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HOMOTOPY MINIMAL PERIODS FOR HYPERBOLIC MAPS ON INFRA-NILMANIFOLDS

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Abstract. In this paper, we show that for every nonnilpotent hyperbolic map f on an infra-nilmanifold, the set $\mathrm{HPer}(f)$ is cofinite in $\mathbb{N}$. This is a generalization of a similar result for expanding maps in Lee and Zhao (J. Math. Soc. Japan **59**(1) (2007), 179–184). Moreover, we prove that for every nilpotent map f on an infra-nilmanifold, $\mathrm{HPer}(f)=\{1\}$.

§1. Infra-nilmanifolds

Let $f: X\to X$ be a map on a topological space X. We say that $x\in X$ is a periodic point of f if $f^n(x)=x$ for some positive integer n. If this is the case, we say that this positive integer n is the pure period of x if $f^l(x)\neq x$ for all $l<n$. In this paper, we study these periodic points when X is an infra-nilmanifold and we show that for a large class of maps f on such manifolds, there exists a positive integer m such that any map g homotopic to f admits points of pure period k for any $k\in[m,+\infty)$. In the first section, we recall the necessary background on the class of infra-nilmanifolds and their maps. In the next section, we give a more detailed description of the theory of fixed and periodic points. The third and last section is devoted to the proof of our main result.

Every infra-nilmanifold is modeled on a connected and simply connected nilpotent Lie group. Given such a Lie group G, we consider its group of affine transformations $\mathrm{Aff}(G)=G\rtimes \mathrm{Aut}(G)$, which admits a natural left action on the Lie group G:

$$\forall (g,\alpha)\in \mathrm{Aff}(G),\quad \forall h\in G:\ {}^{(g,\alpha)}h=g\alpha(h).$$

Note that when G is abelian, G is isomorphic to $\mathbb{R}^n$ for some n and $\mathrm{Aff}(G)$ is the usual affine group $\mathrm{Aff}(\mathbb{R}^n)$ with its usual action on the affine space $\mathbb{R}^n$.

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Let $p : \mathrm{Aff}(G) = G \rtimes \mathrm{Aut}(G) \to \mathrm{Aut}(G)$ denote the natural projection onto the second factor.

DEFINITION 1.1. A subgroup $\Gamma \subseteq \mathrm{Aff}(G)$ is called almost-crystallographic if and only if $p(\Gamma)$ is finite and $\Gamma \cap G$ is a uniform and discrete subgroup of G. The finite group $F = p(\Gamma)$ is called the holonomy group of Γ.

The action of Γ on G is properly discontinuous and cocompact and when Γ is torsion-free, this action becomes a free action, from which we can conclude that the resulting quotient space $\Gamma\backslash G$ is a compact manifold with fundamental group Γ.

DEFINITION 1.2. A torsion-free almost-crystallographic group $\Gamma \subseteq \mathrm{Aff}(G)$ is called an almost-Bieberbach group, and the corresponding manifold $\Gamma\backslash G$ is called an infra-nilmanifold (modeled on G).

When the holonomy group is trivial, Γ will be a lattice in G and the corresponding manifold $\Gamma\backslash G$ is a nilmanifold. When G is abelian, Γ will be called a Bieberbach group and $\Gamma\backslash G$ a compact flat manifold. When G is abelian and the holonomy group of Γ is trivial, then Γ is just a lattice in some $\mathbb{R}^n$ and $\Gamma\backslash G$ is a torus.

Now, define the semigroup $\mathrm{aff}(G) = G \rtimes \mathrm{Endo}(G)$, where $\mathrm{Endo}(G)$ is the set of continuous endomorphisms of G. Note that $\mathrm{aff}(G)$ acts on G in a similar way as $\mathrm{Aff}(G)$, that is, any element $(\delta, \mathfrak{D})$ of $\mathrm{aff}(G)$ can be seen as a self-map of G:

$$(\delta, \mathfrak{D}) : G \to G : h \mapsto \delta\mathfrak{D}(h)$$

and we refer to $(\delta, \mathfrak{D})$ as an affine map of G. One of the nice features of infra-nilmanifolds is that any map on a infra-nilmanifold is homotopic to a map which is induced by an affine map of G. One can prove this by using the following result by Lee.

THEOREM 1.3. (Lee [18]) *Let G be a connected and simply connected nilpotent Lie group and suppose that $\Gamma, \Gamma' \subseteq \mathrm{Aff}(G)$ are two almost-crystallographic groups modeled on G. Then for any homomorphism $\varphi : \Gamma \to \Gamma'$ there exists an element $(\delta, \mathfrak{D}) \in \mathrm{aff}(G)$ such that*

$$\forall \gamma \in \Gamma : \varphi(\gamma)(\delta, \mathfrak{D}) = (\delta, \mathfrak{D})\gamma.$$

Note that we can consider the equality $\varphi(\gamma)(\delta, \mathfrak{D}) = (\delta, \mathfrak{D})\gamma$ in $\mathrm{aff}(G)$, since $\mathrm{Aff}(G)$ is contained in $\mathrm{aff}(G)$. With this equality in mind, it is easy to

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see that the affine map $(\delta, \mathfrak{D})$ induces a well-defined map

$$\overline{(\delta, \mathfrak{D})} : \Gamma\backslash G \to \Gamma'\backslash G : \Gamma h \to \Gamma'\delta\mathfrak{D}(h),$$

which exactly induces the morphism φ on the level of the fundamental groups.

