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INEQUALITIES FOR SYMMETRIC MEANS, SYMMETRIC HARMONIC MEANS, AND THEIR APPLICATIONS

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In this paper, we establish a number of inequalities involving symmetric means and symmetric harmonic means. We then apply these new inequalities to obtain many geometric inequalities of isoperimetric type for plane polygons.

1. INTRODUCTION

Inequalities are basic in pure and applied mathematics which brought together three distinguished mathematicians Hardy, Littlewood and Pólya to publish the famous book "Inequalities" [**4**]. One of the most well-known inequalities is the arithmetic-geometric mean inequality. Among many important applications of inequalities to other fields, inequalities involving various means, such as symmetric means and symmetric harmonic means are of particular interest. They are used extensively in probability and statistics. In geometry and topology, many basic invariants are also defined in terms of the symmetric means, for instance, the rth mean curvature of a Riemannian submanifold et cetera. Let $\mathbf{R}$ denote the field of real numbers, and $\mathbf{R}_+ = \{x \in \mathbf{R} \,:\, x > 0\}$. For $1 \leqslant r \leqslant n$, the rth *symmetric mean* $P_n^{[r]}(\mathbf{a})$ of $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbf{R}_+^n$ is defined by

$$P_n^{[r]}(\mathbf{a}) = \left\{ \begin{bmatrix} n \\ r \end{bmatrix}^{-1} \sigma_r(a_1, \ldots, a_n) \right\}^{1/r},$$

where

$$\sigma_r(a_1, \ldots, a_n) = \sum_{1 \leqslant i_1 < \ldots < i_r \leqslant n} a_{i_1} a_{i_2} \ldots a_{i_r}$$

is the rth elementary symmetric function of $\mathbf{a}$, and the summation is taken over all possible permutations of $(i_1, \ldots, i_r)$, $1 \leqslant r \leqslant n$. The *arithmetic mean* $A_n(\mathbf{a})$ and the *geometric mean* $G_n(\mathbf{a})$ are simply

$$A_n(\mathbf{a}) = P_n^{[1]}(\mathbf{a}) \qquad \text{and} \qquad G_n(\mathbf{a}) = P_n^{[n]}(\mathbf{a}).$$

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Set $\mathbf{b} = (1/a_1, \ldots, 1/a_n)$. The *harmonic mean* $H_n(\mathbf{a})$ of $\mathbf{a}$ is defined as $H_n(\mathbf{a}) = n/\sigma_1(\mathbf{b})$. It is well-known that

$$A_n(\mathbf{a}) \geqslant G_n(\mathbf{a}) \geqslant H_n(\mathbf{a}),$$

with either equality holding if and only if $a_1 = a_2 = \ldots = a_n$. Furthermore, we have the following symmetric mean inequality.

Lemma 1.1. (See [**3, 7, 9, 10**].) *For $1 \leqslant t < r \leqslant n$, and $\mathbf{a} \in \mathbf{R}_+^n$,*

$$P_n^{[t]}(\mathbf{a}) \geqslant P_n^{[r]}(\mathbf{a}), \tag{1}$$

with equality holding if and only if $a_1 = \ldots = a_n$.

For $\mathbf{a} \in \mathbf{R}_+^n$, set $m(\mathbf{a}) = \min\{a_i : 1 \leqslant i \leqslant n\}$, $\overline{m}(\mathbf{a}) = \max\{a_i : 1 \leqslant i \leqslant n\}$ and define

$$e_n(\mathbf{a}) = 4m(\mathbf{a})\overline{m}(\mathbf{a})/\{m(\mathbf{a}) + \overline{m}(\mathbf{a})\}^2. \tag{2}$$

By the arithmetic-geometric mean inequality (or Lemma 1.1), we have

$$e_n(\mathbf{a}) \leqslant 1, \quad \text{with equality if and only if } a_1 = a_2 = \ldots = a_n. \tag{3}$$

In terms of $e_n(\mathbf{a})$, we can restate a result in [**3**, p.201] as follows in contrast with the inequality $A_n(\mathbf{a}) \geqslant H_n(\mathbf{a})$.

Lemma 1.2. (Kantorovich.) *For $\mathbf{a} \in \mathbf{R}_+^n$,*

$$H_n(\mathbf{a}) \geqslant e_n(\mathbf{a})A_n(\mathbf{a}). \tag{4}$$

Equality holds if and only if $n = 2m$ is even, $a_1 = \ldots = a_m$ and $a_{m+1} = \ldots = a_{2m}$.

In this paper, we shall generalise the inequality (1) and establish inequalities that reverse the direction of (1) in the sense of (4). In fact, we shall prove the following result.

Theorem 1.3 *Let $1 \leqslant t < r \leqslant n$ and $\mathbf{a} \in \mathbf{R}_+^n$. Then*

$$P_n^{[r]}(\mathbf{a}) \geqslant \{e_n(\mathbf{a})\}^{(r-t)/r} P_n^{[t]}(\mathbf{a}).$$

Equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

We apply these inequalities along with some other inequalities for various means that we have proved earlier in [**6**] to establish many new interesting geometric isoperimetric inequalities for plane polygons in section 3.

2. Inequalities for Symmetric and Harmonic Means

We shall begin by proving a new inequality which may be viewed as a counterpart of Lemma 1.1.

