

There is a natural partial ordering '$\preceq$' on the space of possible realizations of η. We write $\omega \preceq \omega'$ if and only if $\mathcal{O}_\omega \subset \mathcal{O}_{\omega'}$. An event A is said to be *increasing* if $\mathcal{O}_\omega \in A$ implies that $\mathcal{O}_{\omega'} \in A$ for all $\omega \preceq \omega'$. A useful property of increasing events is that they are positively correlated. Indeed, if A_1 and A_2 are both increasing events, $\mathbb{P}_\lambda[A_1 \cap A_2] \geq \mathbb{P}_\lambda[A_1] \cdot \mathbb{P}_\lambda[A_2]$. This result is known as FGK (Fortuin–Kasteleyn–Ginibre) inequality and was proved in [25]. For a nice proof using discretization and martingale theory see [23].

2.2. Russo's formula

Russo's differential formula controls how the probability of a monotone event varies under perturbations of the intensity parameter λ, assessing the variation in terms of *pivotal* events. See [21] for a proof.

Proposition 1. *(Russo's formula.) Let A be an increasing event depending only on a bounded subset of* $\mathbb{R}^2$. *Then, for every* $\lambda > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{P}_\lambda[A] = \int_{x\in\mathbb{R}^2} \mathbb{P}_\lambda[\mathrm{Piv}_x(A)]\,\mathrm{d}x, \tag{9}$$

where $\mathrm{Piv}_x(A) := \{\mathcal{O} \notin A\} \cap \{\mathcal{O} \cup B(x, 1) \in A\}$ *is the event that x is pivotal for A.*

2.3. Crossing probabilities and RSW theory

Let $\mathrm{cross}(r, h)$ denote the event that there exists an occupied path inside the rectangle $[-r, r] \times [-h, h]$ between its left and right sides. We write $\mathrm{cross}^*(r, h)$ for the corresponding event for the vacant set. Loosely speaking, the Russo–Seymour–Welsh (RSW) theory states that a lower bound on the crossing probability for a rectangle of aspect ratio ρ implies a lower bound for a rectangle of larger aspect ratio ρ'.

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Proposition 2. *(RSW.) For every* ρ, $\rho' > 0$ *and* $\varepsilon > 0$ *there exists* $\varepsilon' = \varepsilon'(\rho, \rho') > 0$ *such that*

$$\mathbb{P}_\lambda[\mathrm{cross}(\rho n, n)] > \varepsilon \Longrightarrow \mathbb{P}_\lambda[\mathrm{cross}(\rho' n, n)] > \varepsilon' \tag{10}$$

for all $n \geq 1$. *The same holds for* cross*.

RSW bounds for continuum percolation were obtained separately for the occupied and vacant sets in [3, 26] respectively, assuming in both cases heavy restrictions on the radii distribution, and later in [1] under minimal assumptions. A fundamental consequence of the RSW theorem (see again [1]) concerns the abrupt change in crossing probabilities:

- For $\lambda < \lambda_c$ and all $\rho > 0$, there exists $c > 0$ such that $\mathbb{P}_\lambda[\mathrm{cross}(\rho n, n)] \leq \mathrm{e}^{-cn}$ for all $n \geq 1$.
- At criticality, the *box-crossing property* holds. That is, for every $\rho > 0$ there exists $c = c(\rho) > 0$ such that

$$c \leq \mathbb{P}_{\lambda_c}[\mathrm{cross}(\rho n, n)] \leq 1 - c \quad \text{for all } n \geq 1. \tag{11}$$

- For $\lambda > \lambda_c$ and all $\rho > 0$, there exists $c > 0$ such that $\mathbb{P}_\lambda[\mathrm{cross}(\rho n, n)] \geq 1 - \mathrm{e}^{-cn}$ for all $n \geq 1$.

The above results may be translated for the vacant set using the duality observation that

$$\mathbb{P}_\lambda[\mathrm{cross}(\rho n, n)] + \mathbb{P}_\lambda[\mathrm{cross}^*(n, \rho n)] = 1.$$

Indeed, a rectangle is amost surely either crossed horizontally by an occupied path or vertically by a vacant one.

Let us also give a corollary relating the crossing of slightly longer rectangles to that of squares.

Lemma 1. *There exists* $C > 0$ *such that, for any* $R \geq r \geq 1$,

$$\mathbb{P}_{\lambda_c}[\mathrm{cross}(R)] - \mathbb{P}_{\lambda_c}[\mathrm{cross}(R + r, R)] \leq C\frac{r}{R}. \tag{12}$$

This can be shown using the so-called three-arm event in the half-plane. We give a different, more basic, proof.

Proof. For simplicity, assume that R is a multiple of r; the general case may be deduced by the monotonicity in r of the left-hand side of (12). Due to the inclusion of events, the difference in (12) may be written as

$$\mathbb{P}_{\lambda_c}[\mathrm{cross}(R)] - \mathbb{P}_{\lambda_c}[\mathrm{cross}(R + r, R)] = \mathbb{P}_{\lambda_c}[\tilde{\mathcal{C}}(R) \setminus \mathcal{C}(R + r)], \tag{13}$$

where $\mathcal{C}(k)$ is the event that there exists a horizontal occupied crossing of $[0, 2k] \times [-R, R]$, and $\tilde{\mathcal{C}}(k)$ is the translation of this event by $2r$ to the right (see Figure 3).

When $\tilde{\mathcal{C}}(R) \setminus \mathcal{C}(R + r)$ occurs, there exists a vertical vacant crossing of $[0, 2r + 2R] \times [-R, R]$ and a horizontal occupied crossing of $[2r, 2r + 2R] \times [-R, R]$; the vertical crossing avoids the horizontal one by using the strip $[0, 2r] \times [-R, R]$. Now, conditionally on this event, due to the RSW property (Proposition 2), the horizontal crossing may be extended with positive probability into a horizontal occupied crossing of $[2r, 2r + 4R] \times [-R, R]$. This step is

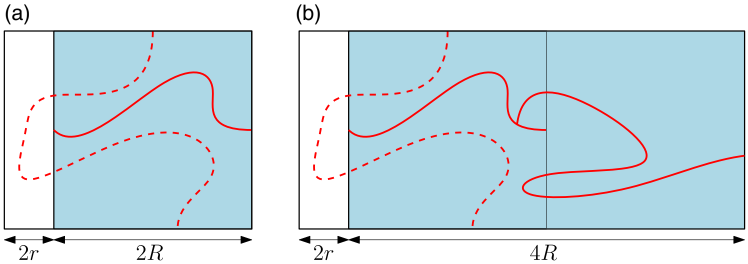

FIGURE 3. (a) A configuration in $\tilde{\mathcal{C}}(R) \setminus \mathcal{C}(R+r)$ contains a horizontal occupied crossing of the blue square $[2r, 2r+2R] \times [-R, R]$, but no crossing of the slightly longer rectangle $[0, 2r+2R] \times [-R, R]$. Occupied crossings are depicted by bold lines, vacant ones by dashed lines. (b) The horizontal crossing of $[2r, 2r+2R] \times [-R, R]$ may be lengthened into one of $[2r, 2r+4R] \times [-R, R]$ at constant cost, due to the RSW theorem. This configuration belongs to $\tilde{\mathcal{C}}(R+kr) \setminus \mathcal{C}(R+(k+1)r)$ for any $0 \leq k < R/r$.

not completely immediate, as it requires proving a separation property by which the horizontal occupied crossing may be taken to end on $\{2R+2r\} \times [-R, R]$, far from the vertical vacant crossing. That is, that the probability of a horizontal occupied crossing is comparable to that of the same event, with prescribed landing areas. This is shown by a gluing argument based on RSW constructions. This type of property is classical and we will not detail its proof.

Observe that, when $[0, 2r+2R] \times [-R, R]$ contains a vertical vacant crossing but $[2r, 2r+4R] \times [-R, R]$ contains an occupied horizontal crossing, all events of the type $\tilde{\mathcal{C}}(R+kr) \setminus \mathcal{C}(R+(k+1)r)$ with $0 \leq k < R/r$ are realized. We conclude that there exists a universal constant $c > 0$ such that, for every $0 \leq k < R/r$,

$$\begin{aligned}\mathbb{P}_{\lambda_c}[\mathcal{C}(R+kr)] - \mathbb{P}_{\lambda_c}[\mathcal{C}(R+(k+1)r)] &= \mathbb{P}_{\lambda_c}\big[\tilde{\mathcal{C}}(R+kr) \setminus \mathcal{C}(R+(k+1)r)\big] \\ &\geq c\, \mathbb{P}_{\lambda_c}[\tilde{\mathcal{C}}(R) \setminus \mathcal{C}(R+r)].\end{aligned}$$

Now sum the above over k to deduce that

$$1 \geq \mathbb{P}_{\lambda_c}[\mathcal{C}(R)] - \mathbb{P}_{\lambda_c}[\mathcal{C}(2R)] \geq c(R/r)\mathbb{P}_{\lambda_c}[\tilde{\mathcal{C}}(R) \setminus \mathcal{C}(R+r)].$$

Apply (13) to obtain the desired inequality. □

2.4. Near-critical percolation

A fundamental question around the phase transition of percolation is the speed at which the model transitions from sub-critical to super-critical behavior as λ increases. In infinite volume the transition occurs instantaneously, but if we limit ourselves to a finite window, say the square of side length L, then the model exhibits critical behavior for an interval of intensities λ around λ_c called the *critical window*.

Alternatively, we can state that, for any given $\lambda \neq \lambda_c$, the model behaves critically at scales up to some $L(\lambda)$, and sub- or super-critically above this scale. The scale $L(\lambda)$ is called the *characteristic length*, and may be shown to be equivalent to the better-known correlation length.

This phenomenon was first proved in [19] for Bernoulli percolation on planar lattices, along with an asymptotic expression for $L(\lambda)$ in terms of the number of pivotals in a box of size n.

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Kesten's study of the near-critical regime produced the so-called scaling relations, which link the algebraic decays of different natural observables of the critical and near-critical models. See also [24] for a more modern exposition of Kesten's result, and [12] for an alternative proof.

The principle of universality suggests that Kesten's results extend to a large variety of percolation models in the plane. They were indeed proven for Voronoi percolation in [29], and appropriate alterations of Kesten's relations were extended to FK (Fortuin–Kasteleyn) percolation in [11].

