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ON THE HOMOLOGY OF FINITE ABELIAN COVERINGS OF LINKS

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ABSTRACT. Let A be a finite abelian group and M be a branched cover of an homology 3-sphere, branched over a link L, with covering group A. We show that $H_1(M; Z[1/|A|])$ is determined as a $Z[1/|A|][A]$-module by the Alexander ideals of L and certain ideal class invariants.

Let $L: \mu S^1 \longrightarrow \Sigma$ be a μ-component link in an homology 3-sphere Σ. The exterior of L is $X(L) = \Sigma - N(L)$, where $N(L)$ is an open regular neighbourhood of the image of L, and the group of L is $\pi L = \pi_1\big(X(L)\big)$. Given an epimorphism $\phi: \pi L \longrightarrow A$, we shall let $X_\phi(L)$ denote the corresponding covering space of $X(L)$; if A is finite $M_\phi(L)$ shall denote the corresponding branched cover of Σ, branched over L. (We shall henceforth assume that A is abelian). The homology groups of finite cyclic covering spaces were among the first invariants used to distinguish knots, and computing these groups for A abelian has remained a problem of continuing interest. In [Sa95] the second author gave precise formulae for the (first) Betti numbers of $X_\phi(L)$ and $M_\phi(L)$. The formulae involved the nullities of intermediate infinite cyclic coverings of sublinks, and thus these Betti numbers are computable from the Alexander ideals of L. In the same paper there are also estimates for the order of the torsion subgroups, which are precise only for very special cases. Here we shall show that if we extend coefficients to invert $|A|$ (the order of A) then these homology groups are determined as modules by the Alexander ideals of L, together with certain Steinitz-Fox-Smythe ideal class invariants. In particular, we may determine the part of the torsion subgroups of order coprime to $|A|$ from the Alexander ideals of L. In the final section we shall consider cyclic branched covers of links, and we shall give examples showing that in general the Alexander polynomials alone do not determine the full torsion subgroup.

1. **Localization away from the order of A.** Given a subring R of the field of rational numbers Q, let $R\Lambda_m = R[t_1, t_1^{-1}, \ldots, t_m, t_m^{-1}]$ and let $\varepsilon: R\Lambda_m \longrightarrow R$ be the augmentation homomorphism, determined by $\varepsilon(t_i) = 1$ for all i. (If $R = Z$ we write Λ_m rather than $Z\Lambda_m$, and if $m = 1$ we drop the subscript). If M is a finitely presentable $R\Lambda_m$-module we shall let $\Delta_i(M)$ denote the greatest common divisor of the elements of its i-th elementary ideal $E_i(M)$. Let ζ_n be a fixed primitive n-th root of unity, with minimal polynomial $\theta_n \in \Lambda$. If $\chi: A \longrightarrow S^1$ is a character of finite order n then $R_\chi = R[\zeta_n] \cong R\Lambda/(\theta_n)$ shall denote the ring of algebraic numbers generated by R and the values of χ.

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If $\phi\colon \pi L \longrightarrow A$ is an epimorphism $H_*\big(X_\phi(L);R\big)$ shall be considered as a module over $R[A]$, and tensor products shall be taken over this ring, unless otherwise indicated. The Alexander module of the link L is the Λ_μ-module $A(L) = H_1\big(X_\alpha(L),\tilde{*};Z\big)$, where $\alpha\colon \pi L \longrightarrow Z^\mu$ is the abelianization and $\tilde{*}$ is the preimage of a basepoint $* \in X(L)$ in the maximal abelian covering space $X_\alpha(L)$. We shall also let L_ϕ denote the sublink of L whose components have meridians mapped nontrivially by ϕ. (Note that if ϕ is trivial then L_ϕ is empty and $X(L_\phi) = \Sigma$).

In the next four lemmas we shall assume that A is a finite abelian group and that $R = Z[1/|A|]$.