Theorem 2.1. *Let $\mathbf{a} \in \mathbf{R}_+^n$ and $1 \leqslant t < r < s \leqslant n$.*

$$P_n^{[r]}(\mathbf{a}) \geqslant \left\{P_n^{[t]}(\mathbf{a})\right\}^{(t(s-r))/(r(s-t))} \left\{P_n^{[s]}(\mathbf{a})\right\}^{(s(r-t))/(r(s-t))}. \tag{5}$$

Equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

PROOF: First we set

$$p_n^{[r]}(\mathbf{a}) = \left\{P_n^{[r]}(\mathbf{a})\right\}^r = \begin{bmatrix} n \\ r \end{bmatrix}^{-1} \sigma_r(a_1, \ldots, a_n), \qquad 1 \leqslant r \leqslant n.$$

It is well-known that if $1 < r < n$, we have (see [**3, 9**])

$$\left\{p_n^{[r]}(\mathbf{a})\right\}^2 \geqslant p_n^{[r-1]}(\mathbf{a}) p_n^{[r+1]}(\mathbf{a}), \tag{6}$$

with equality if and only if $a_1 = a_2 = \ldots = a_n$. We claim that

$$p_n^{[s-k]}(\mathbf{a}) \geqslant \left\{p_n^{[s-k-1]}(\mathbf{a})\right\}^{k/(k+1)} \left\{p_n^{[s]}(\mathbf{a})\right\}^{1/(k+1)}, \qquad k+1 < s \leqslant n. \tag{7}$$

According to (6), (7) is true for $k = 1$. Suppose $2 \leqslant r+1 \leqslant n-2$, and (7) is valid for $k = r$. By induction and (6),

$$\left\{p_n^{[s-r-1]}(\mathbf{a})\right\}^2 \geqslant p_n^{[s-r]}(\mathbf{a}) p_n^{[s-r-2]}(\mathbf{a}) \geqslant \left\{p_n^{[s-r-1]}(\mathbf{a})\right\}^{r/(r+1)} \left\{p_n^{[s]}(\mathbf{a})\right\}^{1/(r+1)} p_n^{[s-r-2]}(\mathbf{a}),$$

and so,

$$\left\{p_n^{[s-r-1]}(\mathbf{a})\right\}^{(r+2)/(r+1)} \geqslant p_n^{[s-r-2]}(\mathbf{a}) \left\{p_n^{[s]}(\mathbf{a})\right\}^{1/(r+1)}.$$

Hence (7) holds for $k = r+1$. To prove the theorem, it suffices to prove the following:

$$p_n^{[r]}(\mathbf{a}) \geqslant \left\{p_n^{[t]}(\mathbf{a})\right\}^{(s-r)/(s-t)} \left\{p_n^{[s]}(\mathbf{a})\right\}^{(r-t)/(s-t)}. \tag{8}$$

Now, by (7) we obtain

$$p_n^{[2]}(\mathbf{a}) \geqslant \left\{p_n^{[1]}(\mathbf{a})\right\}^{(s-2)/(s-1)} \left\{p_n^{[s]}(\mathbf{a})\right\}^{1/(s-1)},$$

that is, (8) holds for $r = 2$. Suppose $3 \leqslant r+1 \leqslant n-1$, and (8) is valid for $k = r$. Then by (7) we get,

$$\begin{aligned} p_n^{[r+1]}(\mathbf{a}) &\geqslant \left\{p_n^{[r]}(\mathbf{a})\right\}^{(s-r-1)/(s-r)} \left\{p_n^{[s]}(\mathbf{a})\right\}^{1/(s-r)} \\ &\geqslant \left[\left\{p_n^{[t]}(\mathbf{a})\right\}^{(s-r)/(s-t)} \left\{p_n^{[s]}(\mathbf{a})\right\}^{(r-t)/(s-t)}\right]^{(s-r-1)/(s-r)} \left\{p_n^{[s]}(\mathbf{a})\right\}^{1/(s-r)} \\ &= \left\{p_n^{[t]}(\mathbf{a})\right\}^{(s-r-1)/(s-t)} \left\{p_n^{[s]}(\mathbf{a})\right\}^{(r+1-t)/(s-t)}. \end{aligned}$$

That is, (8) holds for $k = r+1$. Hence the result follows by induction. □

Now, Theorem 1.3 is simply a corollary of the following theorem.

THEOREM 2.2. *Let* $1 \leqslant t < r \leqslant n$, *and* $\mathbf{a} \in \mathbf{R}_+^n$.

$$P_n^{[r]}(\mathbf{a}) \geqslant \{e_n(\mathbf{a}) A_n(\mathbf{a})\}^{(r-t)/r} \left\{P_n^{[t]}(\mathbf{a})\right\}^{t/r}, \tag{9}$$

with equality if and only if $a_1 = a_2 = \ldots = a_n$.

