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Projective Reconstruction in Algebraic Vision

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Abstract. We discuss the geometry of rational maps from a projective space of an arbitrary dimension to the product of projective spaces of lower dimensions induced by linear projections. In particular, we give an algebro-geometric variant of the projective reconstruction theorem by Hartley and Schaffalitzky.

1 Introduction

Let r be a positive integer and $\boldsymbol{m} = (m_1, \ldots, m_r)$ be a sequence of positive integers. For each $i = 1, \ldots, r$, take a vector space W_i of dimension $m_i + 1$ over a field $\mathbf{k}$, which we assume to be an algebraically closed field of characteristic zero unless otherwise stated.[1] Further, let V be a vector space satisfying $n := \dim V - 1 > m_i$ for any $i = 1, \ldots, r$. A sequence $\boldsymbol{s} = (s_1, \ldots, s_r)$ of surjective linear maps

$$s_i \colon V \longrightarrow W_i, \quad i = 1, \ldots, r$$

induce rational maps

$$\varphi_i \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{m_i}, \quad i = 1, \ldots, r$$

from $\mathbb{P}^n := \mathbb{P}(V)$ to $\mathbb{P}^{m_i} := \mathbb{P}(W_i)$. We call these rational maps *cameras*, with the model of a pinhole camera as a linear projection in mind. Correspondingly, the loci

$$Z_i := \mathbb{P}(\ker s_i) \subset \mathbb{P}^n, \quad i = 1, \ldots, r$$

of indeterminacy are called the *focal loci* of the cameras. The closure X of the image of the rational map

$$\boldsymbol{\varphi} := (\varphi_1, \ldots, \varphi_r) \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{\boldsymbol{m}} := \prod_{i=1}^{r} \mathbb{P}^{m_i}$$

is called the *multiview variety* in the case $n = 3$ and $\boldsymbol{m} = (2^r)$ in [AST13], and we use the same terminology for arbitrary n and $\boldsymbol{m}$. In this section, we assume $|\boldsymbol{m}| := m_1 + \cdots + m_r \geq n + 1$, so that the multiview variety X is a proper subvariety of $\mathbb{P}^{\boldsymbol{m}}$.

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[1] This condition on the field $\mathbf{k}$ allows us to use standard tools in complex algebraic geometry, such as the theorem of Bertini. From the viewpoint of application to computer vision, where the motivation for this paper comes from, the case $\mathbf{k} = \mathbb{R}$ is of particular interest. As we mention later in this section, the reconstruction theorem for $\mathbf{k} = \mathbb{R}$ follows from the reconstruction theorem for $\mathbf{k} = \mathbb{C}$.

Basic properties of multiview varieties are studied in [Li], where formulas for dimensions, multidegrees, and Hilbert polynomials are obtained and the Cohen–Macaulay property is proved.

The *projective reconstruction problem* asks if $\boldsymbol{\varphi}$ is determined uniquely from X, up to the inevitable ambiguity by the action of $\mathrm{PGL}(n+1, \mathbf{k})$. In real-life applications where $\mathbf{k} = \mathbb{R}$, $n = 3$, and $\boldsymbol{m} = (2^r)$, this problem may be phrased as follows. Assume that one is given multiple pictures of one place, taken with various cameras whose positions and angles are not known at the beginning. A *point correspondence* is a collection of points in the pictures, consisting of one point in every picture, which is the image of the same point in the place. The set of point correspondences form an open subset of X. Assume that one can tell sufficiently many point correspondences, say, from the features of the objects, so that one can fix X uniquely. Now the problem is whether one can "reconstruct" the 3-dimensional configuration of objects in the place appearing on the pictures, together with the configuration of the cameras.

Each camera $\varphi_i : \mathbb{P}^n \dashrightarrow \mathbb{P}^{m_i}$ is parametrized by an open subset of the projective space $\mathbb{P}(V^\vee \otimes W_i)$, where the corresponding linear map s_i has the full rank. Let

$$\Phi: \prod_{i=1}^{r} \mathbb{P}\left(V^\vee \otimes W_i\right) \dashrightarrow \mathrm{Hilb}\left(\mathbb{P}^{\boldsymbol{m}}\right) \tag{1.1}$$

be the rational map sending a camera configuration $\boldsymbol{\varphi}$ to the multiview variety X considered as a point in the Hilbert scheme $\mathrm{Hilb}\left(\mathbb{P}^{\boldsymbol{m}}\right)$ of subschemes of $\mathbb{P}^{\boldsymbol{m}}$. The natural right action of $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}(n+1, \mathbf{k})$ on each $\mathbb{P}\left(V^\vee \otimes W_i\right)$ induces the diagonal action on the product $\prod_{i=1}^r \mathbb{P}\left(V^\vee \otimes W_i\right)$. The main result of this paper is the following theorem.

Theorem 1.1 (i) *If* $\boldsymbol{m} \neq (1^{n+1}) := (1, \dots, 1)$, *then a general fiber of* Φ *consists of a single* $\mathrm{PGL}(n+1, \mathbf{k})$*-orbit.*

(ii) *If* $\boldsymbol{m} = (1^{n+1})$, *then a general fiber of* Φ *consists of two* $\mathrm{PGL}(n+1, \mathbf{k})$*-orbits.*

(iii) *If* $|\boldsymbol{m}| \geq 2n-1$, *then* Φ *is dominant onto an irreducible component of* $\mathrm{Hilb}(\mathbb{P}^{\boldsymbol{m}})$.

As we see in Section 6, Theorem 1.1(i),(ii) is an algebro-geometric variant of the projective reconstruction theorem by Hartley and Schaffaltzky [HS09], with a new purely algebro-geometric proof. A result closely related to Theorem 1.1(iii) for $n = 3$ and $\boldsymbol{m} = (2^r)$ is proved in [AST13, Theorem 6.3]. They use the rational map

$$\gamma: G\Big(n+1, \bigoplus_{i=1}^{r} W_i\Big) /\!\!/ \mathbb{G}_m^r \dashrightarrow \mathcal{H}_{n,\boldsymbol{m}}$$

instead of (1.1), where $G(n+1, \bigoplus_{i=1}^r W_i) /\!\!/ \mathbb{G}_m^r$ denotes a GIT quotient of the Grassmannian of $(n+1)$-dimensional subspaces in $\bigoplus_i W_i$ by an algebraic torus, and $\mathcal{H}_{n,\boldsymbol{m}}$ denotes the multigraded Hilbert scheme parametrizing $\mathbb{Z}^r$-homogeneous ideals in the homogeneous coordinate ring of $\mathbb{P}^{\boldsymbol{m}}$. Their result states that the rational map γ dominates an irreducible component of $\mathcal{H}_{n,\boldsymbol{m}}$. The bound $|\boldsymbol{m}| \geq 2n-1$ is sharp in this case. The embedding of the space of cameras into $\mathrm{Hilb}(\mathbb{P}^{\boldsymbol{m}})$ for $\boldsymbol{m} = (2^r)$ is also discussed in [LV].

Note that a real configuration of cameras can naturally be viewed as a complex configuration of cameras, and a pair of real configurations of cameras are related by an action of $\mathrm{PGL}(n+1, \mathbb{R})$ if and only if they are related by an action of $\mathrm{PGL}(n+1, \mathbb{C})$.

It follows that the reconstruction over $\mathbb{C}$ in Theorem 1.1(i) implies the reconstruction over $\mathbb{R}$. See also Remark 4.5 for the fact that Theorem 1.1(ii) also holds over $\mathbb{R}$.

This paper is organized as follows. In Section 2, we collect basic facts about linear projections. The statements (i), (ii), and (iii) in Theorem 1.1 are proved in Sections 3, 4, and 5, respectively. In Section 6, we clarify the relation of Theorem 1.1 to results by Hartley and Schaffalitzky.

2 The Geometry of Linear Projections

Let V and W be vector spaces of dimensions $n+1$ and $m+1$, respectively, with $n > m$, and set $\mathbb{P}^n := \mathbb{P}(V)$ and $\mathbb{P}^m := \mathbb{P}(W)$. Further, let $p\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$ and $q\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ be the projections to the first and the second factors. A surjective linear map $s\colon V \to W$ induces a *linear projection*

$$\varphi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^m,$$

which is a dominant rational map from $\mathbb{P}^n$ to $\mathbb{P}^m$.

The locus $Z := \mathbb{P}(\ker s) \subset \mathbb{P}^n$ of indeterminacy of φ can be eliminated by the blow-up

$$\widetilde{p}\colon \widetilde{X} := \mathrm{Bl}_Z \mathbb{P}^n \longrightarrow \mathbb{P}^n$$

along Z; *i.e.,* there exists a morphism $\widetilde{q}\colon \widetilde{X} \to \mathbb{P}^m$ making the diagram

$$\begin{array}{ccc} & \widetilde{X} & \\ {}^{\widetilde{p}}\swarrow & & \searrow^{\widetilde{q}} \\ \mathbb{P}^n & \overset{\varphi}{\dashrightarrow} & \mathbb{P}^m \end{array}$$

commutative.

The exceptional divisor $E := \widetilde{p}^{-1}(Z) \subset \widetilde{X}$ is the $\mathbb{P}^m$-bundle over Z obtained as the projectivization of the normal bundle $N_{Z/\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus m+1}$ of Z in $\mathbb{P}^n$. For a hyperplane $H \subset \mathbb{P}^n$ containing Z, the closure of the image of H by φ is a hyperplane H' in $\mathbb{P}^m$. Since $\widetilde{p}^* H = \widetilde{q}^* H' + E$, we have

$$\mathcal{O}_{\widetilde{X}}(E) \cong \widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(H) \otimes \widetilde{q}^* \mathcal{O}_{\mathbb{P}^m}(-H') \cong \widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes \widetilde{q}^* \mathcal{O}_{\mathbb{P}^m}(-1). \tag{2.1}$$

The morphism

$$\widetilde{p} \times \widetilde{q}\colon \widetilde{X} \longrightarrow \mathbb{P}^n \times \mathbb{P}^m$$

is an embedding, which allows one to identify $\widetilde{X}$ with the closure of the graph of the rational map φ. Under this embedding, the morphisms $\widetilde{p}$ and $\widetilde{q}$ are restrictions of the projections p and q, respectively.

The Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^m}(-1) \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}^m} \longrightarrow T_{\mathbb{P}^m}(-1) \longrightarrow 0 \tag{2.2}$$

gives $H^0\left(T_{\mathbb{P}^m}(-1)\right) \cong W$, which together with $H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \cong V^\vee$ shows

$$\begin{aligned} H^0\big(p^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes q^* T_{\mathbb{P}^m}(-1)\big) &\cong H^0\big(\mathcal{O}_{\mathbb{P}^n}(1)\big) \otimes H^0\big(T_{\mathbb{P}^m}(-1)\big) \\ &\cong V^\vee \otimes W. \end{aligned} \tag{2.3}$$

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Let $s_{\mathbb{P}}$ be the element of $H^0\left(p^*\mathcal{O}_{\mathbb{P}^n}(1)\otimes q^*T_{\mathbb{P}^m}(-1)\right)$ corresponding to $s\in V^\vee\otimes W$ under the isomorphism (2.3). Since $T_{\mathbb{P}^m}(-1)$ is the universal quotient bundle on $\mathbb{P}^m$, a point $(\ell_1,\ell_2)\in\mathbb{P}^n\times\mathbb{P}^m$ (*i.e.*, a pair of one-dimensional subspaces $\ell_1\subset V$ and $\ell_2\subset W$) is in $s_{\mathbb{P}}^{-1}(0)$ if and only if $s(\ell_1)\subset\ell_2$. In other words, the zero locus of $s_{\mathbb{P}}$ is precisely the graph $\widetilde{X}$ of φ:

$$s_{\mathbb{P}}^{-1}(0)=\widetilde{X}\subset\mathbb{P}^n\times\mathbb{P}^m. \tag{2.4}$$

The pull-back of the Euler sequence (2.2) to $\widetilde{X}$ tensored with $\widetilde{p}^*\mathcal{O}_{\mathbb{P}^n}(1)$ gives

$$\begin{aligned}0\longrightarrow\widetilde{p}^*\mathcal{O}_{\mathbb{P}^n}(1)\otimes\widetilde{q}^*\mathcal{O}_{\mathbb{P}^m}(-1)\longrightarrow\widetilde{p}^*\mathcal{O}_{\mathbb{P}^n}(1)\otimes W&\\ \longrightarrow\widetilde{p}^*\mathcal{O}_{\mathbb{P}^n}(1)\otimes\widetilde{q}^*T_{\mathbb{P}^m}(-1)\longrightarrow 0.&\end{aligned} \tag{2.5}$$

Let $s_{\widetilde{X}}\in H^0(\widetilde{X},\widetilde{p}^*\mathcal{O}_{\mathbb{P}^n}(1)\otimes W)$ be the section corresponding to $s\in V^\vee\otimes W$ under the isomorphism

$$\begin{aligned}H^0\left(\widetilde{X},\widetilde{p}^*\mathcal{O}_{\mathbb{P}^n}(1)\otimes W\right)&\cong H^0\left(\mathbb{P}^n,\widetilde{p}_*\widetilde{p}^*\mathcal{O}_{\mathbb{P}^n}(1)\right)\otimes W\\&\cong H^0\left(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(1)\right)\otimes W\\&\cong V^\vee\otimes W.\end{aligned}$$

The section $s_{\widetilde{X}}$ lies in the image of the injection

$$H^0\left(\widetilde{X},\widetilde{p}^*\mathcal{O}_{\mathbb{P}^n}(1)\otimes\widetilde{q}^*\mathcal{O}_{\mathbb{P}^m}(-1)\right)\hookrightarrow H^0\left(\widetilde{X},\widetilde{p}^*\mathcal{O}_{\mathbb{P}^n}(1)\otimes W\right)$$

induced by (2.5), since its image by the map

$$H^0\left(\widetilde{X},\widetilde{p}^*\mathcal{O}_{\mathbb{P}^n}(1)\otimes W\right)\longrightarrow H^0\left(\widetilde{X},\widetilde{p}^*\mathcal{O}_{\mathbb{P}^n}(1)\otimes\widetilde{q}^*T_{\mathbb{P}^m}(-1)\right)$$

induced by (2.5) is zero by (2.4). It follows from (2.1) and $h^0\left(\mathcal{O}_{\widetilde{X}}(E)\right)=1$ that $s_{\widetilde{X}}^{-1}(0)=E$.

3 Projective Reconstruction in the Case $\boldsymbol{m}\neq(1^{n+1})$

We keep the same notation as in Section 1 and write the projections as $p\colon\mathbb{P}^n\times\mathbb{P}^{\boldsymbol{m}}\to\mathbb{P}^n$, $q_i\colon\mathbb{P}^n\times\mathbb{P}^{\boldsymbol{m}}\to\mathbb{P}^{m_i}$, $\boldsymbol{q}:=(q_1,\dots,q_r)\colon\mathbb{P}^n\times\mathbb{P}^{\boldsymbol{m}}\to\mathbb{P}^{\boldsymbol{m}}$, and $\varpi_i\colon\mathbb{P}^{\boldsymbol{m}}\to\mathbb{P}^{m_i}$. We do not assume $|\boldsymbol{m}|\geq n+1$ unless otherwise stated. The Euler sequence

$$0\longrightarrow\mathcal{O}_{\mathbb{P}^{m_i}}(-1)\longrightarrow W_i\otimes\mathcal{O}_{\mathbb{P}^{m_i}}\longrightarrow T_{\mathbb{P}^{m_i}}(-1)\longrightarrow 0$$

gives $H^0\left(T_{\mathbb{P}^{m_i}}(-1)\right)=W_i$, which together with $H^0(\mathcal{O}_{\mathbb{P}^n}(1))=V^\vee$ shows

$$\begin{aligned}H^0\Big(\mathbb{P}^n\times\mathbb{P}^{\boldsymbol{m}},\bigoplus_{i=1}^r p^*\mathcal{O}_{\mathbb{P}^n}(1)\otimes q_i^*T_{\mathbb{P}^{m_i}}(-1)\Big)&=\bigoplus_{i=1}^r V^\vee\otimes W_i\\&\cong\bigoplus_{i=1}^r\operatorname{Hom}(V,W_i).\end{aligned} \tag{3.1}$$

We abuse notation and identify $\boldsymbol{s}\in\bigoplus_{i=1}^r\operatorname{Hom}(V,W_i)$ with the corresponding global section of $\bigoplus_{i=1}^r p^*\mathcal{O}_{\mathbb{P}^n}(1)\otimes q_i^*T_{\mathbb{P}^{m_i}}(-1)$ on $\mathbb{P}^n\times\mathbb{P}^{\boldsymbol{m}}$. Let $\widetilde{X}\subset\mathbb{P}^n\times\mathbb{P}^{\boldsymbol{m}}$ be the closure of

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the graph of the rational map $\boldsymbol{\varphi}\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{\boldsymbol{m}}$, and set $\widetilde{p} := p|_{\widetilde{X}}$, $\widetilde{q}_i := q_i|_{\widetilde{X}}$, and $\widetilde{\boldsymbol{q}} := \boldsymbol{q}|_{\widetilde{X}}$, so that we have the diagram

$$\begin{array}{ccccc} & & \widetilde{X} & & \\ & \overset{\widetilde{p}}{\swarrow} & & \overset{\widetilde{\boldsymbol{q}}}{\searrow} & \\ \mathbb{P}^n & & \overset{\boldsymbol{\varphi}}{\dashrightarrow} & & X \subset \mathbb{P}^{\boldsymbol{m}}. \end{array}$$

In all the statements, such as lemmas, propositions, and theorems, through the rest of this paper, we assume that the section $\boldsymbol{s}$ is general.

Lemma 3.1 *Let $Z_{\boldsymbol{s}} \subset \mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}$ be the zero locus of the section $\boldsymbol{s}$ in (3.1). For $z \in \mathbb{P}^n$, one has*

$$Z_{\boldsymbol{s}} \cap p^{-1}(z) = \{z\} \times \prod_{i:z\notin Z_i} \varphi_i(z) \times \prod_{i:z\in Z_i} \mathbb{P}^{m_i}. \tag{3.2}$$

Proof Let $v \in V$ be a vector corresponding to $z \in \mathbb{P}^n = \mathbb{P}(V)$. Then $Z_{\boldsymbol{s}} \cap p^{-1}(z) \subset \{z\} \times \mathbb{P}^{\boldsymbol{m}}$ coincides with the zero locus of the section

$$(s_1(v), \dots, s_r(v)) \in \bigoplus_{i=1}^r W_i \cong H^0\Big(\mathbb{P}^{\boldsymbol{m}}, \bigoplus_{i=1}^r \varpi_i^* T_{\mathbb{P}^{m_i}}(-1)\Big).$$

For each i, the zero locus of $s_i(v) \in H^0(\mathbb{P}^{m_i}, T_{\mathbb{P}^{m_i}}(-1))$ is $\varphi_i(z)$ (resp. $\mathbb{P}^{m_i}$) if $s_i(v) \neq 0$ (resp. $s_i(v) = 0$). Since $s_i(v) = 0$ if and only if $z \in Z_i$, we have (3.2). ■

Lemma 3.2 *The zero locus $Z_{\boldsymbol{s}}$ coincides with $\widetilde{X}$.*

Proof Since $\mathbb{P}^n$ is irreducible, the closure $\widetilde{X}$ of the graph of $\boldsymbol{\varphi}$ is irreducible of dimension n. By Lemma 3.1, the generic point of $\widetilde{X}$ and hence $\widetilde{X}$ itself are contained in $Z_{\boldsymbol{s}}$. Euler sequences show that $T_{\mathbb{P}^{m_i}}(-1)$ are globally generated, and hence so is $\bigoplus_{i=1}^r p^*\mathcal{O}_{\mathbb{P}^n}(1) \otimes q_i^* T_{\mathbb{P}^{m_i}}(-1)$. Since $\boldsymbol{s}$ is a general section of a globally generated bundle, the zero of $\boldsymbol{s}$ is smooth of dimension n by a generalization of the theorem of Bertini [Muk92, Theorem 1.10]. Since $Z_{\boldsymbol{s}}$ is smooth and all fibers of $Z_{\boldsymbol{s}} \to \mathbb{P}^n$ are irreducible by Lemma 3.1, $Z_{\boldsymbol{s}}$ is irreducible as well. It follows that $\widetilde{X}$ and $Z_{\boldsymbol{s}}$ are equal, since $\widetilde{X} \subset Z_{\boldsymbol{s}}$ and they are irreducible of the same dimension. ■