LEMMA 1. *Let P be a set of characters $\chi\colon A \longrightarrow S^1$ such that every subgroup $B \leq A$ with A/B cyclic is the kernel of exactly one character in P. Then $R[A] \cong \bigoplus_{\chi\in P} R_\chi$.*

PROOF. For each $\chi \in P$ let $e_\chi = (|A|)^{-1}\Sigma\chi'(a)a$, where the sum runs over all $a \in A$ and all characters χ' with $\operatorname{Ker}\chi' = \operatorname{Ker}\chi$. Then the e_χ are mutually orthogonal idempotents and $\sum_{\chi\in P} e_\chi = 1$. ∎

This lemma corresponds to Lemma 9.2 of [Sa95].

LEMMA 2. *Let $\chi\colon A \longrightarrow S^1$ be a character. Then*

(i) $H_1\big(X_\phi(L);Z\big) \otimes R_\chi \cong H_1\big(X_{\chi\phi}(L);Z\big) \otimes R_\chi$;

(ii) $H_1\big(M_\phi(L);Z\big) \otimes R_\chi \cong H_1\big(M_{\chi\phi}(L_{\chi\phi});Z\big) \otimes R_\chi$.

PROOF. Let C_* be the singular chain complex for $X_\phi(L)$, considered as a complex of free $Z[A]$-modules. Then $C_* \otimes Z[A/\ker\chi]$ is the singular chain complex for $X_{\chi\phi}(L)$ and $C_* \otimes R_\chi = (C_* \otimes Z[A/\ker\chi]) \otimes R_\chi$. Since R_χ is a direct summand of a localization of $Z[A]$ it is a flat $Z[A]$-module, and so (i) follows.

The second assertion follows by use of the transfer. (See Chapter III.2 of [Br]). ∎

This lemma corresponds to step 1 of the proof of Theorem 8.1 of [Sa95].

LEMMA 3. *Let $\chi\colon A \longrightarrow S^1$ be a character. Then*

$$H_1\big(M_{\chi\phi}(L_{\chi\phi});Z\big) \otimes R_\chi \cong H_1\big(X_{\chi\phi}(L_{\chi\phi});Z\big) \otimes R_\chi.$$

PROOF. The homomorphism from $H_1\big(X_{\chi\phi}(L_{\chi\phi});Z\big) \otimes R_\chi$ to $H_1\big(M_{\chi\phi}(L_{\chi\phi});Z\big) \otimes R_\chi$ induced by the inclusion of $X_{\chi\phi}(L_{\chi\phi})$ into $M_{\chi\phi}(L_{\chi\phi})$ is an epimorphism, with kernel generated by lifts of multiples of meridians. Since each meridian of $L_{\chi\phi}$ has nontrivial image under $\chi\phi$ each such generator of this kernel is annihilated by $t^d - 1$, for some proper divisor d of the order of χ. But the images of such terms in R_χ are invertible, and so the kernel is 0. ∎

This is essentially the argument of Section 11 of [Sa95].

LEMMA 4. *Let $\chi\colon A \longrightarrow S^1$ be a character of order $n(\chi)$, and let $\tilde{\chi}\colon \pi L \longrightarrow Z$ be an epimorphism lifting $\chi\phi$. Then*

(i) $H_1\big(X(L);R\big) \cong A(L) \otimes R$;

(ii) $\Big(H_1\big(X_{\chi\phi}(L);Z\big) \otimes R_\chi\Big) \oplus R_\chi \cong A(L) \otimes R_\chi \cong \big(A(L) \otimes_{\tilde{\chi}} R\Lambda\big)/\theta_{n(\chi)}(A(L) \otimes_{\tilde{\chi}} R\Lambda)$, *if $\chi \neq 1$;*

(iii) $A(L) \otimes_{\tilde{\chi}} R\Lambda \cong H_1\big(X_{\tilde{\chi}}(L); R\big) \oplus R\Lambda$. *Moreover* $H_1(X_{\tilde{\chi}}(L); R)$ *has a square presentation matrix as an* $R\Lambda$*-module and* $\tilde{\chi}\big(E_1(L)\big)$ *is a principal ideal, generated by* $\tilde{\chi}\big(\Delta_1(L)\big)$ *if* $\mu = 1$ *and by* $(t-1)\tilde{\chi}\big(\Delta_1(L)\big)$ *if* $\mu > 1$.