PROOF: From the definitions,

$$\{G_n(\mathbf{a})\}^n = H_n(\mathbf{a}) \left\{P_n^{[n-1]}(\mathbf{a})\right\}^{n-1}. \tag{10}$$

Hence, it follows from Lemma 1.2 and Theorem 2.1 that

$$\begin{aligned} G_n(\mathbf{a}) &\geqslant \{e_n(\mathbf{a})\}^{1/n} \left\{P_n^{[n-1]}(\mathbf{a})\right\}^{(n-1)/n} \{A_n(\mathbf{a})\}^{1/n} \\ &\geqslant \{e_n(\mathbf{a})\}^{1/n} \left\{\left[P_n^{[t]}(\mathbf{a})\right]^{t/((n-1)(n-t))} [G_n(\mathbf{a})]^{(n(n-1-t))/((n-1)(n-t))}\right\}^{(n-1)/n} \{A_n(\mathbf{a})\}^{1/n} \end{aligned}$$

Thus,

$$\{G_n(\mathbf{a})\}^{1/(n-t)} \geqslant \{e_n(\mathbf{a})\}^{1/n} \left\{P_n^{[t]}(\mathbf{a})\right\}^{t/(n(n-t))} \{A_n(\mathbf{a})\}^{1/n}.$$

That is,

$$G_n(\mathbf{a}) \geqslant \{e_n(\mathbf{a}) A_n(\mathbf{a})\}^{(n-t)/n} \left\{P_n^{[t]}(\mathbf{a})\right\}^{t/n}. \tag{11}$$

Hence (9) holds for $r = n$. If $r < n$, by (11) and Theorem 2.1 we have

$$\begin{aligned} P_n^{[r]}(\mathbf{a}) &\geqslant \left\{P_n^{[t]}(\mathbf{a})\right\}^{(t(n-r))/(r(n-t))} \{G_n(\mathbf{a})\}^{(n(r-t))/(r(n-t))} \\ &\geqslant \left\{P_n^{[t]}(\mathbf{a})\right\}^{(t(n-r))/(r(n-t))} \left\{[e_n(\mathbf{a})]^{(n-t)/n} \left[P_n^{[t]}(\mathbf{a})\right]^{t/n} [A_n(\mathbf{a})]^{(n-t)/n}\right\}^{(n(r-t))/(r(n-t))} \\ &= \{e_n(\mathbf{a}) A_n(\mathbf{a})\}^{(r-t)/r} \left\{P_n^{[t]}(\mathbf{a})\right\}^{t/r}. \end{aligned}$$

∎

Recall that if we set $\mathbf{b} = (1/a_1, \ldots, 1/a_n)$, then $H_n(\mathbf{a}) = 1/P_n^{[1]}(\mathbf{b})$. Hence we can define the rth *symmetric harmonic mean* $H_{n,r}(\mathbf{a})$ of $\mathbf{a} \in \mathbf{R}_+^n$ by

$$H_{n,r}(\mathbf{a}) = 1\Big/\left(P_n^{[r]}(\mathbf{b})\right), \qquad 1 \leqslant r \leqslant n.$$

Observe that $H_n(\mathbf{a}) = H_{n,1}(\mathbf{a})$ and $H_{n,n}(\mathbf{a}) = G_n(\mathbf{a})$. As an immediate corollary to Lemma 1.1 and Theorem 1.3 we have the following inequalities.

THEOREM 2.3. *Let* $\mathbf{a} \in \mathbf{R}_+^n$ *and* $1 \leqslant t < r \leqslant n$. *Then*

(a) $H_{n,t}(\mathbf{a}) \leqslant H_{n,r}(\mathbf{a})$.

(b) $H_{n,t}(\mathbf{a}) \geqslant \{e_n(\mathbf{a})\}^{(r-t)/r} H_{n,r}(\mathbf{a})$.

Equality holds in either (a) or (b) if and only if $a_1 = a_2 = \ldots = a_n$.

3. GEOMETRIC ISOPERIMETRIC INEQUALITIES

As pointed out by Pólya in his excellent book *Induction and Analogy in Mathematics* [**13**], isoperimetric inequalities and inequalities for means share a lot of common properties. In this section, we shall apply the results about various means in previous sections and some other inequalities that we have obtained earlier in [**6**] to establish geometric

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inequalities for plane polygons. We are able to obtain uncountably many new isoperimetric inequalities which are not only interesting in geometry, but also very important in analysis, as well as in mathematical physics [**2, 11, 12, 14, 15**]. Let $\mathcal{P}_n$ be an n-sided polygon in the plane and $a_1, a_2, \dots, a_n$ denote the lengths of its sides. Denote by $A(\mathcal{P}_n)$ and $L(\mathcal{P}_n) = \sum_{i=1}^{n} a_i$, the area and the perimeter of $\mathcal{P}_n$ respectively. The well-known classical isoperimetric inequality asserts that [**12**, p.1209]:

$$L^2(\mathcal{P}_n)/A(\mathcal{P}_n) \geqslant 4d_n, \quad \text{where} \quad d_n = n \tan\ (\pi/n). \tag{12}$$

Equality holds if and only if $\mathcal{P}_n$ is regular.