We claim that the result in [19] also applies to our model of continuum percolation. Below, we state a consequence of the more general theory of near-critical percolation designed to assist us in the proof of our main results. For $n \geq 1$, write $\alpha_n := 1/\pi_4(n)n^2$, where $\pi_4(n)$ was defined in (2). As illustrated by the next theorem, α_n is the size of the *critical window* at scale n; that is, it is the amount by which the critical parameter should be perturbed to observe off-critical features in a box of size n. We remark that, due to the general bound (20) on the probability of the four-arm event, $\alpha_n \to 0$ as $n \to \infty$.

Theorem 3. *(Crossings in near-critical percolation.) For any $\delta > 0$ there exist positive constants $c(\delta)$, $C(\delta) > 0$, such that, for all $n \geq 1$,*

$$\mathbb{P}_{\lambda_c - C(\delta)\alpha_n}[\text{cross}(n, 2n)] \leq \delta, \qquad \mathbb{P}_{\lambda_c + C(\delta)\alpha_n}[\text{cross}(2n, n)] \geq 1 - \delta, \tag{14}$$

and

$$\big|\mathbb{P}_{\lambda}[\text{cross}(n)] - \mathbb{P}_{\lambda_c}[\text{cross}(n)]\big| \leq \delta \quad \text{when } |\lambda - \lambda_c| < c(\delta)\alpha_n. \tag{15}$$

The technique developed in [19] is easily adapted to continuum percolation when the radii of the disks are fixed (as is the case here), or have compact support in $(0, +\infty)$. Beyond these situations, conditions on the tails of the distribution of radii towards 0 and ∞ are necessary, and the adaptation would require significant additional work.

In the rest of this section, we give an overview of the classical argument in [19] and discuss how it needs to be adapted to continuum percolation in order for it to produce Theorem 3. We start by sketching Kesten's argument in our context.

For $\delta > 0$ define the *characteristic length* at $\lambda > 0$ as

$$L_\delta(\lambda) := \begin{cases} \inf\{n \geq 1 \colon \mathbb{P}_\lambda(\text{cross}(n)) \geq 1 - \delta\} & \text{if } \lambda > \lambda_c, \\ \inf\{n \geq 1 \colon \mathbb{P}_\lambda(\text{cross}^*(n)) \geq 1 - \delta\} & \text{if } \lambda < \lambda_c. \end{cases}$$

Fix δ for the rest of the section and omit it from the notation. We use the notation $\asymp$ to relate two quantities whose ratios are uniformly bounded, with constants that may depend on δ.

A first step in the proof of Theorem 3 is to observe that the RSW theory implies uniform bounds on crossing probabilities 'under the characteristic length'. Indeed, Proposition 2 implies that for any $\rho > 0$ there exists $c = c(\rho) > 0$ such that

$$c < \mathbb{P}_\lambda(\text{cross}(\rho n, n)) < 1 - c \quad \text{for all } \lambda \text{ and } 1 \leq n \leq L(\lambda). \tag{16}$$

This fact will be used implicitly in the following arguments.

Observe now that Russo's formula (9) applied to the event cross(n) reads

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{P}_\lambda[\text{cross}(n)] = \int_{[-n,n]^2} \mathbb{P}_\lambda[\text{Piv}_x(\text{cross}(n))]\,\mathrm{d}x. \tag{17}$$

For points x in the bulk of $[-n, n]^2$, i.e. at a distance of order n from the boundary, the probability of being pivotal may be approximated by that of the four-arm event. A slightly involved

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analysis shows that the bulk provides the major contribution to (17) when $1 \le n \le L(\lambda)$; indeed, for points close to the boundary of $[-n, n]^2$ to be pivotal, they need to exhibit certain arm events in the half-plane, which render their contribution to (17) negligible. Thus we find that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{P}_\lambda[\mathrm{cross}(n)] \asymp n^2\,\mathbb{P}_\lambda[A_4(1, n)] \quad \text{for all } 1 \le n \le L(\lambda). \tag{18}$$

Similar reasoning may be used to bound the logarithmic derivative of $\mathbb{P}_\lambda[A_4(1, n)]$ by $c_0 n^2\,\mathbb{P}_\lambda[A_4(1, n)]$ for some universal constant c_0. Integrating both of these expressions, we conclude that

$$\left|\log\frac{\mathbb{P}_\lambda[A_4(1, n)]}{\mathbb{P}_{\lambda_c}[A_4(1, n)]}\right| \le c_0(\mathbb{P}_\lambda[\mathrm{cross}(n)] - \mathbb{P}_{\lambda_c}[\mathrm{cross}(n)]) \quad \text{for any } \lambda > 0 \text{ and } 1 \le n \le L(\lambda).$$

Now, since the right-hand side here is contained in $[-c_0, c_0]$, we conclude that

$$\mathbb{P}_\lambda[A_4(1, n)] \asymp \mathbb{P}_{\lambda_c}[A_4(1, n)] \quad \text{for any } \lambda > 0 \text{ and } 1 \le n \le L(\lambda). \tag{19}$$

This result is known as the *stability of arm-event probabilities* within the critical window. It is the crucial step in the argument in [19, Theorem 1 and Lemma 8] and in its extensions [11, 29]; see [24, Theorem 23] for a more modern exposition. A more direct proof of (19) is the subject of [12].

Finally, plugging (19) back into (18) and integrating, we find that

$$\mathbb{P}_\lambda[\mathrm{cross}(n)] - \mathbb{P}_{\lambda_c}[\mathrm{cross}(n)] \asymp (\lambda - \lambda_c)n^2\pi_4(n) \quad \text{for any } \lambda > 0 \text{ and } 1 \le n \le L(\lambda).$$

This directly implies Theorem 3. The program described above applies to continuum percolation with only slight additions. When proving (18) for the logarithmic derivative of the four-arm event, the argument showing that the bulk provides the majority of the contribution uses the existence of $c > 0$ such that

$$\mathbb{P}_\lambda[A_4(r, R)] \ge c(r/R)^{2-c} \quad \text{for any } \lambda > 0 \text{ and } r < R \le L(\lambda), \tag{20}$$

which may be colloquially stated as 'the four-arm exponent is strictly smaller than two inside the critical window'. Additionally, it uses that the three-arm probability in the half-plane has the universal exponent 2; the proof of [24, Theorem 23] adapts readily to our setting.

To show (20) we can compare the four-arm event to the five-arm one, which has exponent equal to 2 [20, Lemma 5]. For continuum percolation, an appropriate definition of the five-arm event is necessary for this reasoning to function. For $r < R$, let $A_5(r, R)$ be the event that there exist five disjoint paths $\gamma_1, \ldots, \gamma_5$ in $\overline{B(0, R)} \setminus B(0, r)$, each starting on $\partial B(0, r)$ and ending on $\partial B(0, R)$, distributed in counterclockwise order, with $\gamma_1, \gamma_3, \gamma_5 \in \mathcal{O}$ and $\gamma_2, \gamma_4 \in \mathcal{V}$ and such that there exist two disjoint families of disks in $\mathcal{O}$, the first covering γ_1 and the second covering γ_5. The disks of the two families may overlap, but no disk is allowed to belong to both families.

Using this definition, the same arguments as for Bernoulli percolation on the square lattice (see [20, Lemma 5], [24, Theorem 23], or [29, Proposition 1.13]) show the existence of $c > 0$ such that

$$\mathbb{P}_\lambda[A_5(r, R)] \ge c(r/R)^2, \qquad \mathbb{P}_\lambda[A_4(r, R)] \ge (R/r)^c\,\mathbb{P}_\lambda[A_5(r, R)] \tag{21}$$

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for any $\lambda > 0$ and $r < R \le L(\lambda)$. Let us mention here that the specific definition of the five-arm event A_5 is necessary for the proof of the second inequality. Indeed, this inequality should be understood as 'given the existence of four arms, the existence of a fifth arm comes at a polynomial cost'. To prove this, first explore the interface between the two pairs of prima/dual arms required by $A_4(r, R)$ (say between γ_1 and γ_2 and between γ_3 and γ_4). Then, for $A_5(r, R)$ to be realized, another occupied arm γ_5 should exist between γ_1 and γ_4, which uses none of the disks already explored. This last property ensures that the existence of γ_5 conditionally on the explored arms $\gamma_1, \dots, \gamma_4$ is bounded by the one-arm probability, which in turn is dominated by $(r/R)^c$ for some constant $c > 0$ due to (16).

Finally, (21) implies (20), and Theorem 3 follows.

3. Different couplings: A key observation

For a point process configuration η and $r \ge 0$, set $\mathcal{O}^{(r)} = \bigcup_{x \in \mathrm{supp}(\eta)} B(x, r)$. With this notation, $\mathcal{O} = \mathcal{O}^{(1)}$. Write $\mathcal{O}^{(r)} \in \mathrm{cross}(n)$ for the event that $\mathcal{O}^{(r)}$ contains a horizontal crossing of $[-n, n]^2$. More generally, write $\mathcal{O}^{(r)} \in \mathrm{cross}(m, n)$ for the event that $\mathcal{O}^{(r)}$ contains a path crossing $[-m, m] \times [-n, n]$ horizontally. Finally, set $\mathcal{V}^{(r)} = \mathbb{R}^2 \setminus \mathcal{O}^{(r)}$ and write $\mathcal{V}^{(r)} \in \mathrm{cross}^*(n)$ for the event that $\mathcal{V}^{(r)}$ contains a horizontal crossing of $[-n, n]^2$.

The key to our argument is contained in the following two simple observations.

Lemma 2. *For any $\lambda > 0$,*

$$\mathrm{w}_n^* = 2 \sup\{\varepsilon \ge 0 \colon \mathcal{V}^{(1+\varepsilon)} \in \mathrm{cross}^*(n)\}, \tag{22}$$

$$\mathrm{w}_n \ge 2\sqrt{1 - \inf\{r \le 1 \colon \mathcal{O}^{(r)} \in \mathrm{cross}(n + \sqrt{1 - r^2}, n - 1)\}^2}, \tag{23}$$

where the supremum and infimum are considered equal to 0 and 1, respectively, if the set in question is empty.