PROOF. The link exterior $X(L)$ is homotopy equivalent to a finite 2-complex with one 0-cell and Euler characteristic 0. Hence the relative homology of covering spaces of the pair $\big(X(L), *\big)$ may be computed from chain complexes concentrated in degrees 1 and 2, and so $H_1\big(X_\phi(L), \hat{*}; R\big) \cong A(L) \otimes R[A]$ and $H_1\big(X_{\tilde{\chi}}(L), \hat{*}; R\big) \cong A(L) \otimes R\Lambda$ (where $\hat{*}$ is the preimage of $*$ in $X_\phi(L)$ or $X_{\tilde{\chi}}(L)$, respectively). Therefore the first two assertions and the first part of (iii) follow from the exact sequences of homology for such covering spaces, together with the facts that R_χ is a flat $Z[A]$-module and that $R \otimes R_\chi = 0$ unless $\chi = 1$.

The corresponding cellular chain complex for $X_{\tilde{\chi}}(L)$ (with coefficients R) is a free $R\Lambda$-chain complex C_* with $C_0 \cong R\Lambda$, $C_1 \cong R\Lambda^{a+1}$ and $C_2 \cong R\Lambda^a$, for some $a \geq 0$. Since $\operatorname{Im}\partial_1 = \operatorname{Ker}\varepsilon = (t-1)$ is free of rank 1 and projective $R\Lambda$ modules are free $\operatorname{Ker}\partial_1$ is free of rank a, and so $H_1\big(X_{\tilde{\chi}}(L); R\big)$ has a square presentation matrix. Since $\tilde{\chi}\big(E_1(L)\big) = E_1\big(A(L) \otimes_{\tilde{\chi}} R\Lambda\big) = E_0\Big(H_1\big(X_{\tilde{\chi}}(L); R\big)\Big)$ it is principal.

We may also see this directly. If $\mu = 1$ then $E_1(L) = \big(\Delta_1(L)\big)$, and so $\tilde{\chi}\big(E_1(L)\big) = \Big(\tilde{\chi}\big(\Delta_1(L)\big)\Big)$. If $\mu > 1$ then $E_1(L) = (t_1 - 1, \ldots, t_\mu - 1)\big(\Delta_1(L)\big)$. (See Theorem IV.3 of [Hi]). We may assume that $\tilde{\chi}$ maps the i-th meridian to d_i times a generator of Z. Since $\tilde{\chi}$ is an epimorphism the highest common factor $(d_1, \ldots, d_\mu)$ must be 1 and so $\big((t^{d_1} - 1), \ldots, (t^{d_\mu} - 1)\big) = (t-1)$, as ideals in Λ. Hence $\tilde{\chi}\big(E_1(L)\big) = \big((t^{d_1} - 1), \ldots, (t^{d_\mu} - 1)\big)\Big(\tilde{\chi}\big(\Delta_1(L)\big)\Big) = (t-1)\Big(\tilde{\chi}\big(\Delta_1(L)\big)\Big)$. (Note also that if $\tilde{\chi}\big(\Delta_1(L)\big) \neq 0$ then $\tilde{\chi}\big(E_1(L)\big)$ being principal implies that $H_1\big(X_{\tilde{\chi}}(L); R\big)$ has a square presentation matrix, by Theorem III.9 of [Hi]). ∎

It follows from these lemmas that $H_1\big(X_\phi(L); Z[1/|A|]\big)$ and $H_1\big(M_\phi(L); Z[1/|A|]\big)$ are determined as modules over $Z[1/|A|][A]$ by the homology of cyclic covers associated with the characters $\chi\phi$ and with coefficients $Z[\chi(A), 1/|A|]$, and these are essentially direct summands of quotients of the Alexander module. The coefficient rings are Dedekind domains, and the module theory of such rings is well understood.