On the other hand, if we choose an arbitrary map $f : \Gamma\backslash G \to \Gamma'\backslash G$ between two infra-nilmanifolds and choose a lifting $\tilde{f} : G \to G$ of f, then there exists a morphism $\tilde{f}_* : \Gamma \to \Gamma'$ such that $\tilde{f}_*(\gamma) \circ \tilde{f} = \tilde{f} \circ \gamma$, for all $\gamma \in \Gamma$. By Theorem 1.3, an affine map $(\delta, \mathfrak{D}) \in \mathrm{aff}(G)$ exists which also satisfies $\tilde{f}_*(\gamma) \circ (\delta, \mathfrak{D}) = (\delta, \mathfrak{D}) \circ \gamma$ for all $\gamma \in \Gamma$. Therefore, the induced map $\overline{(\delta, D)}$ and f are homotopic. We call $(\delta, \mathfrak{D})$ an affine homotopy lift of f.

We end this introduction about infra-nilmanifolds with the definition of a hyperbolic map on an infra-nilmanifold. We denote by $\mathfrak{D}_*$ the Lie algebra endomorphism induced by $\mathfrak{D}$ on the Lie algebra $\mathfrak{g}$ associated to G.

Definition 1.4. Let M be an infra-nilmanifold and $f : M \to M$ be a continuous map, with $(\delta, \mathfrak{D})$ as an affine homotopy lift. We say that f is a hyperbolic map if $\mathfrak{D}_*$ has no eigenvalues of modulus 1.

Remark 1.5. The map $\mathfrak{D}$, and hence also $\mathfrak{D}_*$ depends on the choice of the lift $\tilde{f}$. Once the lift $\tilde{f}$ is fixed, and hence the morphism $\tilde{f}_*$ is fixed, the $\mathfrak{D}$ – part of the map $(\delta, \mathfrak{D})$ in Theorem 1.3 is also fixed (although the δ – part is not unique in general). It follows that f determines $\mathfrak{D}$ only up to an inner automorphism of G. But as inner automorphisms have no effect on the eigenvalues of $\mathfrak{D}_*$ (in the case of a nilpotent Lie group G) the notion of a hyperbolic map is well defined.

Two important classes of maps on infra-nilmanifolds which are hyperbolic are the expanding maps and the Anosov diffeomorphisms.

Remark 1.6. Due to [4, Lemma 4.5], it is known that every nowhere expanding map on an infra-nilmanifold only has eigenvalues 0 or eigenvalues of modulus 1. This means that every hyperbolic map for which $\mathfrak{D}_*$ is not nilpotent has an eigenvalue of modulus strictly bigger than 1.

§2. Nielsen theory, dynamical zeta functions and HPer(f)

Let $f : X \to X$ be a self-map of a compact polyhedron X. There are different ways to assign integers to this map f that give information about the fixed points of f. One of these integers is the Lefschetz number $L(f)$

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which is defined as

$$L(f)=\sum_{i=0}^{\dim X}(-1)^i\mathrm{Tr}(f_{*,i}:H_i(X,\mathbb{R})\to H_i(X,\mathbb{R})).$$

In our situation, the space $X=\Gamma\backslash G$ will be a infra-nilmanifold, which is an aspherical space, and hence the (co)homology of the space $X=\Gamma\backslash G$ equals the (co)homology of the group Γ. It follows that in this case we have (see also [13, p. 36])

$$\begin{aligned}L(f)&=\sum_{i=0}^{\dim X}(-1)^i\mathrm{Tr}(f_{*,i}:H_i(\Gamma,\mathbb{R})\to H_i(\Gamma,\mathbb{R}))\\&=\sum_{i=0}^{\dim X}(-1)^i\mathrm{Tr}(f_i^*:H^i(\Gamma,\mathbb{R})\to H^i(\Gamma,\mathbb{R})).\end{aligned}$$

The Lefschetz fixed point theorem states that if $L(f)\neq 0$, then f has at least one fixed point. Because the Lefschetz number is only defined in terms of (co)homology groups, it remains invariant under a homotopy and hence, if $L(f)\neq 0$, the Lefschetz fixed point theorem guarantees that any map homotopic to f also has at least one fixed point.

Another integer giving information on the fixed points of f is the Nielsen number $N(f)$. It is a homotopy-invariant lower bound for the number of fixed points of f. To define $N(f)$, fix a reference lifting $\tilde{f}$ of f with respect to a universal cover $(\tilde{X},p)$ of X and denote the group of covering transformations by $\mathcal{D}$. For $\alpha\in\mathcal{D}$, the sets $p(\mathrm{Fix}(\alpha\circ\tilde{f}))$ form a partition of the fixed point set $\mathrm{Fix}(f)$. These sets are called fixed point classes. By using the fixed point index, we can assign an integer to each fixed point class in such a way that if a nonzero integer is assigned, the fixed point class cannot completely vanish under a homotopy. Such a nonvanishing fixed point class will be called essential and $N(f)$ is defined as the number of essential fixed point classes of f.

By definition, it is clear that $N(f)$ will indeed be a homotopy-invariant lower bound for the number of fixed points of f. Hence, in general, $N(f)$ will give more information about the fixed points of f than $L(f)$. The downside, however, is that Nielsen numbers are often much harder to compute than Lefschetz numbers, because the fixed point index can be a tedious thing to work with. Luckily, on infra-nilmanifolds there exists an algebraic formula to compute $N(f)$, which makes them a convenient class of manifolds to study Nielsen theory on. More information on both $L(f)$ and $N(f)$ can be found in for example, [3, 14, 15].