Remark 2. Lemma 2 provides only a lower bound on the width of occupied crossings. This is for good reason, as the two quantities in (23) are not generally equal. For $\lambda \le \lambda_c$, it is expected that the two quantities are typically equal, but that fails for general values of λ. Indeed, for λ sufficiency large, we can typically find $\mathrm{w}_n > 2$ due to the creation of crossings of $[-n, n]^2$ by 'double paths' of disks.

The use of a slightly longer and thinner rectangle in the right-hand side of (23) rather than simply $\mathrm{cross}(n)$ is for technical reasons explained in the proof below.

Proof. We start with (22). The equality is trivial when $\mathcal{V} \notin \mathrm{cross}^*(n)$, as both terms are equal to 0. Fix η for which $\mathcal{V} \in \mathrm{cross}^*(n)$. Figure 4 may be useful in understanding this proof.

Fix $0 < \varepsilon < \mathrm{w}^*(n)/2$ and let γ be a horizontal crossing of $[-n, n]^2$ for which $\mathrm{dist}(\gamma, \mathcal{O}) > \varepsilon$; the existence of such a path is guaranteed by the definition in (1) of w_n^*. Then $\gamma \in \mathcal{V}^{(1+\varepsilon)}$, and therefore $\mathcal{V}^{(1+\varepsilon)} \in \mathrm{cross}^*(n)$. This allows us to conclude that

$$\mathrm{w}_n^* \le 2 \cdot \sup\{\varepsilon \ge 0 \colon \mathcal{V}^{(1+\varepsilon)} \in \mathrm{cross}^*(n)\}.$$

Conversely, for ε such that $\mathcal{V}^{(1+\varepsilon)} \in \mathrm{cross}^*(n)$, let γ be a horizontal crossing of $[-n, n]$ contained in $\mathcal{V}^{(1+\varepsilon)}$. Then $\mathrm{dist}(\gamma, \mathcal{O}) \ge \varepsilon$, and therefore $\mathrm{w}_n^* \ge 2\varepsilon$. This proves that

$$\mathrm{w}_n^* \ge 2 \cdot \sup\{\varepsilon \ge 0 \colon \mathcal{V}^{(1+\varepsilon)} \in \mathrm{cross}^*(n)\}.$$

Combine the last two displays to obtain the equality in (22).

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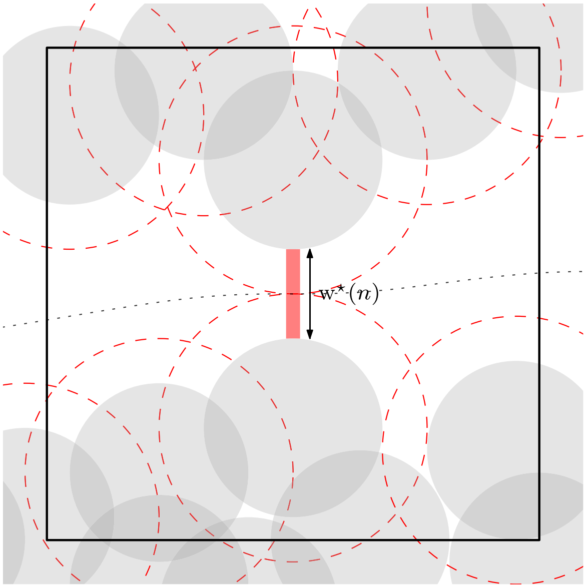

FIGURE 4. Computing the width of a vacant crossing by enlarging the balls.

We move on to the inequality (23), namely that involving w_n. Figure 5 may be useful for this for this part. The inequality is trivial when the right-hand side is equal to 0, and thus without loss of generality we assume it is strictly positive. Fix η and $r < 1$ such that $\mathcal{O}^{(r)} \in \mathrm{cross}(n + \sqrt{1-r^2}, n-1)$. Then there necessarily exists a family $x_0, \ldots, x_k \in \eta \cap [-n, n]^2$ such that $|x_i - x_{i-1}| \leq 2r$ for all $i = 1, \ldots, k$, and with x_0 and x_k at a distance at least $1 - \sqrt{1-r^2}$ from the left and right sides of $[-n, n]^2$, respectively.

Now consider γ to be the shortest path going through the vertices $x_0, \ldots, x_k$ in this order. Furthermore, add to γ the initial horizontal segment going from the left side of $[-n, n]^2$ to x_0 and the final horizontal segment going from x_k to the right side of $[-n, n]^2$. Then γ crosses $[-n, n]^2$ horizontally.

Now, as may be observed in Figure 5,

$$\frac{1}{2}\mathrm{w}_n \geq \mathrm{dist}(\gamma, \mathcal{V}) \geq \mathrm{dist}\left(\gamma, \left[\bigcup_{i=0}^{k} B(x_i, 1)\right]^{\mathrm{c}}\right) \geq \sqrt{1-r^2}.$$

By considering a crossing in $\mathcal{O}^{(r)}$ of a rectangle slightly thinner than $[-n, n]^2$, we ensure that the points $x_0, \ldots, x_k$ do not approach the top and bottom boundaries too closely. We also consider that a slightly longer rectangle as the distance between γ and $\mathcal{V}$ could in principle be attained at the endpoint of γ and be strictly smaller than that between γ and $\mathcal{V} \cap [-n, n]^2$.

Finally, taking the infimum over r, we find (23). □

The second key observation is related to the scaling properties of Poisson point processes.

Lemma 3. *For every $\lambda > 0$ and $r > 0$, the law of $(1/r)\mathcal{O}^{(r)}$ under $\mathbb{P}_\lambda$ is equal to the law of $\mathcal{O}$ under $\mathbb{P}_{\lambda r}$. In particular, $\mathbb{P}_\lambda[\mathcal{O}^{(r)} \in \mathrm{cross}(n)] = \mathbb{P}_{\lambda r}[\mathcal{O} \in \mathrm{cross}(n/r)]$.*

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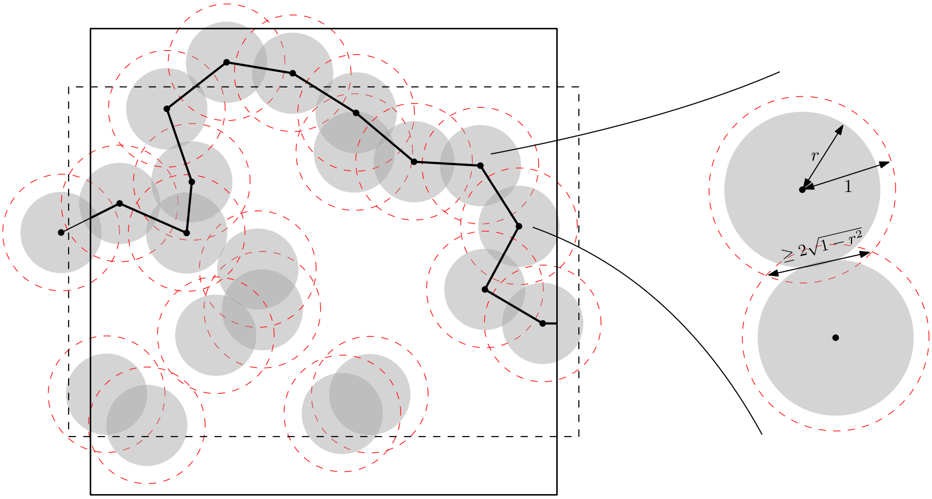

FIGURE 5. When $\mathcal{O}^{(r)} \in \text{cross}(n+\sqrt{1-r^2}, n-1)$, we can identify a chain of points of η, each at a distance at most $2r$ from the previous, contained in $\mathbb{R} \times [-n, n]$, with the first and last within a distance r of the left and right sides of the rectangle, respectively. The path γ (bold black path) is obtained by interpolating linearly between these points, and potentially connecting the first and last points by horizontal lines to the sides of $[-n, n]^2$. The distance from γ to $\left[\bigcup_{i=0}^{k} B(x_i, 1)\right]^c$ is attained at the center of one of the segments $[x_{i-1}, x_i]$ or at the endpoints of γ.

Proof. Fix η. Observe that $(1/r)\mathcal{O}^{(r)} = \bigcup_{x\in(1/r)\text{supp}(\eta)} B(x, 1)$. Now, if η is a Poisson point process of intensity λ, then $(1/r)\text{supp}(\eta)$ is distributed as a Poisson point process of intensity λr. The result follows. □

The two lemmas above combine to prove the following corollary.

Corollary 1. *For any $a \geq 0$ and $\lambda > 0$,*

$$\mathbb{P}_\lambda[\mathrm{w}_n^* \leq 2a] = \mathbb{P}_{\lambda(1+a)}\left[\mathcal{O} \in \text{cross}\left(\frac{n}{1+a}\right)\right], \tag{24}$$

$$\mathbb{P}_\lambda[\mathrm{w}_n \geq 2a] \geq \mathbb{P}_{\lambda\sqrt{1-a^2}}\left[\mathcal{O} \in \text{cross}\left(\frac{n+a}{\sqrt{1-a^2}}, \frac{n-1}{\sqrt{1-a^2}}\right)\right]. \tag{25}$$

Proof. We start with (24). Observe that, due to (22) and the fact that w_n^* has no strictly positive atoms,

$$\mathbb{P}_\lambda[\mathrm{w}_n^* \leq 2a] = \mathbb{P}_\lambda[\mathcal{V}^{(1+a)} \notin \text{cross}^*(n)] = \mathbb{P}_\lambda[\mathcal{O}^{(1+a)} \in \text{cross}(n)] = \mathbb{P}_{\lambda(1+a)}\left[\mathcal{O} \in \text{cross}\left(\frac{n}{1+a}\right)\right],$$

for any $a > 0$. The second equality is due to the duality property and the invariance under rotation by $\pi/2$; the last equality is due to Lemma 3.