Set $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbf{R}_+^n$, $\alpha_i = L(\mathcal{P}_n) - 2a_i$, $1 \leqslant i \leqslant n$. Then we have

$$L(P_n) = \frac{n}{n-2} P_n^{[1]}(\alpha). \tag{13}$$

For $n = 3$ and $n = 4$, the famous Heron formula for a triangle and Brahmagupta formula for a quadrilateral can be expressed as follows [**5, 7, 8**]:

$$\left\{\frac{n}{n-2}\left[A_n(\alpha)\right]^{(n-2)/(2(n-1))}\left[G_n(\alpha)\right]^{n/(2(n-1))}\right\}^2 = 4d_n A(\mathcal{P}_n), \qquad (n = 3). \tag{14}$$

$$\left\{\frac{n}{n-2} P_n^{[n]}(\alpha)\right\}^2 \geqslant 4d_n A(\mathcal{P}_n), \qquad (n = 4). \tag{15}$$

The equality in (15) holds if and only if $\mathcal{P}_4$ is a cyclic quadrilateral, that is, it can be inscribed in a circle. It is a well-known fact that "Of all n-sided plane polygons with given n sides, the cyclic polygon encloses the largest area" [**5, 8**]. Since we are mainly concerned with isoperimetric problem for polygons, from now on, all polygons will be assumed to be cyclic unless otherwise specified. For the simplicity of statement, it is convenient to use α instead of $\mathbf{a} \in \mathbf{R}_+^n$ when we discuss isoperimetric inequalities for polygons by virtue of inequalities for means. Formulas (13), (14), and (15) also motivate us to introduce the following new geometric invariants.

DEFINITION 3.1: The rth *symmetric perimeter* $\mathcal{L}_n^{[r]}(\mathcal{P}_n)$ and the rth *symmetric harmonic perimeter* $\mathcal{L}_{n,r}(\mathcal{P}_n)$ of $\mathcal{P}_n$, $1 \leqslant r \leqslant n$, are defined respectively by

$$\mathcal{L}_n^{[r]}(\mathcal{P}_n) = \frac{n}{n-2} P_n^{[r]}(\alpha), \qquad \text{and} \qquad \mathcal{L}_{n,r}(\mathcal{P}_n) = \frac{n}{n-2} H_{n,r}(\alpha).$$

Set $\mathcal{H}_n(\mathcal{P}_n) = \mathcal{L}_{n,1}(\mathcal{P}_n)$ and call it the *harmonic perimeter* of $\mathcal{P}_n$. By Lemma 1.1 and Theorem 2.3 (a) we have

$$L(\mathcal{P}_n) = \mathcal{L}_n^{[1]}(\mathcal{P}_n) \geqslant \mathcal{L}_n^{[2]}(\mathcal{P}_n) \geqslant \dots \geqslant \mathcal{L}_n^{[n]}(\mathcal{P}_n) = \mathcal{L}_{n,n}(\mathcal{P}_n) \tag{16}$$

and

$$\mathcal{L}_{n,n}(\mathcal{P}_n) \geqslant \mathcal{L}_{n,n-1}(\mathcal{P}_n) \geqslant \dots \geqslant \mathcal{L}_{n,1}(\mathcal{P}_n) = \mathcal{H}_n(\mathcal{P}_n). \tag{17}$$

Now, let us set $e_n(\mathcal{P}_n) = e_n(\alpha)$. By (3), $e_n(\mathcal{P}_n) \leqslant 1$ with equality if and only if $\mathcal{P}_n$ is regular. We present the following generalisation and variations of (12).

THEOREM 3.2. *For $r = 1, 2, \dots, n$, we have*

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(a) $\left\{\mathcal{L}_n^{[r]}(\mathcal{P}_n)\right\}^2 \big/ A(\mathcal{P}_n) \geqslant 4d_n\{e_n(\mathcal{P}_n)\}^{(2(r-1))/r}$.

(b) $\{\mathcal{L}_{n,r}(\mathcal{P}_n)\}^2 \big/ A(\mathcal{P}_n) \geqslant 4d_n\{e_n(\mathcal{P}_n)\}^{(2(2n-r-1))/n}$.

In particular,

$$\{\mathcal{H}_n(\mathcal{P}_n)\}^2 / A(\mathcal{P}_n) \geqslant 4d_n\{e_n(\mathcal{P}_n)\}^{(4(n-1))/n}.$$

Equality holds in (a) (respectively (b)) if and only if $\mathcal{P}_n$ is regular.

PROOF: Apply Theorem 1.3 to $\alpha \in \mathbf{R}_+^n$. ∎

Notice that if $r = 1$, (a) is simply the inequality (12).

Next, we shall briefly review the so-called generalised power means and their inequalities that we have obtained in [**6**]. Then we shall be able to establish a family of isoperimetric inequalities that are much more general than Theorem 3.2. Let $f_i : \mathbf{R}_+^n \longrightarrow \mathbf{R}_+$, $1 \leqslant i \leqslant m$, be distinct functions, and $\mathbf{f} = (f_1, f_2, \dots f_m)$. Let $w_i > 0$, $1 \leqslant i \leqslant m$, and $\triangle(\mathbf{w}) = \triangle(w_1, \dots, w_m)$ be the $(m-1)$-simplex in $\mathbf{R}^m$ with vertices

$$\mathbf{W}_i = (0, \dots, 0, 1/w_i, 0, \dots, 0) \quad \text{where } 1/w_i \text{ is the } i\text{th coordinate}, \quad i = 1, \dots, m.$$