By using the Lefschetz and Nielsen numbers of iterates of f as coefficients, it is possible to define the so-called dynamical zeta functions. The Lefschetz zeta function was introduced by Smale in [21]:

$$L_f(z) = \exp\left(\sum_{k=1}^{+\infty} \frac{L(f^k)}{k} z^k\right).$$

In his paper, Smale also proved that the Lefschetz zeta function is always rational for self-maps on compact polyhedra.

The proof is actually quite straightforward. Let the λ_{ij}'s denote the eigenvalues of $f_*^i : H^i(X,\mathbb{R}) \to H^i(X,\mathbb{R})$, with $j \in \{1,\ldots,\dim(H^i(X,\mathbb{R}))\}$. Because the trace of a matrix is the sum of the eigenvalues, we find

$$L_f(z) = \exp\left(\sum_{k=1}^{+\infty}\left(\sum_{i=0}^{\dim X}(-1)^i \sum_{j=1}^{\dim H^i(X)} \lambda_{ij}^k\right)\frac{z^k}{k}\right).$$

By reordering the terms and by using the fact that

$$\sum_{k=1}^{+\infty} \frac{a^k z^k}{k} = -\log(1-az) \quad \text{for } |z| < |a|^{-1},$$

it is easy to derive that

$$L_f(z) = \prod_{i=0}^{\dim X} \prod_{j=1}^{\dim H^i(X)} (1-\lambda_{ij} z)^{(-1)^{i+1}}. \tag{1}$$

REMARK 2.1. Suppose that Λ is a lattice of a connected and simply connected nilpotent Lie group G and $f:\Lambda\backslash G \to \Lambda\backslash G$ is a self-map of the nilmanifold $\Lambda\backslash G$ with affine homotopy lift $(\delta,\mathfrak{D})$. Let $\mathfrak{D}_*$ be the induced linear map on the Lie algebra $\mathfrak{g}$ of G as before. The main result of [19] states that there are natural isomorphisms

$$H^i(\Lambda,\mathbb{R}) \cong H^i(\Lambda\backslash G,\mathbb{R}) \cong H^i(\mathfrak{g},\mathbb{R}).$$

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The naturality of these automorphisms implies that there is a commutative diagram

$$\begin{array}{ccc} H^i(\Lambda,\mathbb{R}) & \xrightarrow{\cong} & H^i(\mathfrak{g},\mathbb{R}) \\ {\scriptstyle f_i^*}\downarrow & & \downarrow{\scriptstyle \mathfrak{D}_*^i} \\ H^i(\Lambda,\mathbb{R}) & \xrightarrow[\cong]{} & H^i(\mathfrak{g},\mathbb{R}) \end{array}$$

Here $\mathfrak{D}_*^i$ is the map induced by $\mathfrak{D}_*$ on the ith cohomology space of $\mathfrak{g}$. Recall, that the cohomology of $\mathfrak{g}$ is defined as the cohomology of a cochain complex, where the ith term is $\mathrm{Hom}(\bigwedge^i \mathfrak{g},\mathbb{R}) = (\bigwedge^i \mathfrak{g})^*$, the dual space of $\bigwedge^i \mathfrak{g}$. So, $\mathfrak{D}_*^i$ is induced by the dual map of $\bigwedge^i \mathfrak{D}_*$. Since this dual map and $\bigwedge^i \mathfrak{D}_*$ have the same eigenvalues, it follows that the set of eigenvalues of $\mathfrak{D}_*^i$, hence also the set of eigenvalues $\lambda_{i,j}$ of f_i^* in expression (1), is a subset of the set of eigenvalues of $\bigwedge^i \mathfrak{D}_* : \bigwedge^i \mathfrak{g} \to \bigwedge^i \mathfrak{g}$. (This fact is also reflected in the formula obtained in [7, Theorem 23].)

The Nielsen zeta function was introduced by Fel'shtyn in [10, 20] and is defined in a similar way as the Lefschetz zeta function:

$$N_f(z) = \exp\left(\sum_{k=1}^{+\infty} \frac{N(f^k)}{k} z^k\right).$$

It is known that this zeta function does not always have to be a rational function. A counterexample for this can be found in [7], for example, in Remark 7.

For self-maps on infra-nilmanifolds, however, the Nielsen zeta function will always be rational. To prove this, one can exploit the fact that $N(f)$ and $L(f)$ are very closely related. In [5], we defined a subgroup Γ_+ of Γ, which equals Γ or is of index 2 in Γ. The precise definition is not of major significance for the rest of this paper. However, it allowed us to write $N_f(z)$ as a function of $L_f(z)$ if $\Gamma = \Gamma_+$, and as a combination of $L_f(z)$ and $L_{f_+}(z)$ if $[\Gamma : \Gamma_+] = 2$. Here, $f_+ : \Gamma_+\backslash G \to \Gamma_+\backslash G$ is a lift of f to the 2-folded covering space $\Gamma_+\backslash G$ of $\Gamma\backslash G$. The following theorem, together with the fact that Lefschetz zeta functions are always rational, therefore proves the rationality of Nielsen zeta functions for infra-nilmanifolds.