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We now turn to (25). By (23) we have $\{\mathrm{w}_n \geq 2a\} \supset \{\mathcal{O}^{(\sqrt{1-a^2})} \in \mathrm{cross}(n+a, n-1)\}$. Therefore, employing Lemma 3 we find that

$$\begin{aligned}\mathbb{P}_\lambda[\mathrm{w}_n \geq 2a] &\geq \mathbb{P}_\lambda\big[\mathcal{O}^{(\sqrt{1-a^2})} \in \mathrm{cross}(n+a, n-1)\big] \\ &= \mathbb{P}_{\lambda\sqrt{1-a^2}}\left[\mathcal{O} \in \mathrm{cross}\left(\frac{n+a}{\sqrt{1-a^2}}, \frac{n-1}{\sqrt{1-a^2}}\right)\right]. \qquad \square\end{aligned}$$

4. Proof of Theorem 1 and of the lower bounds in Theorem 2

The proofs of Theorem 1 and of the lower bounds in Theorem 2 are easy consequences of Corollary 1. As Corollary 1 or Lemma 2 give no upper bounds on w_n, the upper bounds in (6) and (7) will be harder to prove. They are postponed to the next section.

Recall the notation $\alpha_n = 1/n^2\pi_4(n)$.

Proof of Theorem 1. Let us start with the **critical** case, $\lambda = \lambda_c$. Fix $\delta > 0$. Then, for $C > c > 0$ and $n \geq 1$, due to (24),

$$\begin{aligned}&1 - \mathbb{P}_{\lambda_c}[2c\alpha_n \leq \mathrm{w}_n^* \leq 2C\alpha_n \mid \mathrm{cross}^*(n)] \\ &\quad = \frac{1}{\mathbb{P}_{\lambda_c}[\mathrm{cross}^*(n)]}\big(\mathbb{P}_{\lambda_c}[\mathrm{w}_n^* < 2c\alpha_n] - \mathbb{P}_{\lambda_c}[\mathrm{w}_n^* = 0] + \mathbb{P}_{\lambda_c}[\mathrm{w}_n^* > 2C\alpha_n]\big) \\ &\quad \leq C_1\left(\mathbb{P}_{\lambda_c(1+c\alpha_n)}\left[\mathrm{cross}\left(\frac{n}{1+c\alpha_n}\right)\right] - \mathbb{P}_{\lambda_c}[\mathrm{cross}(n)] + 1 - \mathbb{P}_{\lambda_c(1+C\alpha_n)}\left[\mathrm{cross}\left(\frac{n}{1+C\alpha_n}\right)\right]\right),\end{aligned}$$

where C_1 is a universal constant provided by (11). We used Corollary 1 to relate the bounds on w_n^* to crossing events. Applying Theorem 3, we deduce that c and C may be chosen small and large enough, respectively, independently of n, such that the above is bounded as

$$1 - \mathbb{P}_{\lambda_c}[2c\alpha_n \leq \mathrm{w}_n^* \leq 2C\alpha_n \mid \mathrm{cross}^*(n)] \leq \delta + C_1\left(\mathbb{P}_{\lambda_c}\left[\mathrm{cross}\left(\frac{n}{1+c\alpha_n}\right)\right] - \mathbb{P}_{\lambda_c}[\mathrm{cross}(n)]\right).$$

Specifically, we have used the left-hand side in (14) to find C large enough that, for the last term,

$$1 - \mathbb{P}_{\lambda_c(1+C\alpha_n)}\left[\mathrm{cross}\left(\frac{n}{1+C\alpha_n}\right)\right] \leq \delta/C_1,$$

and (15) to find c small enough that, for the first term,

$$\mathbb{P}_{\lambda_c(1+c\alpha_n)}\left[\mathrm{cross}\left(\frac{n}{1+c\alpha_n}\right)\right] \leq \mathbb{P}_{\lambda_c}\left[\mathrm{cross}\left(\frac{n}{1+c\alpha_n}\right)\right] + \delta/C_1.$$

Finally, employing the continuity of crossing probabilities, since $\alpha_n \to 0$, we see that the last term above may also be rendered arbitrarily smaller than δ, provided that n is large enough.

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We now consider the **sub-critical** case, $\lambda < \lambda_c$. As in the critical case, applying Corollary 1 yields, for any $n \geq 1$ and $C > 0$,

$$\mathbb{P}_\lambda\left[\left|w_n^* - 2\left(\frac{\lambda_c}{\lambda} - 1\right)\right| > 2C\alpha_n \mid \text{cross}^*(n)\right]$$
$$\leq \frac{1}{\mathbb{P}_\lambda[\text{cross}^*(n)]}\left(\mathbb{P}_\lambda\left[w_n^* < 2\left(\frac{\lambda_c - \lambda}{\lambda} - C\alpha_n\right)\right] + \mathbb{P}_\lambda\left[w_n^* > 2\left(\frac{\lambda_c - \lambda}{\lambda} + C\alpha_n\right)\right]\right)$$
$$\leq \frac{1}{\mathbb{P}_\lambda[\text{cross}^*(n)]}\big(\mathbb{P}_{\lambda_c - C\lambda\alpha_n}[\text{cross}(n_-)] + 1 - \mathbb{P}_{\lambda_c + C\lambda\alpha_n}[\text{cross}(n_+)]\big), \tag{26}$$

where

$$n_\pm = \frac{n}{\big(1 + ((\lambda_c - \lambda)/\lambda) \pm C\alpha_n\big)}.$$

Now, due to (14), C may be chosen large enough (depending on λ, but not on n) that

$$\mathbb{P}_{\lambda_c - C\lambda\alpha_n}[\text{cross}(n_-)] \leq \delta, \qquad 1 - \mathbb{P}_{\lambda_c + C\lambda\alpha_n}[\text{cross}(n_+)] \leq \delta$$

for all n large enough. Finally, for all n sufficiently large, the prefactor on the right-hand side of (26) is smaller than 2. Combining the above inequalities leads to the desired conclusion.

Finally, let us consider the **super-critical** case, (5). We will treat the lower and upper bounds on w_n^* separately. We start with the former.

Due to (24), for any $c > 0$ and $n \geq 1$, we have

$$\mathbb{P}_\lambda\left[w_n^* < \frac{2c}{n} \mid \text{cross}^*(n)\right] = \frac{1}{\mathbb{P}_\lambda[\text{cross}^*(n)]}\left(\mathbb{P}_{\lambda(1+c/n)}\left[\text{cross}\left(\frac{n}{1+c/n}\right)\right] - \mathbb{P}_\lambda[\text{cross}(n)]\right)$$
$$\leq \frac{1}{\mathbb{P}_\lambda[\text{cross}(n)^c]}\big(\mathbb{P}_{\lambda(1+c/n)}[\text{cross}(n-c, n) \setminus \text{cross}(n)] + \mathbb{P}_{\lambda(1+c/n)}[\text{cross}(n)] - \mathbb{P}_\lambda[\text{cross}(n)]\big), \tag{27}$$

where the inequality is obtained by adding and subtracting $\mathbb{P}_{\lambda(1+c/n)}[\text{cross}(n)]$ and using the inclusion of rectangles. We will bound separately the first term and the difference of the last two terms appearing in the parentheses above. We start with the latter.

Consider the measure P which consists in choosing a Poisson process η of intensity λ and an independent additional Poisson point process $\tilde{\eta}$ of intensity $c\lambda/n$. Write $\mathcal{O}$ and $\tilde{\mathcal{O}}$ for the occupied sets produced by these two processes. Then

$$\frac{\mathbb{P}_{\lambda(1+c/n)}[\text{cross}(n)] - \mathbb{P}_\lambda[\text{cross}(n)]}{\mathbb{P}_\lambda[\text{cross}(n)^c]} = P[\mathcal{O} \cup \tilde{\mathcal{O}} \in \text{cross}(n) \mid \mathcal{O} \notin \text{cross}(n)]. \tag{28}$$

Now, when $\mathcal{O} \notin \text{cross}(n)$, write $\mathcal{C}$ for the union of the vacant clusters crossing $[-n, n]^2$ vertically. Also, write $\mathcal{C}^{(1)} = \{x + z \colon x \in \mathcal{C},\ |z| \leq 1\}$ for the fattening of $\mathcal{C}$ by 1 and $\mathcal{A}(\mathcal{C}^{(1)})$ for the area covered by $\mathcal{C}^{(1)}$.

Since $\lambda > \lambda_c$, the conditioning on $\mathcal{O} \notin \text{cross}(n)$ is very degenerate, which renders the typical cluster $\mathcal{C}$ very thin. Indeed, a straightforward adaptation of the theory of [7] induces the existence of a constant $C(\delta) > 0$ such that

$$P[\mathcal{A}(\mathcal{C}^{(1)}) \geq C(\delta)n \mid \mathcal{O} \notin \text{cross}(n)] < \delta \quad \text{for all } n. \tag{29}$$

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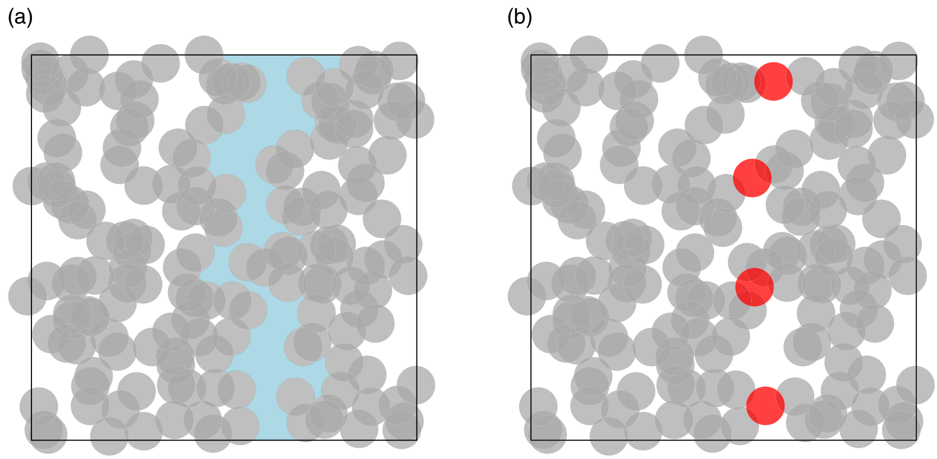

FIGURE 6. (a) In the super-critical regime, when cross(n) fails, the vacant cluster crossing $[-n, n]^2$ vertically is thin; it typically has an area $\mathcal{A}(\mathcal{C})$ of order n. (b) In the same situation, there exists a linear number of places where adding one disk induces an occupied horizontal crossing. The centers of these potential disks form $\mathbb{P}i$.