LEMMA 5. *Let D be a Dedekind domain and M a finitely generated D-module, with D-torsion submodule T. Then $M \cong T \oplus (M/T)$ and M/T is projective of rank* $r = \min\{j | E_j(M) \neq 0\}$. *If $r > 0$ then $M/T \cong D^{r-1} \oplus \wedge_r(M/T)$, and $\wedge_r(M/T)$ is determined by the Steinitz-Fox-Smythe row ideal class derived from any presentation matrix for M. The torsion submodule T is determined by the elementary ideals of M.*

PROOF. Since M/T is torsion free it is projective, by the remark following Lemma 1.5 of [Mi], and so $M \cong T \oplus (M/T)$. The rank of M/T is $r = \min\{j | E_j(M/T) \neq 0\}$. If $r > 0$ then $(M/T) \cong D^{r-1} \oplus J$ for some ideal J of D, by Theorem 1.6 of [Mi]. It is clear that $J \cong \wedge_r(M/T)$ and so the ideal class of J is just the Steinitz-Fox-Smythe row class invariant of M, by Theorem III.12 of [Hi].

If Q is a finitely generated projective complement to M/T with $(M/T) \oplus Q \cong D^{r+s}$, say, then $E_r(M/T)E_s(Q) = (1)$, and so $E_i(M/T) = (1)$ for $i \geq r$. Since $E_i(M)$ is the ideal generated by $\bigcup_{0 \leq j \leq i} E_j(T)E_{i-j}(M/T)$, it follows that $E_i(M) = 0$ for $i < r$ and $E_{r+j}(M) = E_j(T)$ for $j \geq 0$. Now since T is a finitely generated torsion D-module its annihilator $\mathrm{Ann}(T)$ is nonzero, and so is contained in only finitely many maximal ideals $\{m_i | i \in I\}$, by Proposition I.3.6 of [Se]. The localization D_S with respect to the multiplicatively closed set $S = D - \bigcup_{i \in I} m_i$ is a PID, since it is a Dedekind domain with only finitely many maximal ideals. (See the Corollary to Proposition I.3.7 of [Se]). Hence $T = T_S$ is determined as an D_S-module by its elementary ideals $E_i(T_S) = E_i(T)_S$, which are generated by the images of the elementary ideals of H in D_S. ∎

REMARK. Any ideal in a Dedekind domain may be generated by at most two elements. (This follows easily from Lemma 1.10 of [Mi]).

LEMMA 6. *Let R be a subring of Q and let H be a finitely generated $R\Lambda$-module. Then $\bar{H} = H/\theta_n H$ is determined as a module by the elementary ideals of H and an ideal class invariant. In particular, the torsion subgroup T has order $|T| = \left|R\Lambda/\left(E_r(H), \theta_n\right)\right|$, where $r = \min\{j | E_j(H) \not\subset (\theta_n)\}$, and so $|T|$ is divisible in R by $\left|\mathrm{Res}\left(\Delta_r(H), \theta_n\right)\right|$.*

PROOF. The ring $R\Lambda/(\theta_n) = R[\zeta_n]$ is a Dedekind domain, by Theorem 1.4 of [Mi]. The quotient $\bar{H}$ is a finitely generated $R[\zeta_n]$-module, of rank $r = \min\{j | E_j(H) \not\subset (\theta_n)\}$, and its ($Z$-)torsion subgroup T is also its $R[\zeta_n]$-torsion submodule. Since the elementary ideals of $\bar{H}$ are just the images of the elementary ideals of H in $R[\zeta_n]$ the first assertion follows from Lemma 5.

Let $S = R[\zeta_n] - \bigcup_{i \in I} m_i$ be the complement of the maximal ideals dividing $\mathrm{Ann}(T)$. Since T_S and $R[\zeta_n]_S/E_0(T_S)$ are $R[\zeta_n]_S$-torsion modules they have finite composition series. The simple $R[\zeta_n]_S$-modules are the quotients $R[\zeta_n]_S/m_{iS} \cong R[\zeta_n]/m_i$ (for $i \in I$) and the number of simple factors isomorphic to a given simple module $R[\zeta_n]_S/m_{iS}$ in any such composition series for $R[\zeta_n]_S/E_0(T_S)$ is the same as for T_S. Hence $|T| = |T_S| = |R[\zeta_n]_S/E_0(T_S)| = |R[\zeta_n]/E_0(T)| = \left|R\Lambda/\left(E_r(H), \phi_n\right)\right|$.