THEOREM 2.2. [5, Theorem 4.6] *Let $M = \Gamma\backslash G$ be an infra-nilmanifold and let $f : M \to M$ be a self-map with affine homotopy lift $(\delta, \mathfrak{D})$. Let p*

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denote the number of positive real eigenvalues of $\mathfrak{D}_*$ which are strictly greater than 1 and let n denote the number of negative real eigenvalues of $\mathfrak{D}_*$ which are strictly less than -1. Then we have the following table of equations:

	p *even,* n *even*	p *even,* n *odd*	p *odd,* n *even*	p *odd,* n *odd*
$\Gamma = \Gamma_+$	$N_f(z) = L_f(z)$	$N_f(z) = \dfrac{1}{L_f(-z)}$	$N_f(z) = \dfrac{1}{L_f(z)}$	$N_f(z) = L_f(-z)$
$\Gamma \neq \Gamma_+$	$N_f(z) = \dfrac{L_{f_+}(z)}{L_f(z)}$	$N_f(z) = \dfrac{L_f(-z)}{L_{f_+}(-z)}$	$N_f(z) = \dfrac{L_f(z)}{L_{f_+}(z)}$	$N_f(z) = \dfrac{L_{f_+}(-z)}{L_f(-z)}$

Moreover, this theorem also tells us that we can write $N_f(z)$ in a similar form as in equation (1), since every Lefschetz zeta function is of this form. More information about dynamical zeta functions can be found in [7].

Closely related to fixed point theory, is periodic point theory. We call $x \in X$ a periodic point of f if there exists a positive integer n, such that $f^n(x) = x$. Of course, when $f^n(x) = x$, this does not automatically imply that the actual period of x is n. For example, it is immediately clear that every fixed point is also a periodic point of period n, for all $n > 0$. In order to exclude these points, we define the set of periodic points of pure period n:

$$P_n(f) = \{x \in X \parallel f^n(x) = x \text{ and } f^k(x) \neq x, \forall k|n\}.$$

The set of homotopy minimal periods of f is then defined as the following subset of the positive integers:

$$\mathrm{HPer}(f) = \bigcap_{f \simeq g} \{n | P_n(g) \neq \emptyset\}.$$

This set has been studied extensively, for example, in [1] for maps on the torus, in [12] for maps on nilmanifolds and in [9, 17] for maps on infra-nilmanifolds.

Just as Nielsen fixed point theory divides $\mathrm{Fix}(f)$ into different fixed point classes, Nielsen periodic point theory divides $\mathrm{Fix}(f^n)$ into different fixed point classes, for all $n > 0$ and looks for relations between fixed point classes on different levels. This idea is covered by the following definition.

DEFINITION 2.3. Let $f : X \to X$ be a self-map. If $\mathbb{F}_k$ is a fixed point class of f^k, then $\mathbb{F}_k$ will be contained in a fixed point class $\mathbb{F}_{kn}$ of $(f^k)^n$, for all n. We say that $\mathbb{F}_k$ boosts to $\mathbb{F}_{kn}$. On the other hand, we say that $\mathbb{F}_{kn}$ reduces to $\mathbb{F}_k$.

An important definition that gives some structure to the boosting and reducing relations is the following.

DEFINITION 2.4. A self-map $f : X \to X$ will be called essentially reducible if, for all n, k, essential fixed point classes of f^{kn} can only reduce to essential fixed point classes of f^k. A space X is called essentially reducible if every self-map $f : X \to X$ is essentially reducible.

It can be shown that the fixed point classes for maps on infra-nilmanifolds always have this nice structure for their boosting and reducing relations.

THEOREM 2.5. [17] *Infra-nilmanifolds are essentially reducible.*

One of the consequences of having this property, is the following.

THEOREM 2.6. [1] *Suppose that f is essentially reducible and suppose that*

$$N(f^k) > \sum_{p \text{ prime},\, p|k} N(f^{k/p}),$$

then $k \in \mathrm{HPer}(f)$.

The idea of this theorem is actually quite easy to grasp. Because maps on infra-nilmanifolds are essentially reducible, every reducible essential fixed point class on level k will reduce to an essential fixed point class on level $\frac{k}{p}$, with p a prime divisor of k. Therefore, the condition

$$N(f^k) > \sum_{p \text{ prime},\, p|k} N(f^{k/p})$$

actually tells us that there is definitely one irreducible essential fixed point class on level k, which means that there is at least one periodic point of pure period k.

For this paper, this is all we need to know about Nielsen periodic point theory. More information about Nielsen periodic point theory in general can be found in [11, 13] or [14].

§3. HPer(f) for hyperbolic maps on infra-nilmanifolds

3.1 The nonnilpotent case

We begin with the following definition, which tells us something about the asymptotic behavior of the sequence $\{N(f^k)\}_{k=1}^{\infty}$.

DEFINITION 3.1. The asymptotic Nielsen number of f is defined as

$$N^{\infty}(f) = \max\left\{1, \limsup_{k\to\infty} N(f^k)^{1/k}\right\}.$$

By $\mathrm{sp}(A)$ we mean the spectral radius of the matrix or the operator A. It equals the largest modulus of an eigenvalue of A.

THEOREM 3.2. [8, Theorem 4.3] *For a continuous map f on an infra-nilmanifold, with affine homotopy lift $(\delta, \mathfrak{D})$, such that $\mathfrak{D}_*$ has no eigenvalue 1, we have*

$$N^{\infty}(f) = \mathrm{sp}\Big(\bigwedge \mathfrak{D}_*\Big).$$

If $\{\nu_i\}_{i\in I}$ is the set of eigenvalues of $\mathfrak{D}_*$, we know that

$$\mathrm{sp}\Big(\bigwedge \mathfrak{D}_*\Big) = \begin{cases} \prod_{|\nu_i|>1} |\nu_i| & \text{if } \mathrm{sp}(\mathfrak{D}_*) > 1, \\ 1 & \text{if } \mathrm{sp}(\mathfrak{D}_*) \leqslant 1. \end{cases}$$

Therefore, we have the following corollary of Theorem 3.2.