Thus, if $\mathbf{x} = (x_1, \dots, x_m) \in \triangle(\mathbf{w})$, then $\sum_{i=1}^{m} w_i x_i = 1$. For $\mathbf{x} \in \triangle(\mathbf{w})$, $\mathbf{a} \in \mathbf{R}_+^n$, and $r \geqslant 0$, we have defined the *generalised power mean* $L_{n,m}^{[r]}[\mathbf{f}; \mathbf{x}; \mathbf{w}](\mathbf{a})$ in [**6**] by

$$L_{n,m}^{[r]}[\mathbf{f}; \mathbf{x}; \mathbf{w}](\mathbf{a}) = \begin{cases} \prod_{i=1}^{m} \{f_i(\mathbf{a})\}^{w_i x_i} & \text{if } r = 0, \\ \left\{\sum_{i=1}^{m} w_i x_i \, [f_i(\mathbf{a})]^r\right\}^{1/r} & \text{if } r > 0. \end{cases}$$

Moreover, if $0 \leqslant t < r$, $\mathbf{x} \in \triangle(\mathbf{w})$, the following inequality generalises the ordinary power mean inequality, and was proved in [**6**, Theorem 2.5(a)]:

$$L_{n,m}^{[t]}[\mathbf{f}; \mathbf{x}; \mathbf{w}](\mathbf{a}) \leqslant L_{n,m}^{[r]}[\mathbf{f}; \mathbf{x}; \mathbf{w}](\mathbf{a}), \qquad \mathbf{a} \in \mathbf{R}_+^n. \tag{18}$$

Equality holds if and only if $a_1 = a_2 = \dots = a_n$.

We now can define the *mixed symmetric perimeter* $\mathcal{L}_n^{[r]}[\mathbf{x}; \mathbf{w}](\mathcal{P}_n)$ and the *mixed symmetric harmonic perimeter* $\mathcal{L}_{n,r}[\mathbf{x}; \mathbf{w}](\mathcal{P}_n)$ for $\mathcal{P}_n$ with $\mathbf{w} \in \mathbf{R}_+^n$ and $\mathbf{x} \in \triangle(\mathbf{w})$, $0 \leqslant r \leqslant n$, as following:

$$\mathcal{L}_n^{[r]}[\mathbf{x}; \mathbf{w}](\mathcal{P}_n) = \frac{n}{n-2} L_{n,n}^{[r]}[\mathbf{f}; \mathbf{x}; \mathbf{w}](\alpha), \quad \text{where} \quad f_i(\alpha) = P_n^{[i]}(\alpha), \quad 1 \leqslant i \leqslant n. \tag{19}$$

$$\mathcal{L}_{n,r}[\mathbf{x}; \mathbf{w}](\mathcal{P}_n) = \frac{n}{n-2} L_{n,n}^{[r]}[\mathbf{f}; \mathbf{x}; \mathbf{w}](\alpha), \quad \text{where} \quad f_i(\alpha) = H_{n,i}(\alpha), \quad 1 \leqslant i \leqslant n. \tag{20}$$

Clearly, if $\mathbf{x} = \mathbf{W}_i$,

$$\mathcal{L}_n^{[r]}[\mathbf{x}; \mathbf{w}](\mathcal{P}_n) = \mathcal{L}_n^{[r]}(\mathcal{P}_n), \quad \text{and} \quad \mathcal{L}_{n,r}[\mathbf{x}; \mathbf{w}](\mathcal{P}_n) = \mathcal{L}_{n,r}(\mathcal{P}_n).$$

In [**6**], we introduced a dominance relation $\succ$ on $\triangle(\mathbf{w})$ as follows: for $\mathbf{x}$, $\mathbf{x}' \in \triangle(\mathbf{w})$, $\mathbf{x} \succ \mathbf{x}'$ if $\mathbf{x} = \mathbf{x}'$, or there exists an integer k, $1 \leqslant k < m$ such that $x_i \geqslant x_i'$ for $1 \leqslant i \leqslant k$, $x_{k+1} < x'_{k+1}$; and $x_i \leqslant x_i'$ for $k+2 \leqslant i \leqslant m$, if $k+2 \leqslant m$. We proved a fundamental inequality for generalised power means [**6**, Theorem 2.5(b)]. That is, if $\mathbf{x}, \mathbf{x}' \in \triangle(\mathbf{w})$ and $\mathbf{x} \succ \mathbf{x}'$, then

$$L_{n,m}^{[r]}[\mathbf{f}; \mathbf{x}; \mathbf{w}](\mathbf{a}) \geqslant L_{n,m}^{[r]}[\mathbf{f}; \mathbf{x}'; \mathbf{w}](\mathbf{a}).$$

As direct consequences of this result, we have the following inequalities for the mixed symmetric perimeters.

THEROEM 3.3. *Suppose* $\mathbf{x} \succ \mathbf{x}'$; $\mathbf{x}, \mathbf{x}' \in \triangle(\mathbf{w})$, $\mathbf{w} \in \mathbf{R}_+^n$, *and* $0 \leqslant r \leqslant n$. *Then*

(a) $\mathcal{L}_n^{[r]}[\mathbf{x}; \mathbf{w}](\mathcal{P}_n) \geqslant \mathcal{L}_n^{[r]}[\mathbf{x}'; \mathbf{w}](\mathcal{P}_n)$;

(b) $\mathcal{L}_{n,r}[\mathbf{x}; \mathbf{w}](\mathcal{P}_n) \geqslant \mathcal{L}_{n,r}[\mathbf{x}'; \mathbf{w}](\mathcal{P}_n)$.