(The quoted lemma states that, with high probability, the crossing cluster contains a linear number of regeneration points. An additional consequence of the mass gap principle, proved in the same way, is that the gap between successive regeneration points has exponential tails. The cluster is contained in the cones formed by these regeneration points, which delimit a total volume of linear order with exponentially high probability. A more detailed description of the structure of the crossing cluster is given in [9, Section 1.2 and Theorem 3.1] in the context of FK percolation.) See Figure 6 (left) for an illustration. Due to the independence of $\mathcal{O}$ and $\tilde{\mathcal{O}}$, the probability that a disk of $\tilde{\mathcal{O}}$ intersects $\mathcal{C}$ may be computed as

$$P[\tilde{\mathcal{O}} \cap \mathcal{C} \neq \emptyset \mid \mathcal{O}] = 1 - \exp\left[-c\lambda \mathcal{A}(\mathcal{C}^{(1)})/n\right].$$

Fix $c > 0$ sufficiently small that $\exp(-c\lambda C(\delta)) > 1 - \delta$, with $C(\delta)$ the constant given by (29). Then we find that

$$P[\mathcal{O} \cup \tilde{\mathcal{O}} \in \text{cross}(n) \mid \mathcal{O} \notin \text{cross}(n)]$$

$$\leq P[\mathcal{A}(\mathcal{C}^{(1)}) \geq C(\delta)n \mid \mathcal{O} \notin \text{cross}(n)] + P\big[\mathcal{C} \cap \tilde{\mathcal{O}} \neq \emptyset \mid \mathcal{O} \notin \text{cross}(n),\ \mathcal{A}(\mathcal{C}^{(1)}) < C(\delta)n\big] < 2\delta.$$

Combined with (28), this yields

$$\frac{1}{\mathbb{P}_\lambda[\text{cross}(n)^c]}\big(\mathbb{P}_{\lambda(1+c/n)}[\text{cross}(n)] - \mathbb{P}_\lambda[\text{cross}(n)]\big) \leq 2\delta. \tag{30}$$

We now turn to $\mathbb{P}_{\lambda(1+c/n)}[\text{cross}(n-c, n) \setminus \text{cross}(n)]$. For this event to occur, a vacant vertical crossing of $[-n, n]^2$ needs to exist that visits $[-n, -n+c] \times [-n, n]$ or $[n-c, n] \times [-n, n]$. Since any such crossing is essentially straight [9, Theorem C] (i.e. has width of $o(n)$), we find that, for n large enough,

$$\mathbb{P}_{\lambda(1+c/n)}[\text{cross}(n-c, n) \mid \text{cross}(n)^c] \leq \delta.$$

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(The crossing cluster is unique by the positivity of the correlation length; it travels in the vertical direction due to the strict convexity of the Wulff shape, which in the present context is a Euclidean ball. Finally, the horizontal coordinate of its starting point is almost uniform in $[-n, n]$, except for a slight repulsion at the edges of the interval.) Thus,

$$\mathbb{P}_{\lambda(1+c/n)}[\mathrm{cross}(n-c, n) \setminus \mathrm{cross}(n)] \leq \delta\, \mathbb{P}_{\lambda(1+c/n)}[\mathrm{cross}(n)^{\mathrm{c}}] \leq \delta\, \mathbb{P}_{\lambda}[\mathrm{cross}(n)^{\mathrm{c}}], \tag{31}$$

where the second inequality comes from the monotonicity in λ. The term δ in the first inequality could even be replaced $O(1/n)$ by studying the vacant cluster more carefully, but this is unnecessary for our purposes.

Inserting (30) and (31) in (27), we find that

$$\mathbb{P}_{\lambda}\left[\mathrm{w}_n^* < \frac{2c}{n} \mid \mathrm{cross}^*(n)\right] \leq 3\delta, \tag{32}$$

which is the desired lower bound on w_n^*.

We now turn to the upper bound on w_n^*. Using (24), for $C > 0$ and $n \geq 1$,

$$\begin{aligned}\mathbb{P}_{\lambda}\left[\mathrm{w}_n^* > \frac{2C}{n} \mid \mathrm{cross}^*(n)\right] &= \frac{1}{\mathbb{P}_{\lambda}[\mathrm{cross}^*(n)]}\left(1 - \mathbb{P}_{\lambda(1+C/n)}\left[\mathrm{cross}\left(\frac{n}{1+C/n}\right)\right]\right)\\ &\leq \frac{1}{\mathbb{P}_{\lambda}[\mathrm{cross}(n)^{\mathrm{c}}]}\mathbb{P}_{\lambda(1+C/n)}[\mathrm{cross}(n, n-C)^{\mathrm{c}}].\end{aligned}$$

The second inequality is due to the inclusion of rectangles. As in (31),

$$\mathbb{P}_{\lambda(1+C/n)}[\mathrm{cross}(n, n-C)^{\mathrm{c}}] \leq \mathbb{P}_{\lambda(1+C/n)}[\mathrm{cross}(n)^{\mathrm{c}}](1+\delta)$$

for n large enough. Thus,

$$\mathbb{P}_{\lambda}\left[\mathrm{w}_n^* > \frac{2C}{n} \mid \mathrm{cross}^*(n)\right] \leq (1+\delta)\frac{\mathbb{P}_{\lambda(1+C/n)}[\mathrm{cross}(n)^{\mathrm{c}}]}{\mathbb{P}_{\lambda}[\mathrm{cross}(n)^{\mathrm{c}}]}. \tag{33}$$

Using the same notation P, $\mathcal{O}$, and $\tilde{\mathcal{O}}$ as above, with $\tilde{\eta}$ having intensity $C/n\lambda$, we find

$$\mathbb{P}_{\lambda}[\mathrm{cross}(n)^{\mathrm{c}}] - \mathbb{P}_{\lambda(1+C/n)}[\mathrm{cross}(n)^{\mathrm{c}}] = P[\mathcal{O} \notin \mathrm{cross}(n) \text{ but } \mathcal{O} \cup \tilde{\mathcal{O}} \in \mathrm{cross}(n)]. \tag{34}$$

When $\mathcal{O} \notin \mathrm{cross}(n)$, the Ornstein–Zernike theory of [7] states that there typically exists a unique vacant cluster connecting the top and bottom of $\mathrm{cross}(n)$, and that this cluster has a linear number of pivotals. Let us give a precise statement of this fact. For a configuration with $\mathcal{O} \notin \mathrm{cross}(n)$, write $\mathbb{P}i$ for the set of pivotal points, i.e. $\mathbb{P}i := \{x \in \mathbb{R}^2 : \mathcal{O} \cup B(x, 1) \in \mathrm{cross}(n)\}$; see Figure 6(b) for an illustration. Then, an adaptation of [7, Lemma 4.1] to the continuous setting shows the existence of a constant $c(\delta) > 0$ such that $\mathbb{P}_{\lambda}[\mathcal{A}(\mathbb{P}i) > c(\delta)n \mid \mathcal{O} \notin \mathrm{cross}(n)] \geq 1-\delta$, where $\mathcal{A}(\mathbb{P}i)$ denotes the area of $\mathbb{P}i$. (The quoted lemma states that, with exponentially high probability, the crossing cluster contains a linear number of *regeneration* points; each such point may be transformed into a pivotal point by local surgery. The uniqueness of the cluster is due to the positivity of the correlation length.) It follows that

$$\begin{aligned}&P[\mathcal{O} \cup \tilde{\mathcal{O}} \in \mathrm{cross}(n) \mid \mathcal{O} \notin \mathrm{cross}(n)]\\ &\geq P[\mathcal{O} \cup \tilde{\mathcal{O}} \in \mathrm{cross}(n) \mid \mathcal{O} \notin \mathrm{cross}(n) \text{ and } \mathcal{A}(\mathbb{P}i) > c(\delta)n]P[\mathcal{A}(\mathbb{P}i) > c(\delta)n \mid \mathcal{O} \notin \mathrm{cross}(n)]\\ &\geq (1 - \mathrm{e}^{-c(\delta)C\lambda})(1-\delta),\end{aligned}$$

since $1-\mathrm{e}^{-c(\delta)C\lambda}$ bounds from below the probability that $\tilde{\eta}$ contains a pivotal point. Taking $C>0$ large enough, depending on $c(\delta)$ and λ but not on n, we conclude that the above is larger than $1-2\delta$ for all n large enough. Inserting this into (34), we find that $\mathbb{P}_{\lambda(1+C/n)}[\mathrm{cross}(n)^{\mathrm{c}}]\leq 2\delta\,\mathbb{P}_{\lambda}[\mathrm{cross}(n)^{\mathrm{c}}]$. This together with (33) imply that

$$\mathbb{P}_{\lambda}\left[\mathrm{w}_n^*>\frac{2C}{n}\mid \mathrm{cross}^*(n)\right]\leq (1+\delta)2\delta.$$

Finally, this together with (32) yield (5). □

We now turn to the results concerning the maximal width in the occupied set. In this section, we only prove lower bounds for w_n; upper bounds are proved in the next section.

Proof of Theorem 2, lower bounds. Fix $\delta>0$. We will prove in each case that, by taking c small enough, $\mathbb{P}_{\lambda}[\mathrm{w}_n\leq c\,\theta(n)\mid \mathrm{cross}(n)]$ may be rendered smaller than some explicit function of δ that tends to 0 as $\delta\to 0$. The threshold $\theta(n)$ depends on whether λ is smaller than, equal to, or larger than λ_{c}, and is given in Theorem 2.