COROLLARY 3.3. *Let f be a hyperbolic, continuous map on an infra-nilmanifold. Let $(\delta, \mathfrak{D})$ be an affine homotopy lift of f and let $\{\nu_i\}_{i\in I}$ be the set of eigenvalues of $\mathfrak{D}_*$. If $\mathfrak{D}_*$ is not nilpotent, then*

$$N^{\infty}(f) = \prod_{|\nu_i|>1} |\nu_i|.$$

Proof. When $\mathfrak{D}_*$ is not nilpotent, we know by Remark 1.6 that $\mathrm{sp}(\mathfrak{D}_*) > 1$. Because f is hyperbolic, 1 is certainly not an eigenvalue of $\mathfrak{D}_*$ and therefore, we can use the result of Theorem 3.2. □

Because of Theorem 2.2, we know that $N_f(z)$ can be written as the quotient of Lefschetz zeta functions. Since every Lefschetz zeta function on a compact polyhedron is of the form

$$L_f(z) = \prod_{i=1}^{m} (1 - \mu_i z)^{\gamma_i},$$

with $\mu_i \in \mathbb{C}$ and $\gamma_i \in \{1, -1\}$, the same will hold for $N_f(z)$. Also, it is easy to check that

$$N_f(z) = \prod_{i=1}^{n} (1 - \lambda_i z)^{-\varepsilon_i} \Rightarrow N(f^k) = \sum_{i=1}^{n} \varepsilon_i \lambda_i^k,$$

for all $k \in \mathbb{N}$.

In Remark 2.1 we already mentioned the fact that for nilmanifolds the μ_i's appearing in the expression for $L_f(z)$ are eigenvalues of $\bigwedge \mathfrak{D}_*$. We now claim that the same holds for maps on infra-nilmanifolds. Consider an infra-nilmanifold $\Gamma\backslash G$ and a self-map f of $\Gamma\backslash G$ with affine homotopy lift $(\delta, \mathfrak{D})$. Without loss of generality, we may assume that $f = \overline{(\delta, \mathfrak{D})}$. We now fix a fully characteristic subgroup Λ of finite index in Γ that is contained in G (e.g., see [16]). Hence for the induced morphism $f_* : \Gamma \to \Gamma$ we have that $f_*(\Lambda) \subseteq \Lambda$. It follows that $(\delta, \mathfrak{D})$ also induces a map $\hat{f}$ on the nilmanifold $\Lambda\backslash G$ and that $\hat{f}_* = f_{*|\Lambda}$. By [2, Theorem III 10.4] we know that the restriction map induces an isomorphism $\mathrm{res} : H^i(\Gamma, \mathbb{Q}) \to H^i(\Lambda, \mathbb{Q})^{\Gamma/\Lambda}$. As the restriction map is natural, we obtain the following commutative diagram:

$$\begin{array}{ccc} H^i(\Gamma,\mathbb{Q}) & \xrightarrow[\cong]{\mathrm{res}} & H^i(\Lambda,\mathbb{Q})^{\Gamma/\Lambda} \\ \Big\downarrow f_*^i & & \Big\downarrow \hat{f}_*^i \\ H^i(\Gamma,\mathbb{Q}) & \xrightarrow[\cong]{\mathrm{res}} & H^i(\Lambda,\mathbb{Q})^{\Gamma/\Lambda} \end{array}$$

It follows that each of the eigenvalues of f_*^i is also an eigenvalue of $\hat{f}_*^i$. Since the latter ones are all eigenvalues of $\bigwedge^i \mathfrak{D}_*$, by Remark 2.1, it follows that all eigenvalues of f_*^i are also eigenvalues of $\bigwedge^i \mathfrak{D}_*$. This means that the μ_i's appearing in the expression for $L_f(z)$ are eigenvalues of $\bigwedge \mathfrak{D}_*$ and of course, because f_+ has the same affine homotopy lift as f, the same applies to $L_{f_+}(z)$.

By Theorem 2.2, we know that $N_f(z)$ can be written as a combination of $L_f(z)$ and possibly $L_{f_+}(z)$, or as a combination of $L_f(-z)$ and possibly $L_{f_+}(-z)$. In the first case, by the previous discussion we see that all λ_i's in the expression for $N_f(z)$ are eigenvalues of $\bigwedge \mathfrak{D}_*$. In the latter case, all λ_i's are the opposite of eigenvalues of $\bigwedge \mathfrak{D}_*$. This means that we can write

$$N(f^k) = \sum_{i=1}^{n} \varepsilon_i \lambda_i^k,$$

such that all λ_i's or all $-\lambda_i$'s are eigenvalues of $\bigwedge \mathfrak{D}_*$.

LEMMA 3.4. *If f is a nonnilpotent hyperbolic map on an infra-nilmanifold, with $(\delta, \mathfrak{D})$ as affine homotopy lift, it is possible to write*

$$N(f^k) = \sum_{i=1}^{m} a_i \lambda_i^k,$$

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with $a_i \in \mathbb{Z}$*,* $a_1 \geqslant 1$ *and such that*

$$|\lambda_1| = \lambda_1 = \mathrm{sp}\Big(\bigwedge \mathfrak{D}_*\Big) > |\lambda_2| \geqslant \cdots \geqslant |\lambda_m|.$$