We regard the section $\boldsymbol{q}_*\boldsymbol{s} \in H^0\left(\mathbb{P}^{\boldsymbol{m}}, V^\vee \otimes \bigoplus_{i=1}^r \varpi_i^* T_{\mathbb{P}^{m_i}}(-1)\right)$ as a morphism

$$V \otimes \mathcal{O}_{\mathbb{P}^{\boldsymbol{m}}} \longrightarrow \bigoplus_{i=1}^r \varpi_i^* T_{\mathbb{P}^{m_i}}(-1)$$

on $\mathbb{P}^{\boldsymbol{m}}$. It follows from the definition of $Z_{\boldsymbol{s}} = \widetilde{X}$ that

$$\widetilde{\boldsymbol{q}}^{-1}(x) = \mathbb{P}\big(\ker(\boldsymbol{q}_*\boldsymbol{s} \otimes k(x))\big) \times \{x\} \subset \mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}} \tag{3.3}$$

for any $x \in \mathbb{P}^{\boldsymbol{m}}$, where

$$\boldsymbol{q}_*\boldsymbol{s} \otimes k(x)\colon V \longrightarrow \bigoplus_{i=1}^r \varpi_i^* T_{\mathbb{P}^{m_i}}(-1) \otimes k(x)$$

is the induced linear map tensoring by $k(x)$. For $0 \le j \le \min\{n+1, |\boldsymbol{m}|\}$, let $X_j \subset \mathbb{P}^{\boldsymbol{m}}$ be the *j-th degeneracy locus* of $\boldsymbol{q}_*\boldsymbol{s}$ defined as the zero of

$$(3.4) \qquad (\boldsymbol{q}_*\boldsymbol{s})^{\wedge(j+1)}: \bigwedge^{j+1} V \otimes \mathcal{O}_{\mathbb{P}^{\boldsymbol{m}}} \longrightarrow \bigwedge^{j+1} \bigoplus_{i=1}^{r} \varpi_i^* T_{\mathbb{P}^{m_i}}(-1),$$

that is, the locus where the rank of $\boldsymbol{q}_*\boldsymbol{s}$ is at most j. Hence, one has

$$(3.5) \qquad \dim \ker\left(\boldsymbol{q}_*\boldsymbol{s} \otimes k(x)\right) = n + 1 - j$$

for $x \in X_j \backslash X_{j-1}$. In particular, one has

$$(3.6) \qquad X = X_n.$$

As $T_{\mathbb{P}^{m_i}}(-1)$ is globally generated, so is $V^\vee \otimes \bigoplus_{i=1}^r \varpi_i^* T_{\mathbb{P}^{m_i}}(-1)$. Since $\boldsymbol{q}_*\boldsymbol{s}$ is a general section of a globally generated vector bundle, one has $X_j = \varnothing$ or

$$\operatorname{codim}\left(X_j \big/ \mathbb{P}^{\boldsymbol{m}}\right) = (n+1-j)(|\boldsymbol{m}| - j)$$

by [Ott95, Theorem 2.8]. If $|\boldsymbol{m}| \ge n+1$, the dimension of X_{n-1} is at most

$$(3.7) \qquad \begin{aligned} |\boldsymbol{m}| - (n+1-(n-1))(|\boldsymbol{m}| - (n-1)) &= |\boldsymbol{m}| - 2(|\boldsymbol{m}| - n + 1) \\ &= 2n - |\boldsymbol{m}| - 2 \\ &\le n - 3. \end{aligned}$$

Since $\widetilde{\boldsymbol{q}}: \widetilde{X} \to X$ is an isomorphism over $X \backslash X_{n-1}$ by (3.3) and $\widetilde{X}$ is smooth, $X \backslash X_{n-1}$ is smooth. Thus $X = X_n$ is smooth in codimension one. Since X_n is Cohen–Macaulay by [ACGH85, Chapter II], X is normal.

A morphism is said to be *small* if it is an isomorphism in codimension one.

Lemma 3.3 *If* $|\boldsymbol{m}| \ge n+1$, *the morphism* $\widetilde{\boldsymbol{q}}: \widetilde{X} \to X$ *is small.*

Proof It follows from (3.3) that $\widetilde{\boldsymbol{q}}$ is an isomorphism over $X_n \backslash X_{n-1}$ and

$$\begin{aligned} \dim \widetilde{\boldsymbol{q}}^{-1}(X_{n-1}) &= \max\left\{\dim\left(X_j \backslash X_{j-1}\right) + n - j \;\middle|\; 0 \le j \le n-1\right\} \\ &\le \max\left\{|\boldsymbol{m}| - (n+1-j)(|\boldsymbol{m}| - j) + n - j \mid 0 \le j \le n-1\right\} \\ &= \max\left\{j - (n-j)(|\boldsymbol{m}| - j - 1) \mid 0 \le j \le n-1\right\} \\ &= 2n - |\boldsymbol{m}| - 1 \\ &\le n - 2, \end{aligned}$$

hence $\widetilde{\boldsymbol{q}}$ is small. ■

Since $s_i: p^* \mathcal{O}_{\mathbb{P}^n}(-1) \to q_i^* T_{\mathbb{P}^{m_i}}(-1)$ is zero on $\widetilde{X}$, we see by a similar argument in Section 2 that its restriction $s_i|_{\widetilde{X}}$ is a global section of $\widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes \widetilde{q}_i^* \mathcal{O}_{\mathbb{P}^{m_i}}(-1)$ by the exact sequence on $\widetilde{X}$,

$$\begin{aligned} 0 \longrightarrow \widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes \widetilde{q}_i^* \mathcal{O}_{\mathbb{P}^{m_i}}(-1) \longrightarrow \widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes W_i& \\ \longrightarrow \widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes \widetilde{q}_i^* T_{\mathbb{P}^{m_i}}(-1) \longrightarrow 0.& \end{aligned}$$

Let $E_i \subset \widetilde{X}$ be the Cartier divisor defined by this section.

By (2.4), $\mathrm{Bl}_{Z_i} \mathbb{P}^n \subset \mathbb{P}^n \times \mathbb{P}^{m_i}$ is the zero locus of $(s_i)_{\mathbb{P}}$, where we use the notation in Section 2. Since $\widetilde{X}$ is the zero locus of $\boldsymbol{s} = (s_1, \ldots, s_r)$, $\widetilde{p}: \widetilde{X} \to \mathbb{P}^n$ factors

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as $\widetilde{X} \to \mathrm{Bl}_{Z_i} \mathbb{P}^n \to \mathbb{P}^n$ by projections $\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}} \to \mathbb{P}^n \times \mathbb{P}^{m_i} \to \mathbb{P}^n$. As in the last paragraph in Section 2, s_i induces a global section $(s_i)_{\mathrm{Bl}_{Z_i} \mathbb{P}^n}$ on $\mathrm{Bl}_{Z_i} \mathbb{P}^n$ whose zero locus in the exceptional divisor, and $s_i|_{\widetilde{X}}$ is nothing but the pullback of $(s_i)_{\mathrm{Bl}_{Z_i} \mathbb{P}^n}$. Hence, E_i is the total transform of the exceptional divisor of $\mathrm{Bl}_{Z_i} \mathbb{P}^n$. In particular, $E_i = \widetilde{p}^{-1}(Z_i)$ holds.

Lemma 3.4 (i) *The restriction of $\widetilde{p}$ to $\widetilde{X} \setminus \bigcup_{i=1}^r E_i$ is an isomorphism onto $\mathbb{P}^n \setminus \bigcup_{i=1}^r Z_i$.*
(ii) *For each $i = 1, \dots, r$, the divisor E_i is irreducible.*
(iii) *One has $\widetilde{\boldsymbol{q}}(E_1) = \mathbb{P}^{m_1} \times \overline{(\varphi_2, \dots, \varphi_r)(Z_1)}$, and similarly for $\widetilde{\boldsymbol{q}}(E_i)$ for $i = 2, \dots, r$.*

Proof (i) holds, since the rational map $\boldsymbol{\varphi}$ is defined on $\mathbb{P}^n \setminus \bigcup_{i=1}^r Z_i$ and $\widetilde{X}$ is the graph of $\boldsymbol{\varphi}$.

(ii) It follows from Lemma 3.1 that $\widetilde{p}^{-1}(Z_i \setminus \bigcup_{j \neq i} Z_j)$ is irreducible of dimension $n-1$ and $\dim \widetilde{p}^{-1}(Z_i \cap \bigcup_{j \neq i} Z_j) < n-1$. Since E_i is a Cartier divisor, all irreducible components are $n-1$-dimensional, and hence E_i is irreducible.

(iii) By Lemma 3.1 and Lemma 3.2, $\widetilde{p}^{-1}(z) = \widetilde{X} \cap p^{-1}(z)$ coincides with

$$\{z\} \times \mathbb{P}^{m_1} \times \varphi_2(z) \times \cdots \times \varphi_r(z)$$

for all $z \in Z_1 \setminus \bigcup_{j \neq 1} Z_j$. Hence, the image of $\widetilde{p}^{-1}(Z_1 \setminus \bigcup_{j \neq 1} Z_j)$ by $\widetilde{\boldsymbol{q}}$ is $\mathbb{P}^{m_1} \times (\varphi_2, \dots, \varphi_r)(Z_1 \setminus \bigcup_{j \neq 1} Z_j)$. Since $\widetilde{p}^{-1}(Z_1 \setminus \bigcup_{j \neq 1} Z_j)$ is dense in E_1, we have (iii). ∎

From now on, we assume $|\boldsymbol{m}| \geq n+1$. Lemma 3.3 implies that $\widetilde{\boldsymbol{q}}(E_i)$ is a prime Weil divisor on X. We set

$$L_i := \varpi_i^* \mathcal{O}_{\mathbb{P}^{m_i}}(1)|_X \tag{3.8}$$

for $i = 1, \dots, r$.