The final observation is clear, since $\left(E_r(H), \theta_n\right) \subseteq \left(\Delta_r(H), \phi_n\right)$. ∎

If H is a Λ-module with a square presentation matrix P and θ_n does not divide $\Delta_0(H) = \det P$ then $r = 0$ and $H/\theta_n H = T$. In this case $|H/\theta_n H| = \left|\mathrm{Res}\left(\Delta_0(H), \theta_n\right)\right| = |\Pi \Delta_0(\omega)|$, where the product is taken over all primitive n-th roots of unity ω.

THEOREM. *Let L be a μ-component link in an homology 3-sphere Σ and let $\phi\colon \pi L \longrightarrow A$ be an epimorphism to a finite abelian group. Let $R = Z[1/|A|]$. Then*

(i) $H_1\left(X_\phi(L); Z[1/|A|]\right) \cong \oplus_{\chi \in P} H_1\left(X_{\chi\phi}(L); Z\right) \otimes R_\chi$;

(ii) $H_1\left(M_\phi(L); Z[1/|A|]\right) \cong \oplus_{\chi \in P} H_1\left(X_{\chi\phi}(L_{\chi\phi}); Z\right) \otimes R_\chi$.

In particular, $H_1\left(X_\phi(L); Z[1/|A|]\right)$ and $H_1\left(M_\phi(L); Z[1/|A|]\right)$ are determined as modules over $Z[1/|A|][A]$ by the Alexander ideals of L together with the Steinitz-Fox-Smythe row class invariants corresponding to characters $\chi \in P$ such that R_χ is not a PID.

PROOF. The direct sum decompositions follow from Lemmas 1–3, and the further assertions then follow from Lemmas 4 and 6. ∎

Let $\mathrm{null}(L;\chi) = \max\{d \mid \chi\phi\big(E_d(L)\big) = 0\}$ and let $E(L;\chi) = \chi\phi\big(E_{null(L;\chi)+1}(L)\big)$. (Thus $E(L;\chi)$ is an ideal in the ring of cyclotomic integers Z_χ). Let $\varphi(n) = [Q(\zeta_n) : Q]$ be Euler's totient function.

COROLLARY [SA95]. *(i)* $\beta_1\big(X_\phi(L)\big) = \sum_{\chi\in P}\varphi\big(n(\chi)\big)\mathrm{null}(L;\chi)$ *and* $\beta_1\big(M_\phi(L)\big) = \sum_{\chi\in P}\varphi\big(n(\chi)\big)\mathrm{null}(L_{\chi\phi};\chi)$*;*

(ii) Up to powers of primes dividing $|A|$ *we have* $\Big|\mathrm{Tor}\Big(H_1\big(X_\phi(L);Z\big)\Big)\Big| = \prod_{\chi\in P}|Z_\chi/E(L;\chi)|$ *and* $\Big|\mathrm{Tor}\Big(H_1\big(M_\phi(L);Z\big)\Big)\Big| = \prod_{\chi\in P}|Z_\chi/E(L_{\chi\phi};\chi)|$*. Moreover if* χ *has order* n *and* $\tilde\chi\colon\pi L \longrightarrow Z$ *is an epimorphism lifting* $\chi\phi$ *then* $|Z_\chi/E(L;\chi)| = |\Lambda/(\tilde\chi(E(L;\chi)),\theta_n)|$ *and is divisible by* $\Big|\mathrm{Res}\Big(\tilde\chi\big(\Delta_{null(L;\chi)+1}(L)\big),\theta_n\Big)\Big|$. ∎

Note in particular that $H_1\big(M_\phi(L);Z\big)$ is finite if and only if $\chi\phi\big(\Delta_1(L_{\chi\phi})\big) \neq 0$ in Q_χ, for all $\chi \in P$.

The above argument does not work for integral homology, since $Z[A]$ does not decompose as a direct sum of Dedekind domains. However the natural homomorphism from $Z[A]$ to $\oplus_{\chi\in P} Z_\chi$ is injective, and its cokernel F is a finite $Z[A]$-module with exponent dividing $|A|$. The long exact sequence of homology derived from the coefficient module sequence $0 \longrightarrow Z[A] \longrightarrow \oplus_{\chi\in P} Z_\chi \longrightarrow F \longrightarrow 0$ gives rise to the sequence

$$\cdots H_2\big(X(L);F\big) \longrightarrow H_1\big(X_\phi(L);Z\big) \longrightarrow \bigoplus_{\chi\in P} H_1\big(X_{\chi\phi}(L);Z\big)\otimes Z_\chi \longrightarrow H_1\big(X(L);F\big) \longrightarrow 0.$$

