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AN IRREDUCIBLE REPRESENTATION OF sl(2)

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In a recent paper [1] MM. Arnal and Pinczon have classified all complex irreducible representations (ρ, V) of sl(2) having the property (P) that there exists a non-zero element $x \in \mathrm{sl}(2)$ such that $\rho(x)$ admits an eigenvalue. It is the purpose of this note to demonstrate, by example, that there exist irreducible representations of sl(2) which do not have property (P). As usual, we consider sl(2) embedded in its universal enveloping algebra U and identify the representations of sl(2) and U.

By a *Cartan basis* of sl(2), we shall mean a linear basis $\{Y, X, H\}$ satisfying $[X, Y]=H$, $[H, X]=2X$ and $[H, Y]=-2Y$. If $\{Y', X', H'\}$ is a second Cartan basis of sl(2), we have that $H=\alpha X'+\beta Y'+\gamma H'$ where $\alpha\beta+\gamma^2=1$. The corresponding expressions for the elements X and Y in terms of the basis $\{Y', X', H'\}$ must be separated into three types. First, if $\beta\neq 0$, then

$$\text{(1)} \qquad \begin{aligned} Y &= k_1\{\beta^2 Y' - (\gamma-1)^2 X' + \beta(\gamma-1)H'\} \\ X &= k_2\{\beta^2 Y' - (\gamma+1)^2 X' + \beta(\gamma+1)H'\} \end{aligned}$$

where $k_1k_2=-1/4\beta^2$

Secondly, if $\beta=0$ and $\gamma=1$, then

$$\text{(2)} \qquad \begin{aligned} Y &= k_1\left\{Y' - \frac{\alpha^2}{4}X' - \frac{\alpha}{2}H'\right\} \\ X &= k_2 X' \end{aligned}$$

where $k_1k_2 = 1$

Finally, if $\beta=0$ and $\gamma=-1$ then

$$\text{(3)} \qquad \begin{aligned} Y &= k_1 X' \\ X &= k_2\left\{Y' - \frac{\alpha^2}{4}X' + \frac{\alpha}{2}H'\right\} \end{aligned}$$

where $k_1k_2 = 1$

Now fix one Cartan basis $\{Y, X, H\}$ of sl(2) and let M denote a maximal left ideal of U containing Y^2+X-1. Such a maximal left ideal exists since Y^2+X-1 is not invertible in U. We claim that the left regular irreducible representation of sl(2) on U modulo M is not equivalent to any irreducible representation of sl(2) having property (P). To prove this, it suffices to show that for any irreducible

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representation (ρ, V) of sl(2) having property (P), $\rho(Y^2+X-1)$ has zero kernel since, if U/M is equivalent to (ρ, V) we would have ker $\rho(Y^2+X-1)\neq\{0\}$. Using the classification of [1] adapted to our notation, we divide the irreducible representations of sl(2) having property (P) into three cases.

CASE I. Suppose (ρ, V) is a finite dimensional, say dim $V=m+1$, irreducible representations of sl(2). Then V admits a basis $\{v_0, \ldots, v_m\}$ such that

$$\begin{aligned} \rho(H)v_i &= (m-2i)v_i && \text{for all } i = 0, 1, \ldots, m \\ \rho(Y)v_i &= v_{i+1} && \text{for } i = 0, 1, \ldots, m-1 \\ &\quad 0 && \text{for } i = m \\ \rho(X)v_i &= i(m-(i-1))v_{i-1} && \text{for } i = 0, 1, 2, \ldots, m \end{aligned}$$

Take $v=\lambda_0v_0+\cdots+\lambda_mv_m\in\ker\rho(Y^2+X-1)$ then by direct computation

$$0 = \rho(Y^2+X-1)v = (m\lambda_1-\lambda_0)v_0+(2(m-1)\lambda_2-\lambda_1)v_1+\cdots+(\lambda_{m-2}-\lambda_m)v_m$$

Solving this system of equations, we find that $\lambda_i=0$ for all i and hence ker $\rho(Y^2+X-1)=\{0\}$.

CASE II. Suppose (ρ, V) is an infinite dimensional irreducible representation of sl(2) for which there exists a Cartan basis $\{Y', X', H'\}$ with $\rho(H')$ admitting an eigenvalue. Without loss of generality, we may assume the eigenvalues of $\rho(H')$ have no lower bound (they may or may not have an upper bound) and V admits a basis $\{\cdots v_{-1}, v_0, v_1, \ldots\}$ of eigenvectors of $\rho(H')$ with

$$\begin{aligned} \rho(H')v_i &= (\alpha-2i)v_i && \text{for all } i \\ \rho(Y')v_i &= v_{i+1} && \text{for all } i \\ \rho(X')v_i &= \text{non zero multiple of } v_{i-1} && \text{for } i \text{ not minimum} \\ &\quad 0 && \text{for } i \text{ minimum index.} \end{aligned}$$

Let $v=\lambda_iv_i+$terms with lower index be an element of ker $\rho(Y^2+X-1)$. Assuming that the Cartan bases $\{Y, X, H\}$ and $\{Y', X', H'\}$ are related as in (1), we have

$$\rho(Y^2+X-1) = k_1^2\beta^4\rho(Y')^2+\text{terms involving } \rho(Y'X'),\ \rho(X')^2, \text{ etc.}$$

Then $\rho(Y^2+X-1)v$ contains a non-zero multiple of v_{i+2}. Thus ker $\rho(Y^2+X-1)=\{0\}$. If the Cartan bases are related as in (2) or (3), we can, in an analogous manner, verify that ker $\rho(Y^2+X-1) = \{0\}$.

CASE III. Finally, suppose (ρ, V) is an irreducible representation of sl(2) for which there exists a Cartan basis $\{Y', X', H'\}$ with $\rho(X')$ admitting an eigenvalue

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1. Then there exists a basis $\{v_0, v_1, v_2, \ldots\}$ of v with

$$\rho(H')v_i = v_{i+1} \quad \text{for all } i$$
$$\rho(X')v_i = (\rho(H')-2)^i v_0 \quad \text{for all } i$$
$$\rho(Y')v_i = (\rho(H')+2)^i(\gamma v_0 - \tfrac{1}{2}v_1 - \tfrac{1}{4}v_2) \quad \text{for all } i$$

(γ is an arbitrary scalar).

Again let $v=\lambda_0 v_0 + \cdots + \lambda_n v_n$ belong in the kernel of $\rho(Y^2+X-1)$. Assuming that the Cartan bases $\{Y, X, H\}$ and $\{Y', X', H'\}$ are related as in (1), we have

$$\rho(Y^2+X-1) = k_1^2\beta^4\rho(Y')^2 + \text{terms involving } \rho(Y'X'),\ \rho(X')^2, \text{ etc.}$$

Then $\rho(Y^2+X-1)v$ contains a non-zero multiple of v_{n+4} and we conclude that $\ker \rho(Y^2+X-1)=\{0\}$. If the Cartan bases are related as in (2) or (3) by similar considerations, we verify that $\ker \rho(Y^2+X-1)=\{0\}$.

Thus for all irreducible representations (ρ, V) having property (P) we have $\ker \rho(Y^2+X-1)=\{0\}$ and hence the constructed irreducible representation does not have property (P).

REMARK. I have recently learned that MM. Arnal and Pinczon [C.R. Acad. Sc. Paris, t 274 (1972) pp. 248–250] have also constructed irreducible representations of sl(2) which do not have property (P). Their construction arises from a study of the action of the enveloping algebra automorphisms on the equivalence classes of irreducible representations.

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