Lemma 3.5 *The rational map defined by $|\mathcal{O}_X(\widetilde{\boldsymbol{q}}(E_1)) \otimes L_1|$ is inverse to the rational map $\boldsymbol{\varphi}\colon \mathbb{P}^n \dashrightarrow X$ up to* $\mathrm{Aut}(\mathbb{P}^n)$.

Proof For a linear system Λ, we let ϕ_Λ denote the rational map defined by Λ.

Since X is normal and $\widetilde{\boldsymbol{q}}$ is small, we have

$$H^0\big(X, \mathcal{O}_X(\widetilde{\boldsymbol{q}}(E_1)) \otimes L_1\big) = H^0\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(E_1) \otimes \widetilde{q}_1^* \mathcal{O}_{\mathbb{P}^{m_1}}(1)\big)$$

and the rational map $\phi_{|\mathcal{O}_X(\widetilde{\boldsymbol{q}}(E_1)) \otimes L_1|}$ coincides with the composite map

$$\phi_{|\mathcal{O}_{\widetilde{X}}(E_1) \otimes \widetilde{q}_1^* \mathcal{O}_{\mathbb{P}^{m_1}}(1)|} \circ \widetilde{\boldsymbol{q}}^{-1}.$$

Since $\mathcal{O}_{\widetilde{X}}(E_1) \cong \widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes \widetilde{q}_1^* \mathcal{O}_{\mathbb{P}^{m_1}}(-1)$, we have $\mathcal{O}_{\widetilde{X}}(E_1) \otimes \widetilde{q}_1^* \mathcal{O}_{\mathbb{P}^{m_1}}(1) \cong \widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1)$. Thus, the rational map $\phi_{|\mathcal{O}_{\widetilde{X}}(E_1) \otimes \widetilde{q}_1^* \mathcal{O}_{\mathbb{P}^{m_1}}(1)|}$ coincides with $\phi_{|\widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1)|} = \widetilde{p}$ up to $\mathrm{Aut}(\mathbb{P}^n)$. Hence, $\phi_{|\mathcal{O}_X(\widetilde{\boldsymbol{q}}(E_1)) \otimes L_1|} = \phi_{|\mathcal{O}_{\widetilde{X}}(E_1) \otimes \widetilde{q}_1^* \mathcal{O}_{\mathbb{P}^{m_1}}(1)|} \circ \widetilde{\boldsymbol{q}}^{-1}$ coincides with $\widetilde{p} \circ \widetilde{\boldsymbol{q}}^{-1} = \boldsymbol{\varphi}^{-1}$ up to $\mathrm{Aut}(\mathbb{P}^n)$. ∎

Lemma 3.5 shows that we can reconstruct $\boldsymbol{\varphi}$ up to $\mathrm{Aut}(\mathbb{P}^n)$ from X and $\widetilde{\boldsymbol{q}}(E_1)$. Lemma 3.6 shows that $\widetilde{\boldsymbol{q}}(E_1)$ is uniquely determined by $X \subset \mathbb{P}^{\boldsymbol{m}}$ if $|\boldsymbol{m}| \geq n+2$ or $|\boldsymbol{m}| = n+1$ and $m_1 \geq 2$.

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Lemma 3.6 *Assume $|\boldsymbol{m}| \geq n+2$ or $|\boldsymbol{m}| = n+1$ and $m_1 \geq 2$. Then $\widetilde{\boldsymbol{q}}(E_1)$ is the unique Weil divisor on X of the form $\mathbb{P}^{m_1} \times Y$ for some $Y \subset \prod_{i \neq 1} \mathbb{P}^{m_i}$.*

Proof By Lemma 3.4(iii), $\widetilde{\boldsymbol{q}}(E_1)$ is a Weil divisor of such form.

Assume there exists a subvariety $Y \subset \prod_{i\neq 1} \mathbb{P}^{m_i}$ of dimension $n-1-m_1$ such that $D = \mathbb{P}^{m_1} \times Y$ is contained in X and $D \neq \widetilde{\boldsymbol{q}}(E_1)$. Let $\widetilde{X}^\dagger \subset \mathbb{P}^n \times \prod_{i\neq 1} \mathbb{P}^{m_i}$ and $X^\dagger \subset \prod_{i \neq 1} \mathbb{P}^{m_i}$ be the subvarieties obtained from $(s_2, \ldots, s_r)$ in the same way as $\widetilde{X}$ and X. Consider the diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\ \widetilde{\boldsymbol{q}}\ } & X \\ {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \widetilde{X}^\dagger & \xrightarrow{\ \widetilde{\boldsymbol{q}}^\dagger\ } & X^\dagger, \end{array}$$

where $\pi, \tilde{\pi}, \widetilde{\boldsymbol{q}}^\dagger$ are induced by the natural projections. Lemma 3.4 shows that E_1 is the unique divisor contracted by $\tilde{\pi}$. Since $\widetilde{\boldsymbol{q}}$ is small and $D \neq \widetilde{\boldsymbol{q}}(E_1)$, we have a divisor $\widetilde{D}^\dagger \subset \widetilde{X}^\dagger$, which is the strict transform of D by the birational map $\tilde{\pi} \circ \widetilde{\boldsymbol{q}}^{-1}$. Since $D = \mathbb{P}^{m_1} \times Y$, one has $\pi(D) = Y$ and hence $\widetilde{\boldsymbol{q}}^\dagger(\widetilde{D}^\dagger) = Y$.

Since $\dim \widetilde{D}^\dagger - \dim Y = m_1$, it follows from (3.3) and (3.5) for $\widetilde{\boldsymbol{q}}^\dagger$, $\widetilde{X}^\dagger$, and $X^\dagger$ that Y must be contained in $X^\dagger_{n-m_1}$, where we define the degeneracy locus $X^\dagger_{n-m_1}$ in the same way as X_j. On the other hand, the dimension of $X^\dagger_{n-m_1}$ is at most

$$\sum_{i\neq 1} m_i - \big(n+1-(n-m_1)\big)\Big(\sum_{i\neq 1} m_i - (n-m_1)\Big) = \sum_{i \neq 1} m_i - (m_1+1)(|\boldsymbol{m}|-n).$$

Hence, we have

$$n-1-m_1 = \dim Y \leq \dim X^\dagger_{n-m_1} \leq \sum_{i\neq 1} m_i - (m_1+1)(|\boldsymbol{m}|-n),$$

which implies $m_1(|\boldsymbol{m}|-n) \leq 1$. This contradicts the assumption $|\boldsymbol{m}| \geq n+2$ or $|\boldsymbol{m}| = n+1$ and $m_1 \geq 2$. ■

Proof of Theorem 1.1(i) Let X be the multiview variety for general $\boldsymbol{\varphi}$. To show that $X \subset \mathbb{P}^{\boldsymbol{m}}$ determines the rational map $\boldsymbol{\varphi}$ uniquely up to the action of $\mathrm{Aut}(\mathbb{P}^n) \cong \mathrm{PGL}(n+1, \mathbf{k})$, it suffices to see that the inverse $\boldsymbol{\varphi}^{-1}$ is uniquely determined by $X \subset \mathbb{P}^{\boldsymbol{m}}$ up to $\mathrm{PGL}(n+1, \mathbf{k})$.

Assume $|\boldsymbol{m}| \geq n+1$ and $\boldsymbol{m} \neq (1^{n+1}) \coloneqq (1, \ldots, 1)$. Relabeling the indexes of m_i if necessary, we can assume that $|\boldsymbol{m}| \geq n+2$ or $|\boldsymbol{m}| = n+1$ and $m_1 \geq 2$. Then Lemma 3.6 states that $X \subset \mathbb{P}^{\boldsymbol{m}}$ uniquely determines $\widetilde{\boldsymbol{q}}(E_1) \subset X$ without using $\boldsymbol{\varphi}$, $\widetilde{\boldsymbol{q}}$, etc. Hence, $X \subset \mathbb{P}^{\boldsymbol{m}}$ uniquely determines $\boldsymbol{\varphi}^{-1}$ by Lemma 3.5. The inevitable ambiguity by the action of $\mathrm{PGL}(n+1, \mathbf{k})$ comes from the identification of $\phi_{|\mathcal{O}_{\widetilde{X}}(E_1) \otimes \widetilde{q}_1^* \mathcal{O}_{\mathbb{P}^{m_1}}(1)|}$ with $\phi_{|\widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1)|} = \widetilde{p}$. ■

4 Projective Reconstruction in the Case $\boldsymbol{m} = (1^{n+1})$

Assume $r = n+1$ and $m_i = 1$ for any $1 \leq i \leq n+1$. Note that $\mathbb{P}(W_i)$ can be canonically identified with $\mathbb{P}(W_i^\vee)$, since $\dim W_i = 2$. Set

$$V' \coloneqq (\operatorname{coker} \boldsymbol{s})^\vee,$$

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which is $(n+1)$-dimensional, since $\boldsymbol{s} = (s_1, \ldots, s_{n+1})\colon V \to \bigoplus_{i=1}^{n+1} W_i$ is general. The canonical inclusion

$$\boldsymbol{s}'\colon V' \longrightarrow \Big(\bigoplus_{i=1}^{n+1} W_i\Big)^{\vee} = \bigoplus_{i=1}^{n+1} W_i^{\vee} \tag{4.1}$$

defines a hypersurface

$$X' \subset \prod_{i=1}^{n+1} \mathbb{P}(W_i^{\vee}) = (\mathbb{P}^1)^{n+1}$$

in the same way as X. We also define $\widetilde{X}'$, E_i', $\widetilde{\boldsymbol{q}}'$, etc. in the same way as X.