Mayberry and Murasugi give a formula for the order of $H_1\big(M_\phi(L);Z\big)$, when it is finite, without localization. (See Theorem 10.1 of [MM]). In our terms this formula reads approximately as follows: $\big|H_1\big(M_\phi(L);Z\big)\big| = D(\phi)\prod_{\chi\in P}\Big|\mathrm{Res}\Big(\tilde\chi\big(\Delta_1(L_{\chi\phi})\big),\theta_{n(\chi)}\Big)\Big|$, where $D(\phi)$ is an integer defined in [MM] which depends only on the homomorphism from $Z^\mu = (\pi L)^{ab}$ to A induced by ϕ, and $n(\chi)$ is the order of χ.

In [Sa82] it is shown that the order of the cokernel of the natural homomorphism from $H_1\big(X_\alpha(L);Z\big)$ to $H_1\big(M_\phi(L);Z\big)$ is $R(\phi) = \Pi n_i/|A|$, where n_i is the order of the image $\phi(t_i)$ of the i-th meridian. How is this number related to the term $D(\phi)$ of [MM]?

2. **Cyclic branched coverings.**

In this section we shall assume that $A \cong Z/nZ$ and that ϕ maps each meridian of L to a generator of A. (In the terminology of [MM] the covering is meridian cyclic or ξ-cyclic).

Let $\tilde\phi\colon\pi L \longrightarrow Z$ be an epimorphism lifting ϕ, and let $\nu = (t^n - 1)/(t-1)$. The coefficient module sequence $0 \longrightarrow Z \longrightarrow \Lambda/(t^n-1) \longrightarrow \Lambda/(\nu) \longrightarrow 0$ gives rise to an exact sequence $H_1\big(X(L);Z\big) \longrightarrow H_1\big(X_\phi(L);Z\big) \longrightarrow H_1\big(X(L);\Lambda/(\nu)\big) \longrightarrow 0$, where the first map is the transfer. Since ϕ maps each meridian of L to a generator of A the image of the transfer is the submodule generated by the lifts of the n-th powers of the meridians and so $H_1\big(M_\phi(L);Z\big) \cong H_1\big(X(L);\Lambda/(\nu)\big)$. On the other hand the latter group may be described as a quotient of the homology of $X_{\tilde\phi}(L)$, via the short exact sequence of chain complexes $0 \longrightarrow C_* \longrightarrow C_* \longrightarrow C_*/\nu \longrightarrow 0$, where C_* is the singular chain complex for $X_{\tilde\phi}(L)$. Hence $H_1\big(M_\phi(L);Z\big) \cong H_1\big(X_{\tilde\phi}(L);Z\big)/\nu H_1\big(X_{\tilde\phi}(L);Z\big)$. It is finite if and only if

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$\left|\mathrm{Res}\left(\tilde{\phi}\left(\Delta_1(L)\right),\nu\right)\right| \neq 0$, in which case it has order $\left|\mathrm{Res}\left(\tilde{\phi}\left(\Delta_1(L)\right),\nu\right)\right|$ (if $\mu = 1$) or $n\left|\mathrm{Res}\left(\tilde{\phi}(\Delta_1(L)),\nu\right)\right|$ (if $\mu > 1$). (See [Sa79], [Sa81]. Note also that these formulae are special cases of the formula of [MM], in the light of their Theorem 4.5).