If $\mathbf{x} \neq \mathbf{x}'$, *then equality holds in (a) (respectively (b)) if and only if* $\mathcal{P}_n$ *is regular.*

By using these new geometric quantities of $\mathcal{P}_n$, Theorem 3.2 can be generalised even further.

THEOREM 3.4. *For* $\mathbf{w} \in \mathbf{R}_+^n$, *and* $\mathbf{x} \in \triangle(\mathbf{w})$, *we have*

(a) $$\left\{\mathcal{L}_n^{[r]}[\mathbf{x}; \mathbf{w}](\mathcal{P}_n)\right\}^2 / A(\mathcal{P}_n) \geqslant 4d_n\{e_n(\mathcal{P}_n)\}^{2-2\sum_{i=1}^{m} w_i x_i / i}.$$

(b) $$\{\mathcal{L}_{n,r}[\mathbf{x}; \mathbf{w}](\mathcal{P}_n)\}^2 / A(\mathcal{P}_n) \geqslant 4d_n\{e_n(\mathcal{P}_n)\}^{4-(2/n)-(2/n)\sum_{i=1}^{m} i w_i x_i}.$$

Equality holds in (a) or (b) if and only if $\mathcal{P}_n$ *is regular.*

PROOF: (a) By (18) and Theorem 3.2 (a),

$$\begin{aligned}\left\{\mathcal{L}_n^{[r]}[\mathbf{x}; \mathbf{w}](\mathcal{P}_n)\right\}^2 / A(\mathcal{P}_n) &\geqslant \left\{\mathcal{L}_n^{[0]}[\mathbf{x}; \mathbf{w}](\mathcal{P}_n)\right\}^2 / A(\mathcal{P}_n) \\ &= \prod_{i=1}^{n} \left\{\left[\mathcal{L}_n^{[i]}(\mathcal{P}_n)\right]^2 / A(\mathcal{P}_n)\right\}^{w_i x_i} \\ &\geqslant 4d_n\{e_n(\mathcal{P}_n)\}^{2-2\sum_{i=1}^{m} w_i x_i / i}.\end{aligned}$$

(b) Similar. ∎

We can restate (14) and (15) respectively as

$$\left\{\mathcal{L}_3^{[2]}(\mathcal{P}_3)\right\}^2 / A(\mathcal{P}_3) = 4d_3,$$

$$\left\{\mathcal{L}_4^{[4]}(\mathcal{P}_4)\right\}^2 / A(\mathcal{P}_4) \geqslant 4d_4.$$

Thus, we may ask the following questions:

QUESTION 3.5. Can one improve Theorem 3.4 to obtain the following stronger inequalities:

(21) $$\left\{\mathcal{L}_n^{[r]}[\mathbf{x}; \mathbf{w}](\mathcal{P}_n)\right\}^2 / A(\mathcal{P}_n) \geqslant 4d_n;$$

(22) $$\{\mathcal{L}_{n,r}[\mathbf{x};\mathbf{w}](\mathcal{P}_n)\}^2 \big/ A(\mathcal{P}_n) \geqslant 4d_n;$$

with equality in either (21) or (22) if and only if $\mathcal{P}_n$ is regular?

Observe that from (16),(17),(19),(20), together with Theorem 3.3, the mixed symmetric perimeters in (21) and (22) are all smaller than or equal to the ordinary perimeter $L(\mathcal{P}_n)$ of $\mathcal{P}_n$. For example,

$$L^2(\mathcal{P}_n) - \left\{\mathcal{L}_n^{[r]}[\mathbf{x};\mathbf{w}](\mathcal{P}_n)\right\}^2 = B(\mathcal{P}_n) \geqslant 0, \qquad \mathbf{w} \in \mathbf{R}_+^n,\ \mathbf{x} \in \triangle(\mathbf{w}),$$

with equality holding if and only if $\mathcal{P}_n$ is regular. If the answers to the questions were to be affirmative, one of its significances is that we would have a large family of the following *Bonnesen-style inequalities* for plane polygons [**16**]

$$L^2(\mathcal{P}_n) - 4d_n A(\mathcal{P}_n) \geqslant B(\mathcal{P}_n).$$

As a matter of fact, a class of cyclic polygons satisfying (21) and (22) does exist. Let $\mathcal{P}_n = \mathcal{P}_{m,m}$ $(n = 2m)$ denote cyclic $2m$-gons with m sides of length b and remaining m sides of length c. Set

$$q = \frac{(b-c)^2}{2[(b^2+c^2)\cos(\pi/m) + 2bc]}, \qquad \alpha(m) = 1 + q\left(1 - \cos\frac{\pi}{m}\right), \qquad \text{and}$$

$$\beta(m) = \alpha(m) - \frac{q(1+\cos(\pi/m))}{(m-1)^2}.$$

If $m \geqslant 3$, a simple calculation using $\cos\ \theta = \sum_{i=0}^{\infty}(-1)^i\left(\theta^{2i}/(2i)!\right)$ yields

(23) $$1 - \cos\ \frac{\pi}{m} > \frac{1+\cos(\pi/m)}{(m-1)^2}, \qquad m \geqslant 3.$$

Hence, we have

(24) $$\alpha(m) > \beta(m) > 1 \qquad \text{if} \quad m \geqslant 3.$$

LEMMA 3.6. *Let $\mathcal{P}_n = \mathcal{P}_{m,m}$. Then*

(a) $\left\{\mathcal{L}_n^{[1]}(\mathcal{P}_n)\right\}^2 \big/ A(\mathcal{P}_n) = 4d_n\alpha(m)$.