Lemma 4.1 *The hypersurfaces X and X' coincide under the canonical identifications $\mathbb{P}(W_i) = \mathbb{P}(W_i^{\vee})$ for $i = 1, \ldots, n+1$.*

Proof On $\prod_{i=1}^{n+1} \mathbb{P}(W_i)$, we have a diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \bigoplus_{i=1}^{n+1} \varpi_i^* \mathcal{O}_{\mathbb{P}(W_i)}(-1) & \searrow & & & \\
 & & & & \downarrow & & & & \\
0 \longrightarrow & V \otimes \mathcal{O} & \xrightarrow{\ \boldsymbol{s}\ } & & (\bigoplus_{i=1}^{n+1} W_i) \otimes \mathcal{O} & \xrightarrow{(\boldsymbol{s}')^{\vee}} & (V')^{\vee} \otimes \mathcal{O} & \longrightarrow 0. & \\
 & & \searrow & & \downarrow & & & & \\
 & & & & \bigoplus_{i=1}^{n+1} \varpi_i^* T_{\mathbb{P}(W_i)}(-1) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array} \tag{4.2}$$

A point $x \in \prod_{i=1}^{n+1} \mathbb{P}(W_i)$ is contained in $X = X_n$ if and only if the rank of the linear map

$$\boldsymbol{q}_* \boldsymbol{s} \otimes k(x)\colon V \longrightarrow \bigoplus_{i=1}^{n+1} \varpi_i^* T_{\mathbb{P}(W_i)}(-1) \otimes k(x)$$

is at most n; that is, $\boldsymbol{q}_* \boldsymbol{s} \otimes k(x)$ is not injective. By (4.2), this is equivalent to the condition that

$$\boldsymbol{s}(V) \cap \bigoplus_{i=1}^{n+1} \varpi_i^* \mathcal{O}_{\mathbb{P}(W_i)}(-1) \otimes k(x) \neq \{0\},$$

where we take the intersection as subspaces of $\bigoplus_{i=1}^{n+1} W_i$. By (4.2) again, this is equivalent to the condition that the rank of the linear map

$$\bigoplus_{i=1}^{n+1} \varpi_i^* \mathcal{O}_{\mathbb{P}(W_i)}(-1) \otimes k(x) \longrightarrow (V')^{\vee} \tag{4.3}$$

is at most n. Under the identification $\mathbb{P}(W_i^\vee) = \mathbb{P}(W_i)$, the sheaf $T_{\mathbb{P}(W_i^\vee)}(-1)$ is identified with $\mathcal{O}_{\mathbb{P}(W_i)}(1)$. Hence, the rank of the linear map (4.3) is at most n if and only if x is contained in X'. Thus, $X = X'$ holds. ∎

Recall from Lemma 3.4 that $\widetilde{\boldsymbol{q}}(E_1) = \mathbb{P}(W_1) \times \overline{(\varphi_2, \dots, \varphi_r)(Z_1)}$. Similarly, one has $\widetilde{\boldsymbol{q}}'(E_1') = \mathbb{P}(W_1^\vee) \times \overline{(\varphi_2', \dots, \varphi_r')(Z_1')}$.

Lemma 4.2 *The closure $\overline{(\varphi_2, \dots, \varphi_r)(Z_1)} \subset \prod_{i=2}^{n+1} \mathbb{P}(W_i)$ is the $(n-2)$-th degeneracy locus of the composite map*

$$(4.4) \qquad (\ker s_1) \otimes \mathcal{O}_{\prod_{i=2}^{n+1} \mathbb{P}(W_i)} \hookrightarrow V \otimes \mathcal{O}_{\prod_{i=2}^{n+1} \mathbb{P}(W_i)} \xrightarrow{(s_2, \dots, s_{n+1})} \bigoplus_{i=2}^{n+1} \varpi_i^* T_{\mathbb{P}(W_i)}(-1),$$

where we use the same letter ϖ_i for the projection $\prod_{i=2}^{n+1} \mathbb{P}(W_i) \to \mathbb{P}(W_i)$. On the other hand, the closure $\overline{(\varphi_2', \dots, \varphi_r')(Z_1')} \subset \prod_{i=2}^{n+1} \mathbb{P}(W_i^\vee) = \prod_{i=2}^{n+1} \mathbb{P}(W_i)$ is the $(n-1)$-th degeneracy locus of

$$(4.5) \qquad (s_2, \dots, s_{n+1}) \colon V \otimes \mathcal{O}_{\prod_{i=2}^{n+1} \mathbb{P}(W_i)} \longrightarrow \bigoplus_{i=2}^{n+1} \varpi_i^* T_{\mathbb{P}(W_i)}(-1).$$

Proof The first statement follows by applying (3.6) to $(\varphi_2, \dots, \varphi_r)|_{Z_1} \colon Z_1 = \mathbb{P}(\ker s_1) \dashrightarrow \prod_{i=2}^{n+1} \mathbb{P}(W_i)$. Since $s_1 \colon V \to W_1$ is surjective, we have an exact sequence

$$0 \longrightarrow \ker s_1 \longrightarrow \bigoplus_{i=2}^{n+1} W_i \longrightarrow (V')^\vee \longrightarrow 0,$$

which gives a diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \bigoplus_{i=2}^{n+1} \varpi_i^* \mathcal{O}_{\mathbb{P}(W_i)}(-1) & & & & \\
 & & & & \downarrow & \searrow & & & \\
0 & \longrightarrow & (\ker s_1) \otimes \mathcal{O} & \longrightarrow & (\bigoplus_{i=2}^{n+1} W_i) \otimes \mathcal{O} & \longrightarrow & (V')^\vee \otimes \mathcal{O} & \longrightarrow & 0 \\
 & & & \searrow & \downarrow & & & & \\
 & & & & \bigoplus_{i=2}^{n+1} \varpi_i^* T_{\mathbb{P}(W_i)}(-1) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

on $\prod_{i=2}^{n+1} \mathbb{P}(W_i)$. By an argument similar to that in the proof of Lemma 4.1, we see that the $(n-2)$-th degeneracy locus of (4.4) coincides with the $(n-1)$-th degeneracy locus of $\bigoplus_{i=2}^{n+1} \varpi_i^* \mathcal{O}_{\mathbb{P}(W_i)}(-1) \to (V')^\vee \otimes \mathcal{O}$, that is, the $(n-1)$-th degeneracy locus of $V' \otimes \mathcal{O} \to \bigoplus_{i=2}^{n+1} \varpi_i^* \mathcal{O}_{\mathbb{P}(W_i)}(1)$ on $\prod_{i=2}^{n+1} \mathbb{P}(W_i)$.

By replacing X with X', we see that $\overline{(\varphi_2', \dots, \varphi_r')(Z_1')} \subset \prod_{i=2}^{n+1} \mathbb{P}(W_i)$ is the $(n-1)$-th degeneracy locus of (4.5), since V' and $\varpi_i^* \mathcal{O}_{\mathbb{P}(W_i)}(1)$ are replaced by V and $\varpi_i^* T_{\mathbb{P}(W_i)}(-1)$, respectively. ■

We note that (4.5) does not depend on s_i, hence neither does $\overline{(\varphi_2', \dots, \varphi_r')(Z_1')}$.

Lemma 4.3 *One has* $\widetilde{\boldsymbol{q}}(E_1) \neq \widetilde{\boldsymbol{q}}'(E_1')$.

Proof It suffices to see that $\overline{(\varphi_2, \dots, \varphi_r)(Z_1)} \neq \overline{(\varphi_2', \dots, \varphi_r')(Z_1')}$. Take a general point $y \in \overline{(\varphi_2', \dots, \varphi_r')(Z_1')}$. By Lemma 4.2, the rank of

$$(s_2, \dots, s_{n+1})_y \colon V \longrightarrow \bigoplus_{i=2}^{n+1} \varpi_i^* T_{\mathbb{P}(W_i)}(-1) \otimes k(y)$$

is $n-1$, since $s_2, \dots, s_{n+1}$ and y are general. Hence, $\ker(s_2, \dots, s_{n+1})_y \subset V$ is two-dimensional. Then $\ker(s_2, \dots, s_{n+1})_y \cap \ker s_1 = \{0\} \subset V$ since $\ker s_1 \subset V$ is of codimension two and general. This means that (4.4) has rank $n-1$ at y. By Lemma 4.2, we have $y \notin \overline{(\varphi_2, \dots, \varphi_r)(Z_1)}$. ■

Lemmas 4.3 and 4.4 show that we have exactly two reconstructions.

Lemma 4.4 *The exceptional locus of the birational morphism* $X \to \prod_{i=2}^{n+1} \mathbb{P}(W_i)$ *is the union of* $\widetilde{\boldsymbol{q}}(E_1)$ *and* $\widetilde{\boldsymbol{q}}'(E_1')$.

Proof Since $X \subset \mathbb{P}(W_1) \times \prod_{i=2}^{n+1} \mathbb{P}(W_i)$, the exceptional locus of $X \to \prod_{i=2}^{n+1} \mathbb{P}(W_i)$ is

$$\mathbb{P}(W_1) \times \left\{ y \in \prod_{i=2}^{n+1} \mathbb{P}(W_i) \;\middle|\; \mathbb{P}(W_1) \times \{y\} \subset X \right\} \subset X.$$

Hence, we need to show

$$\left\{ y \in \prod_{i=2}^{n+1} \mathbb{P}(W_i) \;\middle|\; \mathbb{P}(W_1) \times \{y\} \subset X \right\} = \overline{(\varphi_2, \dots, \varphi_r)(Z_1)} \cup \overline{(\varphi_2', \dots, \varphi_r')(Z_1')}. \tag{4.6}$$

Since

$$\widetilde{\boldsymbol{q}}(E_1) = \mathbb{P}(W_1) \times \overline{(\varphi_2, \dots, \varphi_r)(Z_1)},$$
$$\widetilde{\boldsymbol{q}}'(E_1') = \mathbb{P}(W_1^\vee) \times \overline{(\varphi_2', \dots, \varphi_r')(Z_1')},$$

the inclusion $\supset$ in (4.6) is clear. To show the converse inclusion, we take $y \notin \overline{(\varphi_2, \dots, \varphi_r)(Z_1)} \cup \overline{(\varphi_2', \dots, \varphi_r')(Z_1')}$ and show $\mathbb{P}(W_1) \times \{y\} \not\subset X$. By Lemma 4.2, the linear map

$$(s_2, \dots, s_{n+1})_y \colon V \longrightarrow U \coloneqq \bigoplus_{i=2}^{n+1} \varpi_i^* T_{\mathbb{P}(W_i)}(-1) \otimes k(y)$$

has rank n and the restriction $(s_2, \dots, s_{n+1})_y|_{\ker s_1}$ has rank $n-1$. Recall that the dimensions of V, U, and $\ker s_1$ are $n+1$, n, and $n-1$, respectively. Hence, $\ker(s_2, \dots, s_{n+1})_y \subset V$ is one-dimensional and $\ker(s_2, \dots, s_{n+1})_y \cap \ker s_1 = \{0\} \subset V$. Let $K \subset W_1$ be the