We start with the **critical** case (7). Then, for $c>0$ and $n\geq 1$, due to (25),

$$\begin{aligned}
&\mathbb{P}_{\lambda_{\mathrm{c}}}[\mathrm{w}_n\leq 2\sqrt{c\alpha_n}\mid \mathrm{cross}(n)]\\
&=\frac{1}{\mathbb{P}_{\lambda_{\mathrm{c}}}[\mathrm{cross}(n)]}(\mathbb{P}_{\lambda_{\mathrm{c}}}[\mathrm{cross}(n)]-\mathbb{P}_{\lambda_{\mathrm{c}}}[\mathrm{w}_n>2\sqrt{c\alpha_n}])\\
&\leq\frac{1}{\mathbb{P}_{\lambda_{\mathrm{c}}}[\mathrm{cross}(n)]}\left(\mathbb{P}_{\lambda_{\mathrm{c}}}[\mathrm{cross}(n)]-\mathbb{P}_{\lambda_{\mathrm{c}}\sqrt{1-c\alpha_n}}\left[\mathrm{cross}\left(\frac{n+\sqrt{c\alpha_n}}{\sqrt{1-c\alpha_n}},\frac{n-1}{\sqrt{1-c\alpha_n}}\right)\right]\right)\\
&\leq\frac{1}{\mathbb{P}_{\lambda_{\mathrm{c}}}[\mathrm{cross}(n)]}\big(\mathbb{P}_{\lambda_{\mathrm{c}}}[\mathrm{cross}(n)]-\mathbb{P}_{\lambda_{\mathrm{c}}(1-c\alpha_n)}[\mathrm{cross}(n+3cn\alpha_n,n-1)]\big),
\end{aligned}\tag{35}$$

provided that n is large enough that $(1-c\alpha_n)^{-1/2}>1+c\alpha_n$ and $n\sqrt{c\alpha_n}\geq 1$. The last inequality uses the monotonicities of $\mathbb{P}_{\lambda}(\mathrm{cross}(a,b))$ in λ, a, and b.

Now, by arguments analogous to those used to prove (15) in Theorem 3, we deduce that c may be chosen such that, for all n,

$$\mathbb{P}_{\lambda_{\mathrm{c}}(1-c\alpha_n)}[\mathrm{cross}(n+3c\alpha_n,n-1)]\geq\mathbb{P}_{\lambda_{\mathrm{c}}}[\mathrm{cross}(n+3cn\alpha_n,n-1)]-\delta\geq\mathbb{P}_{\lambda_{\mathrm{c}}}[\mathrm{cross}(n)]-2\delta,$$

with the second inequality due to Lemma 1, n large enough (depending on c), and we have used the a priori estimates on the four-arms event, see (20). This property is classic in Bernoulli percolation, and its proof is analogous in our case.

Combining the above with (35), and using the RSW inequality (10), we conclude that, for c small enough and all n larger than some threshold,

$$\mathbb{P}_{\lambda_{\mathrm{c}}}[\mathrm{w}_n\leq 2\sqrt{c\alpha_n}\mid \mathrm{cross}(n)]\leq C_0\delta,\tag{36}$$

where C_0 is a universal constant.

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We continue with the **super-critical** case (8). Fix $\lambda > \lambda_c$. As in the critical case, applying (25) yields, for $C > 0$ and all n large enough,

$$\begin{aligned}
&\mathbb{P}_\lambda\left[\mathrm{w}_n \geq 2\sqrt{1-\left(\frac{\lambda_c}{\lambda}+\frac{C}{\lambda}\alpha_n\right)^2} \mid \mathrm{cross}(n)\right] \\
&\quad \geq \frac{1}{\mathbb{P}_\lambda[\mathrm{cross}(n)]}\mathbb{P}_{\lambda_c+C\alpha_n}\left[\mathrm{cross}\left(\frac{\lambda(n+\sqrt{1-((\lambda_c/\lambda)+(C/\lambda)\alpha_n)^2})}{\lambda_c+C\alpha_n}, \frac{\lambda(n-1)}{\lambda_c+C\alpha_n}\right)\right] \\
&\quad \geq \frac{1}{\mathbb{P}_\lambda[\mathrm{cross}(n)]}\mathbb{P}_{\lambda_c+C\alpha_n}[\mathrm{cross}(2c(\lambda)n, c(\lambda)n)],
\end{aligned}$$

where $c(\lambda)$ is a constant depending only on λ. Theorem 3 states that C may be chosen large enough that the second term on the right-hand side of the above is larger than $1-\delta$ for all n. The first term on the right-hand side above converges to 1 as $n \to \infty$ due to the choice of λ being super-critical. Thus, for C chosen sufficiently large and all n large enough,

$$\mathbb{P}_\lambda\left[\mathrm{w}_n \geq 2\sqrt{1-\left(\frac{\lambda_c}{\lambda}+\frac{C}{\lambda}\alpha_n\right)^2} \mid \mathrm{cross}(n)\right] \geq 1-2\delta,$$

as claimed.

Finally, let us analyze the **sub-critical** case (6). The strategy is the same as for the super-critical case (5) for the vacant set. Fix $\lambda < \lambda_c$. Due to (25), for any $c > 0$ and $n \geq 1$, we have

$$\begin{aligned}
&\mathbb{P}_\lambda\left[\mathrm{w}_n < 2\sqrt{\frac{c}{n}} \mid \mathrm{cross}(n)\right] \\
&\leq \frac{1}{\mathbb{P}_\lambda[\mathrm{cross}(n)]}\left(\mathbb{P}_\lambda[\mathrm{cross}(n)] - \mathbb{P}_{\lambda\sqrt{1-(c/n)}}\left[\mathrm{cross}\left(\frac{n+\sqrt{c/n}}{\sqrt{1-(c/n)}}, \frac{n-1}{\sqrt{1-(c/n)}}\right)\right]\right) \\
&\leq \frac{1}{\mathbb{P}_\lambda[\mathrm{cross}(n)]}\left(\mathbb{P}_\lambda[\mathrm{cross}(n)] - \mathbb{P}_{\lambda\sqrt{1-(c/n)}}[\mathrm{cross}(n)]\right. \\
&\qquad\left. + \mathbb{P}_{\lambda\sqrt{1-(c/n)}}[\mathrm{cross}(n)] - \mathbb{P}_{\lambda\sqrt{1-(c/n)}}\left[\mathrm{cross}\left(\frac{n+\sqrt{c/n}}{\sqrt{1-(c/n)}}, \frac{n-1}{\sqrt{1-(c/n)}}\right)\right]\right).
\end{aligned} \tag{37}$$

We will bound separately the two differences appearing in the parentheses above, starting with the first one.

Consider the measure P which consists in choosing a Poisson process η of intensity $\lambda\sqrt{1-c/n}$ and an independent additional Poisson point process $\tilde{\eta}$ of intensity $\lambda(1-\sqrt{1-c/n})$. Write $\mathcal{O}$ and $\tilde{\mathcal{O}}$ for the occupied sets produced by these two processes. Then

$$\mathbb{P}_\lambda[\mathrm{cross}(n)] - \mathbb{P}_{\lambda\sqrt{1-c/n}}[\mathrm{cross}(n)] = P[\mathcal{O} \notin \mathrm{cross}(n) \text{ but } \mathcal{O}\cup\tilde{\mathcal{O}} \in \mathrm{cross}(n)]. \tag{38}$$

Now, when $\mathcal{O}\cup\tilde{\mathcal{O}} \in \mathrm{cross}(n)$, write $\mathcal{C}$ for the union of the occupied clusters crossing $[-n, n]^2$ horizontally. Since $\lambda < \lambda_c$, $\mathcal{C}$ is typically formed of a single, thin cluster. As before, [7] implies that this cluster is of linear 'volume', both in area and in the number of disks belonging to it. Thus, there exists a constant $C(\delta) > 0$ such that

$$\mathbb{P}_\lambda\left[|(\eta\cup\tilde{\eta})\cap\mathcal{C}| \geq C(\delta)n \mid \mathcal{O}\cup\tilde{\mathcal{O}} \in \mathrm{cross}(n)\right] < \delta \quad \text{for all } n. \tag{39}$$

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Now, under P and conditionally on $\eta \cup \tilde{\eta}$, each point of $\eta \cup \tilde{\eta}$ belongs to η with a probability $\sqrt{1-c/n}$. Thus, whenever the event in (39) occurs,

$$P[\tilde{\eta} \cap \mathcal{C} = \emptyset \mid \eta \cup \tilde{\eta}] \geq \left(1 - \frac{c}{n}\right)^{C(\delta)n/2} \geq 1 - \delta, \tag{40}$$

provided that $c > 0$ is chosen sufficiently small. Inserting (39) and (40) in (38), we find that

$$\mathbb{P}_{\lambda(1-c/n)}[\mathrm{cross}(n)] \geq (1 - 2\delta)\mathbb{P}_{\lambda}[\mathrm{cross}(n)]. \tag{41}$$

We now turn to the second difference in (37). Assuming that $c > 0$ is sufficiently small and n sufficiently large, we have

$$\begin{aligned}\mathbb{P}_{\lambda(1-c/n)}\left[\mathrm{cross}\left(\frac{n+\sqrt{c/n}}{\sqrt{1-c/n}}, \frac{n-1}{\sqrt{1-c/n}}\right)\right] &\geq \mathbb{P}_{\lambda(1-c/n)}[\mathrm{cross}(n+2c, n-1)] \\ &\geq (1-\delta)\mathbb{P}_{\lambda(1-c/n)}[\mathrm{cross}(n+2c, n)] \\ &\geq (1-2\delta)\mathbb{P}_{\lambda(1-c/n)}[\mathrm{cross}(n)].\end{aligned}$$

The first inequality is due to the inclusion of rectangles and some basic algebra. The second is obtained in the same way as (31) and is valid for n large enough; the occupied component producing $\mathrm{cross}(n+2c, n-1)$ avoids approaching the top and bottom sides of the rectangles with high probability. The third is valid for c small enough (independent of n) and is based on the same reasoning as Lemma 1, namely that, for all $k \leq 1/c$,

$$\mathbb{P}_{\lambda(1-c/n)}\big[\mathrm{cross}(n+2(k+1)c, n) \setminus \mathrm{cross}(n+2kc, n)\big] \geq c_0 \mathbb{P}_{\lambda(1-c/n)}\big[\mathrm{cross}(n+2c, n) \setminus \mathrm{cross}(n)\big],$$

where $c_0 > 0$ is some constant depending only on λ, not on n or c. Finally, the above combined with (41) shows that

$$\begin{aligned}\mathbb{P}_{\lambda(1-c/n)}[\mathrm{cross}(n)] - \mathbb{P}_{\lambda(1-c/n)}\left[\mathrm{cross}\left(\frac{n+\sqrt{c/n}}{\sqrt{1-c/n}}, \frac{n-1}{\sqrt{1-c/n}}\right)\right] &\leq 2\delta\mathbb{P}_{\lambda(1-c/n)}[\mathrm{cross}(n)] \\ &\leq 2\delta\mathbb{P}_{\lambda}[\mathrm{cross}(n)], \end{aligned} \tag{42}$$

where the second inequality comes from the monotonicity in λ. Putting (41) and (42) together, we conclude that

$$\mathbb{P}_{\lambda}\big[\mathrm{w}_n < 2\sqrt{c/n} \mid \mathrm{cross}(n)\big] \leq 4\delta, \tag{43}$$

as claimed.