Proof. By previous arguments, we know that it is possible to write

$$N(f^k) = \sum_{i=1}^{n} \varepsilon_i \lambda_i^k,$$

where all λ_i's or all $-\lambda_i$'s are eigenvalues of $\bigwedge \mathfrak{D}_*$. By grouping the λ's that appear more than once and by changing the order, we obtain the desired form

$$N(f^k) = \sum_{i=1}^{m} a_i \lambda_i^k,$$

with $a_i \in \mathbb{Z}$ and $|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_m|$. There is a unique eigenvalue of $\bigwedge \mathfrak{D}_*$ of maximal modulus, namely the product

$$\prod_{|\lambda_i|\geqslant 1} \lambda_i = \mu_1.$$

Note that the product is real, because for every $\lambda \notin \mathbb{R}$, we know that if $|\lambda| > 1$, then $|\overline{\lambda}| > 1$ and both are eigenvalues of $\bigwedge \mathfrak{D}_*$, because $\mathfrak{D}_*$ is a real matrix. It is unique because f is hyperbolic and $\mathfrak{D}_*$ has no eigenvalues of modulus 1.

Because of Theorem 3.2, we know that $N^\infty(f) = \mathrm{sp}(\bigwedge \mathfrak{D}_*) = |\mu_1|$. Suppose now that μ_1 or $-\mu_1$ does not appear as one of the λ's in the expression of $N(f^k)$. Then, it should still hold that

$$1 = \limsup_{k\to\infty} \left(\frac{\sum_{i=1}^{m} a_i \lambda_i^k}{\mu_1^k}\right)^{1/k}.$$

Let $a_{\max} = \max\{|a_i|\}$, then it is easy to derive that for all k:

$$\frac{\sum_{i=1}^{m} a_i \lambda_i^k}{\mu_1^k} \leqslant \sum_{i=1}^{m} |a_i| \left|\frac{\lambda_i}{\mu_1}\right|^k \leqslant m a_{\max} \left|\frac{\lambda_1}{\mu_1}\right|^k.$$

So, we would have that

$$1 \leqslant \limsup_{k\to\infty} \left(m a_{\max} \left|\frac{\lambda_1}{\mu_1}\right|^k\right)^{1/k} = \left|\frac{\lambda_1}{\mu_1}\right| < 1,$$

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where the last inequality follows from the fact that μ_1 is the unique eigenvalue of maximal modulus. Moreover, an easy argument shows that $a_1 < 0$ or $\lambda_1 < 0$ cannot occur in the expression of $N(f^k)$, because otherwise $N(f^k)$ would be negative for sufficiently large k. As we have already proved that $a_1 = 0$ is impossible, we know that $a_1 \geqslant 1$ and that $\mathrm{sp}(\bigwedge \mathfrak{D}_*)$ will appear as one of the λ's in the expression for $N(f^k)$. □

REMARK 3.5. The fact that $\mathrm{sp}(\bigwedge \mathfrak{D}_*)$ has to appear in the expression for $N(f^k)$ was proved in a more general setting in [9].

LEMMA 3.6. *If f is a hyperbolic map on an infra-nilmanifold, then $N(f^k) \neq 0$ for all $k > 0$.*

Proof. Let $(\delta, \mathfrak{D})$ be an affine homotopy lift of f and let F be the holonomy group of the infra–nilmanifold. By [16], we know that

$$N(f^k) = \frac{1}{\#F} \sum_{\mathfrak{A} \in F} |\det(I - \mathfrak{A}_* \mathfrak{D}_*^k)|.$$

Because all the terms make a nonnegative contribution to this sum, we know that

$$N(f^k) \geqslant \frac{1}{\#F} |\det(I - \mathfrak{D}_*^k)| = \frac{1}{\#F} \prod_{i=1}^{n} |1 - \mu_i^k| > 0,$$

where the μ_i are all the eigenvalues of $\mathfrak{D}_*$. The last inequality follows from the fact that f is hyperbolic and so there are no eigenvalues of modulus 1. □

From now on, we consider f to be a hyperbolic map on an infra-nilmanifold and $N(f^k)$ to be of the form

$$N(f^k) = \sum_{i=1}^{m} a_i \lambda_i^k,$$

with $a_i \in \mathbb{Z}$, $a_1 \geqslant 1$ and such that

$$|\lambda_1| = \lambda_1 = \mathrm{sp}\Big(\bigwedge \mathfrak{D}_*\Big) > |\lambda_2| \geqslant \cdots \geqslant |\lambda_m|.$$

For the sake of clarity, we keep using this notation in the rest of this paragraph.

LEMMA 3.7. *For all μ such that $\lambda_1 > \mu > 1$, there exists $k_0 \in \mathbb{N}$, such that for all $k \geqslant k_0$ and for all $n \in \mathbb{N}$, we have the following inequality:*

$$N(f^{k+n}) > \mu^n N(f^k).$$

Proof. Let $1 > \varepsilon > 0$, such that

$$\frac{\lambda_1 - \mu}{\lambda_1 + \mu} \geqslant \varepsilon > 0.$$

Note that this implies that

$$\lambda_1 \frac{1-\varepsilon}{1+\varepsilon} \geqslant \mu.$$

Now, choose $k_0 \in \mathbb{N}$ such that, for all $i \in \{2, \ldots, m\}$,

$$\left|\frac{a_i}{a_1}\right| \left|\frac{\lambda_i}{\lambda_1}\right|^{k_0} < \frac{\varepsilon}{m}.$$

Because of Lemma 3.4, we know that $|\lambda_1| > |\lambda_i|$, for all these i's, so the inequality will hold for k_0 sufficiently large.