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image of $\ker(s_2, \ldots, s_{n+1})_y$ by $s_1 \colon V \to W_1$. Then we have a diagram

$$\begin{CD}
0 @>>> \ker(s_2,\ldots,s_{n+1})_y \otimes \mathcal{O} @>>> V \otimes \mathcal{O} @>{(s_2,\ldots,s_{n+1})_y}>> U \otimes \mathcal{O} @>>> 0 \\
@. @V{\wr}VV @V{\boldsymbol{s}|_{\mathbb{P}(W_1)\times\{y\}}}VV @| @. \\
@. K \otimes \mathcal{O} @>{(c,0)}>> T_{\mathbb{P}(W_1)}(-1) \oplus (U \otimes \mathcal{O}) @>>> U \otimes \mathcal{O} @>>> 0
\end{CD}$$

on $\mathbb{P}(W_1) \times \{y\}$, where $c \colon K \otimes \mathcal{O} \hookrightarrow W_1 \otimes \mathcal{O} \to T_{\mathbb{P}(W_1)}(-1)$ is the canonical map. Then $(c, 0)$ is injective outside of the point $x_0 \in \mathbb{P}(W_1)$ corresponding to the one-dimensional subspace $K \subset W_1$. Hence, the rank of $\boldsymbol{s}|_{\mathbb{P}(W_1)\times\{y\}}$ is $n+1$ at any $x_1 \neq x_0 \in \mathbb{P}(W_1)$, which means that $(\mathbb{P}(W_1) \setminus \{x_0\}) \times \{y\}$ is not contained in X. Thus, y is not contained in the left-hand side of (4.6). ■

Proof of Theorem 1.1(ii) Lemmas 4.3 and 4.4 show that $\widetilde{\boldsymbol{q}}(E_1) \subset X$ is one of the exceptional prime divisors of the birational morphism $X \to \prod_{i=2}^{n+1} \mathbb{P}(W_i)$. If we choose one of such divisors, we can reconstruct $\boldsymbol{\varphi}^{-1}$ or $\boldsymbol{\varphi}'^{-1}$ by Lemma 3.5 as in the proof of Theorem 1.1(i). ■

Remark 4.5 If φ is defined over $\mathbb{R}$, so is φ'. This follows from the construction of $\boldsymbol{s}'$ in (4.1).

We have the diagram

$$\begin{array}{ccccccc}
 & & \widetilde{X} & \dashrightarrow & \widetilde{X}' & & \\
 & \swarrow^{\widetilde{p}} & & {}_{\widetilde{\boldsymbol{q}}}\searrow \quad \swarrow^{\widetilde{\boldsymbol{q}}'} & & \searrow^{\widetilde{p}'} & \\
\mathbb{P}(V) & \overset{\boldsymbol{\varphi}}{\dashrightarrow} & & X = X' & & \overset{\boldsymbol{\varphi}'}{\dashleftarrow} & \mathbb{P}(V').
\end{array}$$

In the rest of this section, we describe the birational map $\boldsymbol{\varphi}'^{-1} \circ \boldsymbol{\varphi} \colon \mathbb{P}(V) \dashrightarrow \mathbb{P}(V')$. Recall the definition of L_i from (3.8).

Lemma 4.6 *The divisor $\widetilde{\boldsymbol{q}}(E_1) + \widetilde{\boldsymbol{q}}'(E'_1)$ on X is linearly equivalent to $-L_1 + \sum_{i=2}^{n+1} L_i$.*

Proof Since the exceptional locus of the birational morphism

$$(\varpi_2, \ldots, \varpi_{n+1})|_X \colon X \longrightarrow \prod_{i=2}^{n+1} \mathbb{P}(W_i) \tag{4.7}$$

is $\widetilde{\boldsymbol{q}}(E_1) \cup \widetilde{\boldsymbol{q}}'(E'_1)$ by Lemma 4.4, we can write

$$K_X = (\varpi_2, \ldots, \varpi_{n+1})^* K_{\prod_{i=2}^{n+1} \mathbb{P}(W_i)} + a\widetilde{\boldsymbol{q}}(E_1) + a'\widetilde{\boldsymbol{q}}'(E'_1) \tag{4.8}$$

for some integers a and a'. By the birational map

$$(\varphi_2, \ldots, \varphi_{n+1}) \colon \mathbb{P}(V) \dashrightarrow \prod_{i=2}^{n+1} \mathbb{P}(W_i),$$

the birational morphism (4.7) can be identified with the blow-up $\mathrm{Bl}_{Z_1} \mathbb{P}(V) \to \mathbb{P}(V)$ over the generic point of Z_1. Hence the integer a in (4.8), which is the coefficient of $\widetilde{\boldsymbol{q}}(E_1) = \mathbb{P}(W_1) \times \overline{(\varphi_2, \ldots, \varphi_r)(Z_1)}$, is one. Similarly, one has $a' = 1$, and

$$\widetilde{\boldsymbol{q}}(E_1) + \widetilde{\boldsymbol{q}}'(E_1') = K_X - (\varpi_2, \ldots, \varpi_{n+1})^* K_{\prod_{i=2}^{n+1} \mathbb{P}(W_i)}$$

holds. By (3.4), the divisor $X = X_n \subset \prod_{i=1}^{n+1} \mathbb{P}(W_i)$ is the zero locus of

$$(\boldsymbol{q}_* \boldsymbol{s})^{\wedge(n+1)} : \mathcal{O}_{\prod_{i=1}^{n+1} \mathbb{P}(W_i)} \simeq \bigwedge^{n+1} V \otimes \mathcal{O}_{\prod_{i=1}^{n+1} \mathbb{P}(W_i)} \longrightarrow \bigwedge^{n+1} \bigoplus_{i=1}^{n+1} \varpi_i^* T_{\mathbb{P}(W_i)}(-1)$$
$$\simeq \bigotimes_{i=1}^{n+1} \varpi_i^* \mathcal{O}_{\mathbb{P}(W_i)}(1).$$

Thus, X is linearly equivalent to $\sum_{i=1}^{n+1} \varpi_i^* \mathcal{O}_{\mathbb{P}(W_i)}(1)$ on $\prod_{i=1}^{n+1} \mathbb{P}(W_i)$. Hence we have $K_X = -\sum_{i=1}^{n+1} L_i$ by the adjunction formula. Since

$$(\varpi_2, \ldots, \varpi_{n+1})^* K_{\prod_{i=2}^{n+1} \mathbb{P}(W_i)} = -2 \sum_{i=2}^{n+1} L_i,$$

one has

$$\widetilde{\boldsymbol{q}}(E_1) + \widetilde{\boldsymbol{q}}'(E_1') = K_X - (\varpi_2, \ldots, \varpi_{n+1})^* K_{\prod_{i=2}^{n+1} \mathbb{P}(W_i)} = -L_1 + \sum_{i=2}^{n+1} L_i,$$

and Lemma 4.6 is proved. ∎

For each $i = 1, \ldots, n+1$, let $F_i \subset \widetilde{X}$ be the strict transform of the divisor $\widetilde{\boldsymbol{q}}'(E_i') \subset X' = X$.

Lemma 4.7 *The divisor F_1 is linearly equivalent to $\widetilde{p}^* \mathcal{O}_{\mathbb{P}(V)}(n-1) - \sum_{i=2}^{n+1} E_i$.*

Proof This is an immediate consequence of Lemma 4.6 and the linear equivalences $\widetilde{q}_i^* \mathcal{O}_{\mathbb{P}(W_i)}(1) \sim \widetilde{p}^* \mathcal{O}_{\mathbb{P}(V)}(1) - E_i$ for $1 \leq i \leq n+1$. ∎

Corollary 4.8 (i) *The birational map ${\boldsymbol{\varphi}'}^{-1} \circ \boldsymbol{\varphi}: \mathbb{P}(V) \dashrightarrow \mathbb{P}(V')$ is obtained by the linear system $|\mathcal{O}_{\mathbb{P}(V)}(n) \otimes I_{\bigcup_{i=1}^{n+1} Z_i}|$.*

(ii) *For each i, the image $\widetilde{p}(F_i) \subset \mathbb{P}(V)$ is the unique hypersurface of degree $n-1$ containing Z_j for all $j \in \{1, \ldots, n+1\} \setminus \{i\}$.*

(iii) *The birational map ${\boldsymbol{\varphi}'}^{-1} \circ \boldsymbol{\varphi}$ contracts the hypersurface $\widetilde{p}(F_i)$ to Z_i' for each i.*

Proof (i) By Lemma 3.5, the birational map ${\boldsymbol{\varphi}'}^{-1}$ is obtained by the linear system $|\mathcal{O}_X(\widetilde{\boldsymbol{q}}'(E_1')) \otimes L_1|$, which is identified with $|\mathcal{O}_{\widetilde{X}}(F) \otimes \widetilde{q}_1^* \mathcal{O}_{\mathbb{P}(W_1)}(1)|$ by $\widetilde{\boldsymbol{q}}$. Since

$$F_1 + \widetilde{q}_1^* \mathcal{O}_{\mathbb{P}(W_1)}(1) \sim \widetilde{p}^* \mathcal{O}_{\mathbb{P}(V)}(n-1) - \sum_{i=2}^{n+1} E_i + \widetilde{p}^* \mathcal{O}_{\mathbb{P}(V)}(1) - E_1$$
$$= \widetilde{p}^* \mathcal{O}_{\mathbb{P}(V)}(n) - \sum_{i=1}^{n+1} E_i$$

follows from Lemma 4.7, $|\mathcal{O}_{\widetilde{X}}(F) \otimes \widetilde{q}_1^* \mathcal{O}_{\mathbb{P}(W_1)}(1)|$ is identified with $|\mathcal{O}_{\mathbb{P}(V)}(n) \otimes I_{\bigcup_{i=1}^{n+1} Z_i}|$ by $\widetilde{p}$. Hence, ${\boldsymbol{\varphi}'}^{-1} \circ \boldsymbol{\varphi} = {\boldsymbol{\varphi}'}^{-1} \circ \widetilde{\boldsymbol{q}} \circ \widetilde{p}^{-1}$ is obtained by $|\mathcal{O}_{\mathbb{P}(V)}(n) \otimes I_{\bigcup_{i=1}^{n+1} Z_i}|$.