(b) $\left\{\mathcal{L}_n^{[n]}(\mathcal{P}_n)\right\}^2 \big/ A(\mathcal{P}_n) = 4d_n\beta(m)$.

(c) $\{\mathcal{H}_n(\mathcal{P}_n)\}^2 \big/ A(\mathcal{P}_n) = 4d_n\{\beta(m)\}^2/\alpha(m)$.

PROOF: In [**8**], Macnab has proved that

$$A(\mathcal{P}_n) = \frac{m}{4\sin(\pi/m)}\left[\left(b^2+c^2\right)\cos\frac{\pi}{m} + 2bc\right].$$

Hence the results follow directly from the definitions. ∎

LEMMA 3.7. *If $m \geqslant 10$, then $\{\beta(m)\}^2 \geqslant \alpha(m)$.*

PROOF: Set

$$\phi(m) = \frac{-1+\cos\dfrac{\pi}{m}+\dfrac{2(1+\cos(\pi/m))}{(m-1)^2}}{\left[1-\cos\dfrac{\pi}{m}-\dfrac{(1+\cos(\pi/m))}{(m-1)^2}\right]^2}.$$

Obviously, $\{\beta(m)\}^2/\alpha(m) \geqslant 1$ if and only if $q \geqslant \phi(m)$. Since $q > 0$, it suffices to verify that $\phi(m) < 0$ if $m \geqslant 10$. But

$$\phi(m) < 0 \Longleftrightarrow \cos\frac{\pi}{m} < \frac{m^2-2m-1}{m^2-2m+3}. \tag{25}$$

If $m \geqslant 10$,

$$\frac{4}{m^2-2m+3}+\frac{1}{4!}\left(\frac{\pi}{m}\right)^4 < \frac{\pi^2}{2m^2},$$

and so,

$$\cos\frac{\pi}{m} < 1-\frac{1}{2!}\left(\frac{\pi}{m}\right)^2+\frac{1}{4!}\left(\frac{\pi}{m}\right)^4 \leqslant 1-\frac{4}{m^2-2m+3} = \frac{m^2-2m-1}{m^2-2m+3}.$$

This proves the Lemma. ∎

Now, the answer to Question 3.5 is affirmative for $\mathcal{P}_{m,m}$ if $m \geqslant 10$. In fact, we have the following result.

THEOREM 3.8. *Let $m \geqslant 10$, $0 \leqslant r \leqslant n$, $\mathbf{w} \in \mathbf{R}_+^n$, and $\mathbf{x} \in \triangle(\mathbf{w})$. Then the isoperimetric inequalities (21) and (22) hold for polygon $\mathcal{P}_n = \mathcal{P}_{m,m}$.*

PROOF: By (16), (17) and Theorem 3.3, it suffices to show that

$$\{\mathcal{H}_n(\mathcal{P}_n)\}^2/A(\mathcal{P}_n) \geqslant 4d_n, \tag{26}$$

with equality if and only if $\mathcal{P}_n$ is regular. As a matter of fact, from Lemmas 3.6, 3.7 we see that it is certainly true. ∎

REMARK. A special case of inequality (21) in Theorem 3.8 for $r = 0$ was proved in [**7**]. For $\mathcal{P}_n = \mathcal{P}_{m,m}$, $m \geqslant 10$, we can improve inequalities (21) and (22).

THEOREM 3.9. *Let $m \geqslant 10$, $0 \leqslant r \leqslant n$, $\mathbf{w} \in \mathbf{R}_+^n$, $\mathbf{x} \in \triangle(\mathbf{w})$, and $\mathcal{P}_n = \mathcal{P}_{m,m}$.*

(a) $$\left\{\mathcal{L}_n^{[r]}[\mathbf{x};\mathbf{w}](\mathcal{P}_n)\right\}^2 / A(\mathcal{P}_n) \geqslant 4d_n\,[\alpha(m)]^{\sum_{i=1}^{n} w_i x_i((n-2)i+n)/(i(n-1))}.$$

(b) $$\{\mathcal{L}_{n,r}[\mathbf{x};\mathbf{w}](\mathcal{P}_n)\}^2 / A(\mathcal{P}_n) \geqslant 4d_n\,[\alpha(m)]^{2-n\sum_{i=1}^{n}(w_i x_i)/i}.$$

Equality holds in (a) or (b) if and only if $\mathcal{P}_n$ is regular.

PROOF: (a) By Lemma 3.6(b) and Lemma 3.7 we have

$$\left\{\mathcal{L}_n^{[n]}(\mathcal{P}_n)\right\}^2 / A(\mathcal{P}_n) \geqslant 4d_n\{\alpha(m)\}^{1/2}. \tag{27}$$

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It follows from (4) that

$$\left\{\mathcal{L}_n^{[r]}(\mathcal{P}_n)\right\}^2 / A(\mathcal{P}_n) \geqslant \left\{\mathcal{L}_n^{[1]}(\mathcal{P}_n)\right\}^{2(n-r)/r(n-1)} \left\{\mathcal{L}_n^{[n]}(\mathcal{P}_n)\right\}^{2n(r-1)/r(n-1)} / A(\mathcal{P}_n),$$

hence by Lemma 3.6 and (27), we have

$$\left\{\mathcal{L}_n^{[r]}(\mathcal{P}_n)\right\}^2 / A(\mathcal{P}_n) \geqslant 4d_n\{\alpha(m)\}^{(r(n-2)+n)/r(n-1)}. \tag{28}$$

Repeating the proof of Theorem 3.4 using (28) and (18) completes the proof.