Now, consider the fraction

$$\frac{N(f^{k+n})}{N(f^k)} = \frac{a_1\lambda_1^{k+n} + \sum_{i=2}^m a_i\lambda_i^{k+n}}{a_1\lambda_1^k + \sum_{i=2}^m a_i\lambda_i^k} = \frac{\lambda_1^n + \sum_{i=2}^m \frac{a_i}{a_1}\frac{\lambda_i}{\lambda_1}^k \lambda_i^n}{1 + \displaystyle\sum_{i=2}^m \frac{a_i}{a_1}\frac{\lambda_i}{\lambda_1}^k}.$$

Note that $N(f^k) \neq 0$, according to Lemma 3.6, so the fraction is well defined. It is now easy to see that this equality implies the following inequalities:

$$\frac{N(f^{k+n})}{N(f^k)} \geqslant \frac{\lambda_1^n - \left|\sum_{i=2}^m \frac{a_i}{a_1}\frac{\lambda_i}{\lambda_1}^k\right| \lambda_1^n}{1 + \left|\sum_{i=2}^m \frac{a_i}{a_1}\frac{\lambda_i}{\lambda_1}^k\right|} > \lambda_1^n \frac{1-\varepsilon}{1+\varepsilon} \geqslant \lambda_1^n \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^n \geqslant \mu^n.$$

∎

COROLLARY 3.8. *There exists ν, such that $\lambda_1 > \nu > 1$ and an $l_0 \in \mathbb{N}$, such that for all $l \geqslant l_0$ and for all $k < l$:*

$$N(f^l) > \nu^{l-k} N(f^k).$$

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Proof. Fix μ as in Lemma 3.7 and let k_0 be the resulting integer from this lemma. Note that Lemma 3.7 actually tells us that the sequence $\{N(f^k)\}_{k=1}^{\infty}$ will be strictly increasing from a certain point onwards. Because all Nielsen numbers are integers, this means that there will exist $l_0 \geqslant k_0$, such that $N(f^{l_0}) > N(f^l)$, for all $l < l_0$, so also for all $l < k_0$.

Now, let us define the following number

$$\tau = \min\left\{ \left(\frac{N(f^{l_0})}{N(f^l)}\right)^{1/(l_0-l)} \| l < l_0 \right\}.$$

It is clear that $\tau > 1$. Let $\nu = \min\{\mu, (1+\tau)/2\}$. Clearly, $\lambda_1 > \nu > 1$ and, for all $k < l_0$, we have the following inequalities:

$$\frac{N(f^{l_0})}{N(f^k)} \geqslant \tau^{l_0-k} > \nu^{l_0-k}.$$

Because of Lemma 3.7 and the fact that $\mu \geqslant \nu$, we know this inequality also applies to all $l \geqslant l_0$. □

THEOREM 3.9. *If f is a hyperbolic map on an infra-nilmanifold, with affine homotopy lift $(\delta, \mathfrak{D})$, such that $\mathfrak{D}_*$ is not nilpotent, then there exists an integer m_0, such that*

$$[m_0, +\infty) \subset \mathrm{HPer}(f).$$

Proof. Choose ν and l_0 as in Corollary 3.8. Since

$$\lim_{k\to\infty} \frac{\nu^{2^{k-1}}}{k} = +\infty,$$

we know there exists a k_0, such that $\nu^{2^{k-1}} > k$ for all $k \geqslant k_0$. Define $m_0 = \max\{2^{k_0}, 2l_0 + 1\}$.

Now, suppose that $m \geqslant m_0$ and m is even. Let K denote the number of different prime divisors of m. As $m \geqslant 2l_0 + 1$, we know that $m/2 > l_0$ and hence the result of Corollary 3.8 applies. Therefore, we have the following inequalities

$$\sum_{p \text{ prime}, p|m} N(f^{m/p}) \leqslant K \cdot N(f^{m/2}) < \frac{K}{\nu^{m/2}} \cdot N(f^m).$$

By Theorem 2.6, it now suffices to show that

$$\frac{K}{\nu^{m/2}} \leqslant 1.$$

Because K denotes the number of different prime divisors of m, we certainly know that $m > 2^K$. By the definition of m_0, we also know that $m \geqslant 2^{k_0}$. If $K \geqslant k_0$, then

$$\nu^{m/2} > \nu^{2^{K-1}} > K,$$

which is sufficient. If $k_0 > K$, we have that

$$\nu^{m/2} \geqslant \nu^{2^{k_0-1}} > k_0 > K.$$

So, when $m \geqslant m_0$ is even, $m \in \mathrm{HPer}(f)$.

When $m \geqslant m_0$ is odd, a similar argument holds. Let K again be the number of different prime divisors of m and note that $m \geqslant 2l_0 + 1$ implies that $(m-1)/2 \geqslant l_0$. Again, by using Corollary 3.8, we obtain the following inequalities:

$$\sum_{p \text{prime}, p|m} N(f^{m/p}) \leqslant K \cdot N(f^{(m-1)/2}) < \frac{K}{\nu^{(m+1)/2}} \cdot N(f^m).$$

Again, $m > 2^K$ and by definition $m \geqslant 2^{k_0}$. When $K \geqslant k_0$,

$$\nu^{(m+1)/2} > \nu^{(2^K+1)/2} > \nu^{2^{K-1}} > K.$$

When $k_0 > K$, the same reasoning gives us

$$\nu^{(m+1)/2} \geqslant \nu^{(2^{k_0}+1)/2} > \nu^{2^{k_0-1}} > k_0 > K.$$

This concludes the proof of this theorem. □

Remark 3.10. Having obtained Lemma 3.4, it is also possible to prove our main theorem in an alternative way, by following the approach of [8, Section 6].