(ii) It suffices to show this statement for $i = 1$. The linear system on $\mathbb{P}(V)$ consisting of divisors of degree $n-1$ containing $Z_2, \dots, Z_{n+1}$ is identified with $|\widetilde{p}^* \mathcal{O}_{\mathbb{P}(V)}(n-1) - \sum_{i=2}^{n+1} E_i|$ on $\widetilde{X}$ by $\widetilde{p}$. By Lemma 4.7, we have $|\widetilde{p}^* \mathcal{O}_{\mathbb{P}(V)}(n-1) - \sum_{i=2}^{n+1} E_i| = |F_1|$, which in turn is identified with the linear system $|E_1'|$ on $\widetilde{X}'$. The linear system $|E_1'|$ is 0-dimensional, since E_1' is an exceptional divisor. Hence, $\widetilde{p}(F_1)$ is the unique such divisor.

(iii) This statement holds since each $E_i' \subset \widetilde{X}'$ is contracted to Z_i'. ∎

5 Dominance for $|\boldsymbol{m}| \geq 2n-1$

We use the same notation as in Section 3.

Lemma 5.1 *The rational map*

$$\widetilde{\Phi}: \prod_{i=1}^{r} \mathbb{P}(V^\vee \otimes W_i) \dashrightarrow \mathrm{Hilb}(\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}), \qquad \boldsymbol{\varphi} = (\varphi_1, \dots, \varphi_r) \longmapsto [\widetilde{X}]$$

is birational onto an irreducible component.

Proof Since we can recover $\boldsymbol{\varphi}$ from the graph $\widetilde{X}$ of $\boldsymbol{\varphi}$, the rational map $\widetilde{\Phi}$ is generically injective. Hence, it suffices to show that for general $\boldsymbol{\varphi}$, $\mathrm{Hilb}(\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}})$ is smooth at $\widetilde{\Phi}(\boldsymbol{\varphi})$ and the dimension of $\mathrm{Hilb}(\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}})$ at $\widetilde{\Phi}(\boldsymbol{\varphi})$ is equal to that of $\prod_{i=1}^r \mathbb{P}(V^\vee \otimes W_i)$.

Consider the differential

$$(d\widetilde{\Phi})_{\boldsymbol{\varphi}}: T_{\boldsymbol{\varphi}}\Big(\prod_{i=1}^{r} \mathbb{P}(V^\vee \otimes W_i)\Big) \longrightarrow T_{[\widetilde{X}]} \mathrm{Hilb}(\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}). \tag{5.1}$$

Let $\mathrm{Im}(\widetilde{\Phi}) \subset \mathrm{Hilb}(\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}})$ be the closure of the image of $\widetilde{\Phi}$ with reduced structure. Then $(d\widetilde{\Phi})_{\boldsymbol{\varphi}}$ factors as

$$T_{\boldsymbol{\varphi}}\Big(\prod_{i=1}^{r} \mathbb{P}(V^\vee \otimes W_i)\Big) \longrightarrow T_{[\widetilde{X}]}\mathrm{Im}(\widetilde{\Phi}) \subset T_{[\widetilde{X}]} \mathrm{Hilb}(\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}).$$

We note that if $\boldsymbol{\varphi}$ is general, $\mathrm{Im}(\widetilde{\Phi})$ is smooth at $[\widetilde{X}]$ and $T_{\boldsymbol{\varphi}}(\prod_{i=1}^r \mathbb{P}(V^\vee \otimes W_i)) \to T_{[\widetilde{X}]}\mathrm{Im}(\widetilde{\Phi})$ is surjective. Thus, for general $\boldsymbol{\varphi}$, we have

$$\mathrm{rank}(d\widetilde{\Phi})_{\boldsymbol{\varphi}} = \dim \mathrm{Im}(\widetilde{\Phi}) \leq \dim_{[\widetilde{X}]} \mathrm{Hilb}(\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}) \leq \dim T_{[\widetilde{X}]} \mathrm{Hilb}(\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}).$$

If $(d\widetilde{\Phi})_{\boldsymbol{\varphi}}$ is surjective, $\mathrm{rank}(d\widetilde{\Phi})_{\boldsymbol{\varphi}} = \dim T_{[\widetilde{X}]} \mathrm{Hilb}(\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}})$, and hence

$$\begin{aligned} \mathrm{rank}(d\widetilde{\Phi})_{\boldsymbol{\varphi}} &= \dim \mathrm{Im}(\widetilde{\Phi}) = \dim_{[\widetilde{X}]} \mathrm{Hilb}(\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}) \\ &= \dim T_{[\widetilde{X}]} \mathrm{Hilb}(\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}) \end{aligned} \tag{5.2}$$

holds. From the last equality of (5.2), $\mathrm{Hilb}(\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}})$ is smooth at $[\widetilde{X}] = \widetilde{\Phi}(\boldsymbol{\varphi})$. Furthermore, $\dim_{[\widetilde{X}]} \mathrm{Hilb}(\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}) = \dim \mathrm{Im}(\widetilde{\Phi}) = \dim \prod_{i=1}^r \mathbb{P}(V^\vee \otimes W_i)$, since $\widetilde{\Phi}$ is generically injective. Thus, this lemma follows from the surjectivity of $(d\widetilde{\Phi})_{\boldsymbol{\varphi}}$ for general $\boldsymbol{\varphi}$.

In the rest of the proof, we show the surjectivity of $(d\widetilde{\Phi})_{\boldsymbol{\varphi}}$ for general $\boldsymbol{\varphi}$. Recall that $\widetilde{X} \subset \mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}$ is the zero locus of a general section $\boldsymbol{s} \in H^0(\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}, \mathcal{E})$,

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where $\mathcal{E} = \bigoplus_{i=1}^r p^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes q_i^* T_{\mathbb{P}^{m_i}}(-1)$. Hence, $\widetilde{X}$ is smooth with $\operatorname{codim}(X, \mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}) = \operatorname{rank} \mathcal{E}$, and the normal bundle $N_{\widetilde{X}/\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}}$ is isomorphic to $\mathcal{E}|_{\widetilde{X}} = \bigoplus_{i=1}^r \widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes \widetilde{q}_i^* T_{\mathbb{P}^{m_i}}(-1)$. We note that the isomorphism $N^\vee_{\widetilde{X}/\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}} = I_{\widetilde{X}/\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}} / I^2_{\widetilde{X}/\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}} \cong \mathcal{E}^\vee|_{\widetilde{X}}$ is induced from the surjective homomorphism $\mathcal{E}^\vee \twoheadrightarrow I_{\widetilde{X}/\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}} \subset \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}}$ defined by $\boldsymbol{s}$.

From the Euler sequences on $\mathbb{P}^{m_i}$'s, we have an exact sequence on $\widetilde{X}$:

$$0 \longrightarrow \bigoplus_{i=1}^r \widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes \widetilde{q}_i^* \mathcal{O}_{\mathbb{P}^{m_i}}(-1) \longrightarrow \bigoplus_{i=1}^r \widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes W_i \\ \longrightarrow \bigoplus_{i=1}^r \widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes \widetilde{q}_i^* T_{\mathbb{P}^{m_i}}(-1) \longrightarrow 0.$$

Since $\widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes \widetilde{q}_i^* \mathcal{O}_{\mathbb{P}^{m_i}}(-1) \simeq \mathcal{O}_{\widetilde{X}}(E_i)$ and $h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(E_i)) = 0$, we see that the linear map

$$\bigoplus_{i=1}^r V^\vee \otimes W_i = H^0\Big(\widetilde{X}, \bigoplus_{i=1}^r \widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes W_i\Big) \longrightarrow H^0\Big(\widetilde{X}, \bigoplus_{i=1}^r \widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes \widetilde{q}_i^* T_{\mathbb{P}^{m_i}}(-1)\Big)$$

is surjective with the kernel $\bigoplus_{i=1}^r H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(E_i)) = \bigoplus_{i=1}^r \mathbf{k} s_i$. The induced isomorphism

$$\bigoplus_{i=1}^r V^\vee \otimes W_i / \mathbf{k} s_i \xrightarrow{\sim} H^0\Big(\widetilde{X}, \bigoplus_{i=1}^r \widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes \widetilde{q}_i^* T_{\mathbb{P}^{m_i}}(-1)\Big)$$

of vector spaces is identified with the differential (5.1) under the isomorphisms

$$T_{\boldsymbol{\varphi}}\Big(\prod_{i=1}^r \mathbb{P}(V^\vee \otimes W_i)\Big) \cong \Big(\bigoplus_{i=1}^r (V^\vee \otimes W_i / \mathbf{k} s_i) \otimes (\mathbf{k} s_i)^\vee\Big) \xrightarrow{\sim} \bigoplus_{i=1}^r V^\vee \otimes W_i / \mathbf{k} s_i$$

and

$$T_{[\widetilde{X}]} \operatorname{Hilb}(\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}) \cong H^0(\widetilde{X}, N_{\widetilde{X}/\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}}) \xrightarrow{\sim} H^0\Big(\widetilde{X}, \bigoplus_{i=1}^r \widetilde{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes \widetilde{q}_i^* T_{\mathbb{P}^{m_i}}(-1)\Big)$$

induced by the s_i's. This is a special case of the following general fact: Let Y be a projective variety, E be a vector bundle on Y, s be a global section of E, and $Z = s^{-1}(0)$ be the zero locus of s. When Z is a complete intersection (*i.e.,* when $\operatorname{codim} Z = \operatorname{rank} E$), the differential of

$$\mathbb{P}(H^0(E)) \dashrightarrow \operatorname{Hilb}(Y) : s \longmapsto [Z]$$

can be identified with the natural morphism

$$H^0(E)/\mathbf{k} s \to H^0(E|_Z). \qquad \blacksquare$$

Proof of Theorem 1.1(iii) Take general $\boldsymbol{\varphi}$. We study the tangent space of $\operatorname{Hilb}(\mathbb{P}^{\boldsymbol{m}})$ at $[X]$, which is isomorphic to $H^0(X, N_{X/\mathbb{P}^{\boldsymbol{m}}})$. By $|\boldsymbol{m}| \geq 2n-1$ and (3.7), $\widetilde{\boldsymbol{q}} : \widetilde{X} \to X$ is an isomorphism in this case. The diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{\widetilde{X}} & \longrightarrow & T_{\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}}|_{\widetilde{X}} & \longrightarrow & N_{\widetilde{X}/\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}} & \longrightarrow & 0 \\ & & \downarrow \wr & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widetilde{\boldsymbol{q}}^* T_X & \longrightarrow & \widetilde{\boldsymbol{q}}^*(T_{\mathbb{P}^{\boldsymbol{m}}}|_X) & \longrightarrow & \widetilde{\boldsymbol{q}}^* N_{X/\mathbb{P}^{\boldsymbol{m}}} & \longrightarrow & 0 \end{array}$$