5. Remaining proofs

In this section we prove the upper bounds (6) and (7) on w_n in the sub-critical and critical cases. These could not be proved using the techniques of the previous section due to the missing upper bound on w_n in (25).

The method presented here could most likely be used to obtain all the results in Theorems 1 and 2, but is less elegant than that of Section 4 and would require significantly more work.

The arguments in this section use some fine properties of critical and sub-critical percolation, which we will state explicitly, but not prove. Proofs are available in the literature for

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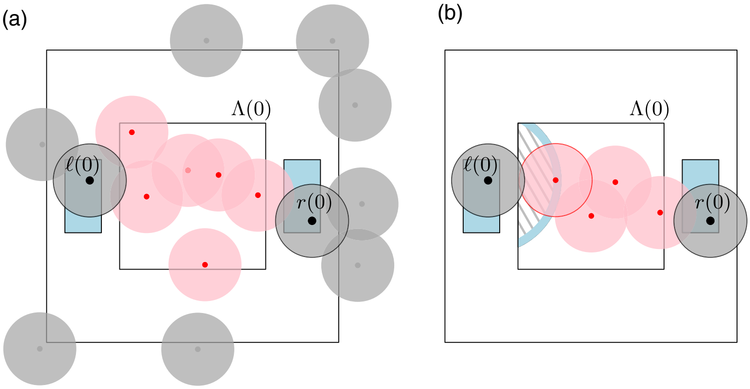

FIGURE 7. (a) A thin connected point. The only two disks with centers in $[-4, 4]^2 \setminus \Lambda(0)$ are marked in bold; their centers belong to the two blue regions on the side of $\Lambda(0)$. They are connected by the pink disks with centers in $\Lambda(0)$. (b) For $0 < \mathrm{w}(0) < 2a$, it suffices to have no point of η in the hashed part of $\Lambda(0)$ and a point of η in the blue region, which is then connected to $r(0)$ by other disks centered in $\Lambda(0)$. The blue region has area of order a^2.

Bernoulli percolation on the square lattice (references will be provided), and these may be adapted directly to our setting. We start with a series of definitions.

For $x \in \mathbb{R}^2$, write $\Lambda(x) = [-2, 2]^2 + x$ for the square of side length 4 centered at the point x. For $n \geq 1$ and points $x_1, \dots, x_k \in \mathbb{R}^2$, we say that $(x_1, \dots, x_k)$ is a pivotal chain for cross(n) if $\mathcal{O} \setminus \bigcup_{i=1}^{k} \Lambda(x_i) \notin \mathrm{cross}(n)$, but the set is minimal for this property, in that, for any $X \subsetneq \{1, \dots, k\}$, $\mathcal{O} \setminus \bigcup_{i \in X} \Lambda(x_i) \in \mathrm{cross}(n)$.

Call 0 a *thin point* if there exist exactly two disks with centers in $[-4, 4]^2 \setminus \Lambda(0)$ and if these disks have centers in $[-3.5, -2.5] \times [-1, 1]$ and $[2.5, 3.5] \times [-1, 1]$, respectively. Call these centers $\ell(0)$ and $r(0)$, respectively. See Figure 7 for an illustration.

Notice that the property that 0 is thin imposes no restriction on the disks inside $\Lambda(0)$, nor on those outside of $[-4, 4]^2$. In particular, there may exist more than two disks intersecting $[-4, 4]^2 \setminus \Lambda(0)$. However, no disk centered inside $\Lambda(0)$ can intersect disks centered outside $[-4, 4]^2$.

We call 0 a thin connected point if the two disks with centers in $[-4, 4]^2 \setminus \Lambda(0)$ are connected by the occupied set formed of the disks with centers in $\Lambda(0)$. Write $\mathrm{w}(0)$ for the maximal width of the occupied connection between $\ell(0)$ and $r(0)$ produced by the disks in $\Lambda(0)$. Set $\mathrm{w}(0) = 0$ if no such connection exists. The following lemma is a simple but key observation.

Lemma 4. *Fix $\lambda > 0$. There exists $c_0 = c_0(\lambda) > 0$ depending on λ such that, for any $a \in [0, 1]$, $\mathbb{P}_\lambda[0 < \mathrm{w}(0) < 2a \mid 0$ thin and η outside $\Lambda(0)] \geq c_0 a^2$.*

In particular, applying the above with $a = 1$ shows that thin points are connected with positive probability.

Proof. The proof follows from a simple geometrical construction. For $\{0 < \mathrm{w}(0) < 2a\}$ to occur, it suffices that there exists a disk in $\Lambda(0)$ at a distance between $1 - a^2$ and 1 from $\ell(0)$ which is connected to $r(0)$ by disks inside $\Lambda(0)$, and that there exists no other point of η in $\Lambda(0)$ at a distance at most 1 from $\ell(0)$; see Figure 7. The existence of the first point occurs with a

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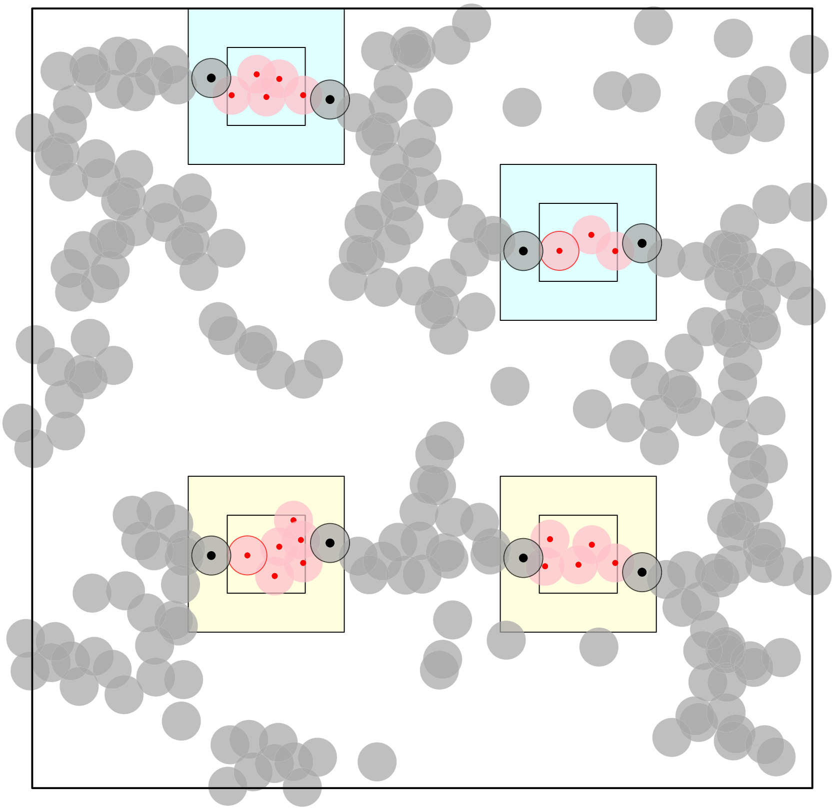

FIGURE 8. A situation in $\mathsf{PivCh}_n(m, K)$ with two sets X_1, X_2 (the corresponding boxes are blue and yellow, respectively) and two points x in each set. All thin points are connected. The overall width of the crossing is small if there exists a point x in each set X_1 and X_2 with $\mathrm{w}(x)$ small (see the top right and bottom left boxes).

probability at least c_0a^2 (for some positive constant c_0 that depends on λ); all other conditions are satisfied with positive probability. □

The definitions of thin and connected thin point apply by translation to any point $x \in \mathbb{R}^2$. Write $\ell(x)$, $r(x)$, and $\mathrm{w}(x)$ for the associated notions.

For $n, m, K \geq 1$, let $\mathsf{PivCh}_n(m, K)$ be the event that there exist $1 \leq k \leq K$ and disjoint sets $X_1, \ldots, X_k \subset (8\mathbb{Z})^2$ such that

- $|X_i| \geq m$ for each i,
- any $x \in \bigcup_{i=1}^k X_i$ is thin and connected,
- for any $x_1 \in X_1$, $x_k \in X_k$, $x_1, \ldots, x_k$ is a pivotal chain for $\mathrm{cross}(n)$.

Observe that here the points of each set X_i are required to be placed on a fixed lattice, at a large distance from each other. Also note that $\mathsf{PivCh}_n(m, K) \subset \mathrm{cross}(n)$. See Figure 8 for an illustration of $\mathsf{PivCh}_n(m, K)$.

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5.1. Critical occupied case: Upper bound

It is a property of critical percolation that, even when cross(n) occurs, there exists a chain of large vacant clusters that cross $[-n, n]^2$ vertically, with the clusters almost touching each other at many points. Moreover, these clusters are few in number and are all of large size. The following lemma is a more formal restatement of this.

Lemma 5. *For any $\delta > 0$, there exist $c > 0$ and $K \geq 1$ such that, for all n large enough,* $\mathbb{P}\big[\mathsf{PivCh}_n(cn^2\pi_4(n), K) \mid \mathrm{cross}(n)\big] \geq 1 - \delta$.

The lemma is a consequence of the RSW theorem and of the a priori bounds on the four- and six-arm probabilities, which may be proven similarly to the Bernoulli case; see, e.g., (20) and the display below. Lemma 5 may be proved in the same way as [13, Theorem 7.5]. We should mention that [13, Theorem 7.5] does produce pivotal chains, but not with points belonging to the lattice $(8\mathbb{Z})^2$, nor points that are thin. To obtain thin points aligned to the lattice, one needs to use the separation of arms for the four-arm event. This is a tedious but standard approach which we will not detail.