Remark 3.11. Note that our proof also applies to every essentially irreducible map f (on any manifold) for which there exists $\mu > 1$ and $k_0 \in \mathbb{N}$, such that for all $k \geqslant k_0$ and for all $n \in \mathbb{N}$, we have that

$$N(f^{k+n}) > \mu^n N(f^k).$$

This condition is therefore sufficient for $\mathrm{HPer}(f)$ to be cofinite in $\mathbb{N}$.

3.2 The nilpotent case

For the sake of completeness, in this section we also treat the case where $\mathfrak{D}_*$ is nilpotent.

The following two theorems can be found in [6].

THEOREM 3.12. *Let $\Gamma \subseteq \mathrm{Aff}(G)$ be an almost-Bieberbach group with holonomy group $F \subseteq \mathrm{Aut}(G)$. Let $M = \Gamma\backslash G$ be the associated infra-nilmanifold. If $f : M \to M$ is a map with affine homotopy lift $(\delta, \mathfrak{D})$, then*

$$R(f) = \infty \text{ if and only if } \exists \mathfrak{A} \in F \text{ such that } \det(I - \mathfrak{A}_*\mathfrak{D}_*) = 0.$$

THEOREM 3.13. *Let f be a map on an infra-nilmanifold such that $R(f) < \infty$, then*

$$N(f) = R(f).$$

PROPOSITION 3.14. *When f is a hyperbolic map on an infra-nilmanifold with affine homotopy lift $(\delta, \mathfrak{D})$ such that $\mathfrak{D}_*$ is nilpotent then, for all k,*

$$N(f^k) = R(f^k) = 1.$$

Proof. By combining Theorems 3.12 and 3.13 we know that every fixed point class of f^k is essential if and only if for all $\mathfrak{A} \in F$ (where F is the holonomy group of our infra-nilmanifold), it is true that

$$\det(I - \mathfrak{A}_*\mathfrak{D}_*^k) \neq 0.$$

By [4, Lemma 3.1], we know that there exists $\mathfrak{B} \in F$, and an integer l, such that

$$(\mathfrak{B}_*\mathfrak{D}_*^k)^l = \mathfrak{D}_*^{lk} \qquad \text{and} \qquad \det(I - \mathfrak{A}_*\mathfrak{D}_*^k) = \det(I - \mathfrak{B}_*\mathfrak{D}_*^k).$$

Note that $\det(I - \mathfrak{B}_*\mathfrak{D}_*^k) = 0$ implies that $\mathfrak{B}_*\mathfrak{D}_*^k$ has an eigenvalue 1, but this would mean that $\mathfrak{D}_*^{lk}$ has an eigenvalue 1, which is in contradiction with the hyperbolicity of our map. Therefore, $R(f^k) = N(f^k)$.

Note that $\mathfrak{D}_*$ only has eigenvalue 0. The fact that there exists $\mathfrak{B} \in F$ and an integer l such that

$$(\mathfrak{B}_*\mathfrak{D}_*^k)^l = \mathfrak{D}_*^{lk} \qquad \text{and} \qquad \det(I - \mathfrak{A}_*\mathfrak{D}_*^k) = \det(I - \mathfrak{B}_*\mathfrak{D}_*^k),$$

implies that $\mathfrak{B}_*\mathfrak{D}_*^k$ only has eigenvalue 0. As a consequence

$$\det(I - \mathfrak{A}_*\mathfrak{D}_*^k) = \det(I - \mathfrak{B}_*\mathfrak{D}_*^k) = 1,$$

for all $\mathfrak{A} \in F$. By applying the main formula from [16], an easy computation shows that $N(f^k) = 1$. ∎

In [8], we find the following proposition.

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PROPOSITION 3.15. *If* $\overline{(\delta, \mathfrak{D})} : M \to M$ *is a continuous map on an infra-nilmanifold, induced by an affine map, then every nonempty fixed point class is path-connected and*

(1) *Every essential fixed point class of* $\overline{(\delta, \mathfrak{D})}$ *consists of exactly one point.*
(2) *Every nonessential fixed point class of* $\overline{(\delta, \mathfrak{D})}$ *is empty or consists of infinitely many points.*

THEOREM 3.16. *If* f *is a hyperbolic map on an infra-nilmanifold with affine homotopy lift* $(\delta, \mathfrak{D})$ *such that* $\mathfrak{D}_*$ *is nilpotent, then*

$$\mathrm{HPer}(f) = \{1\}.$$

Proof. Let $\overline{(\delta, \mathfrak{D})}$ be the induced map of $(\delta, \mathfrak{D})$ on the infra-nilmanifold. It suffices to show that $\mathrm{Per}(\overline{(\delta, \mathfrak{D})}) = \{1\}$, because $N(f) = 1$ immediately implies that $1 \in \mathrm{HPer}(f)$.

By Propositions 3.15 and 3.14, we know that $\mathrm{Fix}(\overline{(\delta, \mathfrak{D})}^k)$ consists of precisely one point, for all $k > 0$. Because, for all $k > 0$, it holds that

$$\mathrm{Fix}(\overline{(\delta, \mathfrak{D})}) \subset \mathrm{Fix}(\overline{(\delta, \mathfrak{D})}^k),$$

we know that $\mathrm{Fix}(\overline{(\delta, \mathfrak{D})}^k) = \mathrm{Fix}(\overline{(\delta, \mathfrak{D})})$, for all $k > 0$. From this, it follows that $\overline{(\delta, \mathfrak{D})}$ only has periodic points of pure period 1. ∎

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