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induces an exact sequence

$$0 \longrightarrow \widetilde{p}^* T_{\mathbb{P}^n} \longrightarrow N_{\widetilde{X}/\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}} \longrightarrow \widetilde{\boldsymbol{q}}^* N_{X/\mathbb{P}^{\boldsymbol{m}}} \longrightarrow 0$$

on $\widetilde{X}$. Since

$$H^0(\widetilde{X}, \widetilde{p}^* T_{\mathbb{P}^n}) \cong H^0(\mathbb{P}^n, T_{\mathbb{P}^n}) \cong V^\vee \otimes V/\mathbf{k}\,\mathrm{id}_V$$

for $\mathrm{id}_V \in \mathrm{Hom}(V, V) \cong V^\vee \otimes V$ and $h^1(\widetilde{X}, \widetilde{p}^* T_{\mathbb{P}^n}) = 0$, we have an exact sequence

$$(5.3) \qquad 0 \longrightarrow V^\vee \otimes V/\mathbf{k}\,\mathrm{id}_V \longrightarrow \bigoplus_{i=1}^{r} V^\vee \otimes W_i/\mathbf{k}s_i \xrightarrow{\ d\ } H^0(X, N_{X/\mathbb{P}^{\boldsymbol{m}}}) \longrightarrow 0,$$

where the middle term

$$H^0(\widetilde{X}, N_{\widetilde{X}/\mathbb{P}^n \times \mathbb{P}^{\boldsymbol{m}}}) \cong \bigoplus_{i=1}^{r} V^\vee \otimes W_i/\mathbf{k}s_i$$

can be identified with $T_{\boldsymbol{\varphi}}\left(\prod_{i=1}^r \mathbb{P}(V^\vee \otimes W_i)\right)$ as in the proof of Lemma 5.1. Then the map d in (5.3) can be identified with $(d\Phi)_{\boldsymbol{\varphi}}$, and $V^\vee \otimes V/\mathbf{k}\,\mathrm{id}_V$ can be identified with the tangent space of the $\mathrm{PGL}(n+1, \mathbf{k})$-orbit of $\boldsymbol{\varphi}$. By Theorem 1.1(i) and (ii), a general fiber of Φ consists of at most two $\mathrm{PGL}(n+1, \mathbf{k})$-orbits. Note that the dimension of the $\mathrm{PGL}(n+1, \mathbf{k})$-orbits is equal to that of $\mathrm{PGL}(n+1, \mathbf{k})$, since $\boldsymbol{\varphi}$ is birational. Hence, we have $\dim \mathrm{Im}(\Phi) = \dim \prod_{i=1}^r \mathbb{P}(V^\vee \otimes W_i) - \dim \mathrm{PGL}(n+1, \mathbf{k})$. Thus, $\dim \mathrm{Im}(\Phi)$ is equal to $h^0(X, N_{X/\mathbb{P}^{\boldsymbol{m}}})$ by (5.3), which is the dimension of the tangent space of $\mathrm{Hilb}(\mathbb{P}^{\boldsymbol{m}})$ at $[X]$. This means that $\mathrm{Hilb}(\mathbb{P}^{\boldsymbol{m}})$ is smooth at $[X]$ of dimension $h^0(X, N_{X/\mathbb{P}^{\boldsymbol{m}}}) = \dim \mathrm{Im}(\Phi)$. Hence, Φ is dominant onto an irreducible component. ∎

Remark 5.2 In the case $n = 3$ and $\boldsymbol{m} = (2^r)$, the condition $|\boldsymbol{m}| \geq 2n-1$ is $2r \geq 5$, that is, $r \geq 3$. As in [AST13, Section 6], the closure of the image of Φ is a cubic hypersurface in an irreducible component $\mathbb{P}^8 = \mathbb{P}(W_1^\vee \otimes W_2^\vee) \subset \mathrm{Hilb}(\mathbb{P}^{\boldsymbol{m}})$ for $r = 2$, so this is sharp in this case. We do not know whether the condition $|\boldsymbol{m}| \geq 2n-1$ is sharp or not in general.

6 Grassmann Tensors as Chow Forms

We recall the standard methods for projective reconstruction from a geometric point of view. Define an index set

$$B(n, \boldsymbol{m}) := \left\{ \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r \,\middle|\, 1 \leq \alpha_i \leq m_i \text{ for any } i = 1, \ldots, r \text{ and } \sum_{i=1}^{r} \alpha_i = n+1 \right\}.$$

For an n-dimensional variety $X \subset \mathbb{P}^{\boldsymbol{m}}$ and $\boldsymbol{\alpha} \in B(n, \boldsymbol{m})$, let

$$\mathcal{Z}(X) \subset G(\boldsymbol{m} - \boldsymbol{\alpha}, \mathbb{P}^{\boldsymbol{m}}) := \prod_{i=1}^{r} G\big(m_i - \alpha_i, \mathbb{P}(W_i)\big)$$

denote the set of points in $G(\boldsymbol{m} - \boldsymbol{\alpha}, \mathbb{P}^{\boldsymbol{m}})$ corresponding to r-tuples $(U_1, \ldots U_r)$ of linear subvarieties $U_i \subset \mathbb{P}(W_i)$ of codimension α_i such that

$$X \cap \prod_{i=1}^{r} U_i \neq \varnothing.$$

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Here, $G(m_i - \alpha_i, \mathbb{P}(W_i))$ is the Grassmannian of $(m_i - \alpha_i)$-planes in $\mathbb{P}(W_i)$, which is embedded in $\mathbb{P}(\bigwedge^{m_i+1-\alpha_i} W_i) = \mathbb{P}(\bigwedge^{\alpha_i} W_i^{\vee})$ by the Plücker embedding for $i = 1, \ldots, r$.

Theorem 6.1 ([HS09, Theorem 3.1], [NHT14, Theorem 2]) *Assume that $\boldsymbol{\varphi}$ is generic. For $\alpha \in B(n, \boldsymbol{m})$ and the multiview variety $X \subset \mathbb{P}^{\boldsymbol{m}}$, the set $\mathcal{Z}(X) \subset G(\boldsymbol{m} - \boldsymbol{\alpha}, \mathbb{P}^{\boldsymbol{m}})$ is a hypersurface defined by the multilinear equation*

$$\sum_{\sigma_1,\ldots,\sigma_r} A^{\sigma_1,\ldots,\sigma_r} p^1_{\sigma_1} \cdots p^r_{\sigma_r} = 0,$$

in $G(\mathbb{P}^{\boldsymbol{m}}, \boldsymbol{\alpha})$, where

$$p^i = \left[p^i_{\sigma_i} \mid \sigma_i = (\sigma_{i,1}, \ldots, \sigma_{i,\alpha_i}), 1 \le \sigma_{i,1} < \cdots < \sigma_{i,\alpha_i} \le m_i + 1 \right]$$

is the Plücker coordinates of the Grassmannian $G(m_i - \alpha_i, \mathbb{P}(W_i))$ for each $i = 1, \ldots, r$. Moreover, the tensor

$$A \coloneqq (A^{\sigma_1,\ldots,\sigma_r}) \in \bigotimes_{i=1}^{r} \bigwedge^{\alpha_i} W_i,$$

is uniquely determined up to scalar from sufficiently many subspace correspondences.[2]

The tensor A, called the *Grassmann tensor of profile α* for a camera configuration $\boldsymbol{\varphi}$ in computer vision, is an analog of the *Chow form* of the multiview variety $X \subset \mathbb{P}^{\boldsymbol{m}}$ (see, *e.g.*, [GKZ08, Section 3.2]). Note that the Grassmann tensors of profile $(2,2)$, $(2,1,1)$, and $(1,1,1,1)$ are classically known as the fundamental matrix, the trifocal tensor, and the quadrifocal tensor, respectively.

The *projective reconstruction theorem* by Hartley and Schaffalitzky can be stated as follows.

Theorem 6.2 ([HS09, Section 5]) *Assume that $\boldsymbol{\varphi}$ is generic. Fix $\alpha \in B(n, \boldsymbol{m})$ and let A be a Grassmann tensor of profile α.*

(i) *If $\boldsymbol{m} \neq (1^{n+1})$, then $\boldsymbol{\varphi}$ is uniquely determined by A up to the actions of* $\mathrm{PGL}(n+1, \mathbf{k})$.
(ii) *If $\boldsymbol{m} = (1^{n+1})$, then two candidates of $\boldsymbol{\varphi}$ are obtained by A up to the actions of* $\mathrm{PGL}(n+1, \mathbf{k})$.

Since the multiview variety X determines its associated hypersurface $\mathcal{Z}(X)$ and the Grassmannian tensor A, Theorem 6.2 implies Theorem 1.1(i) and (ii).

An algorithm for the projective reconstruction for general n and $\boldsymbol{m}$ using Grassmann tensors has been implemented in [HS09], and applied to the analysis of dynamic scenes [WS02, HV08]. In contrast, our proof of the projective reconstruction theorem is based on the analysis of divisors on the multiview variety, and it is an interesting problem to see if it can lead to a new algorithm for the reconstruction.

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[2]Just as a point on the multiview variety X is called a point correspondence, a point in $\mathcal{Z}(X)$ is called a *subspace correspondence*.

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