Proof of Theorem (2) *critical case* (7). Recall that the lower bound on w_n was proved in (36). We will focus here on the upper bound. Fix $\delta > 0$ and let $K \geq 1$ and $c > 0$ be the constants provided by Lemma 5. Let n be large enough that Lemma 5 applies.

When $\mathsf{PivCh}_n(cn^2\pi_4(n), K)$ occurs, let $\mathcal{X} = (X_1, \dots, X_k)$ be the first family of sets of $(8\mathbb{Z})^2$ that satisfies the properties of $\mathsf{PivCh}_n(cn^2\pi_4(n), K)$ according to some arbitrary order. The properties of $\mathcal{X}$ impose that any occupied path crossing $[-n, n]^2$ horizontally crosses all $[\Lambda(x)]_{x\in X_i}$ for some $i \leq k$. As such, we find that

$$\mathrm{w}_n \leq \max_{1\leq i\leq k} \min_{x\in X_i} \mathrm{w}(x).$$

Observe that the thin points of $(8\mathbb{Z})^2$ may be determined by knowing η outside of $\bigcup_{x\in(8\mathbb{Z})^2} \Lambda(x)$. Furthermore, for any family $(X_1, \dots, X_k)$ of sets of $(8\mathbb{Z})^2$, we can determine whether it satisfies the properties of $\mathsf{PivCh}_n(cn^2\pi_4(n), K)$ by knowing η outside of $\bigcup_{x\in(8\mathbb{Z})^2} \Lambda(x)$, inside all $\Lambda(x)$ for points $x \in (8\mathbb{Z})^2$ which are not thin, and by knowing which thin points of $(8\mathbb{Z})^2$ are connected.

It follows that, for any possible realization $\mathcal{X}_0$ of $\mathcal{X}$, the law of η knowing $\mathcal{X} = \mathcal{X}_0$ and the process η outside of $\Lambda(\mathcal{X}_0) := \bigcup_{x\in\mathcal{X}_0} \Lambda(x)$ is simply that of a Poisson point process on $\Lambda(\mathcal{X}_0)$ conditioned on each $x \in \bigcup_i X_i$ being connected. In particular, Lemma 4 shows that, for any $x \in \bigcup_i X_i$,

$$\mathbb{P}_{\lambda_c}\big[\mathrm{w}(x) < 2a \mid \mathcal{X} = \mathcal{X}_0 \text{ and } \eta \text{ outside } \Lambda(\mathcal{X}_0)\big] \geq c_0 a^2 \quad \text{for all } a \in [0, 1]. \tag{44}$$

Apply this to $a = \sqrt{C\alpha_n}$ for some large constant C to deduce that, for each i,

$$\mathbb{P}_{\lambda_c}\Big[\min_{x\in X_i} \mathrm{w}(x) \geq 2\sqrt{C\alpha_n} \mid \mathcal{X} = \mathcal{X}_0 \text{ and } \eta \text{ outside } \Lambda(\mathcal{X}_0)\Big] < (1 - c_0 C\alpha_n)^{c/\alpha_n} < \frac{\delta}{K}.$$

The first inequality is a direct consequence of (44), the fact that the restrictions of η to the different $[\Lambda(x)]_{x\in X_i}$ are independent, and that $|X_i| \geq c/\alpha_n$. The second inequality is ensured by taking C sufficiently large (depending on c and c_0, but not on n). We conclude from the above that

$$\mathbb{P}_{\lambda_c}\Big[\max_{1\leq i\leq k} \min_{x\in X_i} \mathrm{w}(x) \geq \sqrt{C\alpha_n} \mid \mathcal{X} = \mathcal{X}_0 \text{ and } \eta \text{ outside } \Lambda(\mathcal{X}_0)\Big] < \delta.$$

Now apply this to all realizations $\mathcal{X}$ producing $\mathsf{PivCh}_n(cn^2\pi_4(n), K)$, then integrate to obtain

$$\begin{aligned}\mathbb{P}_{\lambda_c}[\mathrm{w}_n \geq 2\sqrt{C\alpha_n}] &\leq \mathbb{P}_{\lambda_c}[\mathrm{w}_n \geq 2\sqrt{C\alpha_n} \text{ and } \mathsf{PivCh}_n(cn^2\pi_4(n), K)] \\ &\quad + \mathbb{P}_{\lambda_c}[\mathrm{cross}(n) \setminus \mathsf{PivCh}_n(cn^2\pi_4(n), K)] \\ &\leq 2\delta,\end{aligned}$$

with the second inequality also due to Lemma 5. Together with the upper bound (36) already proved, this implies (7). □

5.2. Sub-critical occupied case: Upper bound

In sub-critical percolation cross(n) is very unlikely to occur. Moreover, when it does, the occupied cluster crossing $[-n, n]^2$ horizontally is very thin and contains a linear number of pivotals, of which a linear number will be connected thin points. Concretely, the following statement may be deduced from [7, Lemma 4.1] as explained in Section 4.

Lemma 6. *Fix* $\lambda < \lambda_c$. *There exists* $c_1 = c_1(\lambda) > 0$ *depending only on* λ *such that* $\mathbb{P}_\lambda\big[\mathsf{PivCh}_n(c_1 n, 1) \mid \mathrm{cross}(n)\big] \geq 1 - \mathrm{e}^{-c_1 n}$.

Proof of Theorem 2, sub-critical case (6). Recall that the lower bound on w_n was proved in (43). We will focus here on the upper bound. Fix $\lambda < \lambda_c$ and $\delta > 0$, and let $c_1 > 0$ be the constant provided by Lemma 6. Let n be large enough that $\mathrm{e}^{-c_1 n} < \delta$.

The proof is similar to that of Section 5.1. When $\mathsf{PivCh}_n(c_1 n, 1)$ occurs, let $\mathcal{X}$ be the maximal set of $(8\mathbb{Z})^2$ that satisfies the properties of $\mathsf{PivCh}_n(c_1 n, 1)$. Since we are considering situations where each point of $\mathcal{X}$ is pivotal, we can define such a maximal set (this was not the case in Section 5.1, where pivotal chains were considered). Then, since any occupied path crossing $[-n, n]^2$ horizontally crosses all $[\Lambda(x)]_{x\in\mathcal{X}}$, we find that $\mathrm{w}_n \leq \min_{x\in\mathcal{X}} \mathrm{w}(x)$.

The same argument as in Section 5.1 shows that, for any potential realization $\mathcal{X}_0$ of $\mathcal{X}$, the law of η knowing $\mathcal{X} = \mathcal{X}_0$ and the process η outside of $\Lambda(\mathcal{X}_0) := \bigcup_{x\in\mathcal{X}_0} \Lambda(x)$ is simply that of a Poisson point process on $\Lambda(\mathcal{X}_0)$ conditioned on each $x \in \bigcup_i X_i$ being connected. In particular, Lemma 4 shows that, for any $x \in \bigcup_i X_i$,

$$\mathbb{P}_\lambda\big[\mathrm{w}(x) < 2a \mid \mathcal{X} = \mathcal{X}_0 \text{ and } \eta \text{ outside } \Lambda(\mathcal{X}_0)\big] \geq c_0 a^2 \quad \text{for all } a \in [0, 1].$$

Apply this to $a = \sqrt{C/n}$ for some large constant C to deduce that, for any $\mathcal{X}_0$,

$$\mathbb{P}_\lambda\Big[\min_{x\in\mathcal{X}} \mathrm{w}(x) \geq 2\sqrt{C/n} \mid \mathcal{X} = \mathcal{X}_0 \text{ and } \eta \text{ outside } \Lambda(\mathcal{X}_0)\Big] < (1 - c_0 C/n)^{c_1 n} < \delta. \tag{45}$$

The second inequality is ensured by taking C sufficiently large (depending on c_0 and c_1, but not on n).

Now apply (45) to all realizations $\mathcal{X}$ producing $\mathsf{PivCh}_n(c_1 n, 1)$, then integrate to obtain

$$\begin{aligned}\mathbb{P}_\lambda[\mathrm{w}_n \geq 2\sqrt{C/n}] &\leq \mathbb{P}_\lambda[\mathrm{w}_n \geq 2\sqrt{C/n} \text{ and } \mathsf{PivCh}_n(c_1 n, 1)] + \mathbb{P}_\lambda[\mathrm{cross}(n) \setminus \mathsf{PivCh}_n(c_1 n, 1)] \\ &\leq 2\delta\mathbb{P}_\lambda[\mathrm{cross}(n)],\end{aligned}$$

with the second inequality also due to Lemma 6 and the choice of a large enough n. Together with the upper bound (43) already proved, this implies (6). □

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6. Questions

In closing, let us discuss some related open questions.

The most natural question is probably to extend the results beyond non-compactly supported radii distributions. The results surely fail when the tails of the distribution of radii are too heavy, but for quickly decaying distributions they should remain valid.

The second question that comes to mind is whether this analysis may be performed for randomly placed sets of any shape, rather than disks. For such sets, Corollary 1, which is the cornerstone of the proofs of Section 4, ceases to hold. The method used in Section 5, based on the study of pivotal points, appears more robust, and may be used for general shapes. It would therefore be interesting to adapt this method to prove all results. Some problems may arise for lower bounds on w_n and w_n^*, as the points where these minimal widths are reached are not always pivotals.

A third question is related to the difference between the results for the occupied and vacant set. In the critical case, w_n is of the same order as $\sqrt{\mathrm{w}_n^*}$; the same phenomenon happens when comparing (3) to (8) and (5) to (6). This difference appears to be due to the round shape of the disks, but a quick computation based on the method of Section 5 seems to suggest that the same is true when disks are replaced with squares. Is this phenomenon more general?

Finally, note that the super-critical case of Theorem 2 only offers a lower bound on w_n. Indeed, the upper bound

$$\mathbb{P}_\lambda\left[\mathrm{w}_n \leq 2\sqrt{1-\left(\frac{\lambda_c}{\lambda}-\frac{C}{\lambda n^2 \pi_4(n)}\right)^2} \mid \mathrm{cross}(n)\right] \geq 1-\delta$$

fails for λ sufficiently large, due to the phenomenon explained in Remark 2. Still, we can expect it to hold for λ sufficiently close to λ_c. Is this the case?

We close with a thought. This study appears to be specific to continuum percolation, with no apparent correspondence in the discrete. Is there one?

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Competing interests

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