Suppose that $\tilde{\phi}\left(\Delta_1(L)\right) \neq 0$ and let H be the image of $H_1\left(X_\alpha(L);Z\right)$ in $H_1\left(X_{\tilde{\phi}}(L);Z\right)$. The cokernel of the inclusion of H into $H_1\left(X_{\tilde{\phi}}(L);Z\right)$ is $Z^{\mu-1} = \left(\Lambda/(t-1)\right)^{\mu-1}$ (by an iterated Wang sequence argument). Hence $H_1\left(M_\phi(L);Z\right)$ is an extension of $(Z/nZ)^{\mu-1} = \left(\Lambda/(t-1,\nu)\right)^{\mu-1}$ by $H/\nu H$. Since $H_1\left(X_{\tilde{\phi}}(L);Z\right)$ has a square presentation matrix with nonzero determinant it is a torsion Λ-module of projective dimension ≤ 1. Hence H is also a torsion Λ-module of projective dimension ≤ 1 and so also has a square presentation matrix with nonzero determinant, by Theorem III.9 of [Hi]. If $\nabla_{\tilde{\phi}}(L)$ is the latter determinant then $\nabla_{\tilde{\phi}}(L) = \tilde{\phi}\left(\Delta_1(L)\right)/(t-1)^{\mu-1}$, by Lemma III.5 of [Hi]. If $n = p^r$ is a prime power then $\mathrm{Res}\left(\nabla_{\tilde{\phi}}(L),\nu\right) \equiv \varepsilon\left(\nabla_{\tilde{\phi}}(L)\right)^{p^r-p}$ modulo (p), and so $H/\nu H$ is finite of order prime to p if $\left(\varepsilon\left(\nabla_{\tilde{\phi}}(L)\right),p\right) = 1$. Moreover if $s_1 < s_2$ then $(\theta_{p^{s_1}},\theta_{p^{s_2}}) = (\theta_{p^{s_1}},p)$. Hence $H/\nu H \cong \oplus_{s=1}^{r}(H/\theta_{p^s}H)$ and Lemma 5 applies to give the structure of these summands.

If ϕ maps all meridians to the same generator of A then $\tilde{\phi}$ is (up to sign) the total linking number homomorphism and $\nabla(L)$ is the Hosokawa polynomial of L. (In the terminology of [MM] the covering is strictly cyclic). In this case the formula for the order of $H_1\left(M_\phi(L);Z\right)$ was first given in [Fo] for knots in S^3, and in [HK] for links in S^3.

If L is a knot (*i.e.*, $\mu = 1$) then $H = A(L)$ is the Alexander module of L, $\nabla(L) = \Delta_1(L)$ and $\varepsilon\left(\nabla(L)\right) = \pm 1$, and $H_1\left(X_\phi(L);Z\right) \cong H_1\left(M_\phi(L);Z\right) \oplus Z$. If we specialize further to the case when $n = p$ is a prime then $H_1\left(M_\phi(L);Z\right)$ is a finite $Z[\zeta_p]$-module of order prime to p. When $p = 2$ it is easily seen that any finite group of odd order may be realised. (Cyclic groups of odd order may be realized by knots with cyclic knot module; the general case follows on taking connected sums). The modules that arise from knots in this way when p is odd have been determined in [Da]. His description involves consideration of the linking pairing on the torsion of $H_1\left(M_\phi(L);Z\right)$, on which the covering group acts isometrically, and triviality of an ideal class invariant of the torsion, deriving from the fact that $M_\phi(L)$ is homotopy equivalent to a finite complex on which A acts cellularly. He gives also criteria for the case $\mu > 1$ and $n = p$ an odd prime, modulo p-primary torsion; here the ideal class invariant for the torsion may be nontrivial, but is determined by the Steinitz-Fox-Smythe invariant for the torsion-free part.

If the first homology of such a cyclic branched cover of S^3 is finite then its order is determined by the Alexander polynomial of the branch set. However the polynomials need not determine the structure. For example, the Alexander ideals for the knots 6_1 and 9_{46} are $E_1(6_1) = E_1(9_{46}) = (2t^2 - 5t + 2)$, $E_2(6_1) = (1)$, $E_2(9_{46}) = (3, t+1)$ and $E_j(6_1) = E_j(9_{46}) = (1)$ for all $j \geq 3$. Hence these knots each have first Alexander polynomial $2t^2 - 5t + 2$ and higher Alexander polynomials 1, but the first homology groups of the 2-fold branched covers are $Z/9Z$ and $(Z/3Z)^2$, respectively.

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Weber has given examples to show that in general the Alexander polynomials alone do not even determine the prime divisors of the order of the torsion of $H_1\big(M_n(K); Z[1/n]\big)$, if this group is infinite [We]. It remains open to what extent the Alexander ideals determine the torsion.

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