(b) From the definition, we can see that

$$\left\{\mathcal{L}_n^{[n]}(\mathcal{P}_n)\right\}^n = \{\mathcal{L}_{n,r}(\mathcal{P}_n)\}^r \left\{\mathcal{L}_n^{[n-r]}(\mathcal{P}_n)\right\}^{n-r}. \tag{29}$$

By Lemma 3.6, $\left\{\mathcal{L}_n^{[n]}(\mathcal{P}_n)\right\}^2 = \mathcal{L}_n^{[1]}(\mathcal{P}_n)\mathcal{H}_n(\mathcal{P}_n)$, and $\mathcal{L}_n^{[1]}(\mathcal{P}_n) \geqslant \mathcal{L}_n^{[n-1]}(\mathcal{P}_n)$ by Lemma 1.1, hence

$$\begin{aligned}
\{\mathcal{L}_{n,r}(\mathcal{P}_n)\}^2 / A(\mathcal{P}_n) &= \left\{\mathcal{L}_n^{[n]}(\mathcal{P}_n)\right\}^{2n/r} / \left\{\mathcal{L}_n^{[n-r]}(\mathcal{P}_n)\right\}^{2(n-r)/r} A(\mathcal{P}_n) \\
&\geqslant \left\{\mathcal{L}_n^{[1]}(\mathcal{P}_n)\mathcal{H}_n(\mathcal{P}_n)\right\}^{n/r} / \left\{\mathcal{L}_n^{[1]}(\mathcal{P}_n)\right\}^{2(n-r)/r} A(\mathcal{P}_n) \\
&= \{\mathcal{H}_n(\mathcal{P}_n)\}^{n/r} \left\{\mathcal{L}_n^{[1]}(\mathcal{P}_n)\right\}^{2-n/r} / A(\mathcal{P}_n).
\end{aligned}$$

Thus, by (26), we get

$$\{\mathcal{L}_{n,n}(\mathcal{P}_n)\}^2 / A(\mathcal{P}_n) \geqslant 4d_n\{\alpha(m)\}^{2-n/r}. \tag{30}$$

From (30) and (18),

$$\{\mathcal{L}_{n,r}[\mathbf{x};\mathbf{w}](\mathcal{P}_n)\}^2 / A(\mathcal{P}_n) \geqslant \left\{\frac{n}{n-2} L_n^{[0]}[\mathbf{f};\mathbf{x};\mathbf{w}](\alpha)\right\}^2 / A(\mathcal{P}_n) \geqslant 4d_n\{\alpha(m)\}^{2-n\sum_{i=1}^{n}(w_i x_i)/i}.$$

∎

The isoperimetric inequalities (21) and (22) hold for some other types of polygons. For instance, if $\mathcal{P}_{21} = \mathcal{P}_{10,11}$, that is, $a_1 = a_2 = \ldots = a_{10}$ and $a_{11} = a_{12} = \ldots = a_{21}$, then (26) still holds. However, inequality (26) is not always true due to the following results.

THEOREM 3.10. *Let $\mathcal{P}_n = \mathcal{P}_{m,m}$.*

(a) If $3 \leqslant m \leqslant 8$, then

$$\{\mathcal{H}_n(\mathcal{P}_n)\}^2 / A(\mathcal{P}_n) \leqslant 4d_n,$$

with equality holding if and only if $\mathcal{P}_n$ is regular.

(b) For $m = 9$, we have

(i) $\{\mathcal{H}_{18}(\mathcal{P}_{18})\}^2 / A(\mathcal{P}_{18}) \leqslant 4d_{18}$, if $q \leqslant \phi(9)$.

(ii) $\{\mathcal{H}_{18}(\mathcal{P}_{18})\}^2 / A(\mathcal{P}_{18}) \geqslant 4d_{18}$, if $q \geqslant \phi(9)$.

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Equality holds if and only if $\mathcal{P}_{18}$ is regular and $q = \phi(9)$.

PROOF: (a) Since $\{\beta(m)\}^2 \leqslant \alpha(m)$ for $3 \leqslant m \leqslant 8$ because $\phi(3) = 16$, $\phi(4) = 8.116\ldots$, $\phi(5) = 5.788\ldots$, $\phi(6) = 4.348\ldots$, $\phi(7) = 3.078\ldots$, and $\phi(8) = 1.7705\ldots$.

(b) $\phi(9) = 0.3422\ldots$, and $\{\beta(m)\}^2 \leqslant \alpha(m)$ if and only if $q \leqslant \phi(9)$. ∎

One of the importances of the geometric isoperimetric inequalities is that they are closely related to the eigenvalue problems for the Laplace operator [**12, 14, 15**]. To conclude this paper, we shall point out that our new geometric findings can be used to reformulate the famous Pólya conjecture for polygons, and prove the conjecture in some special cases.

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