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Characterizations of Three Classes of Zero-Divisor Graphs

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Abstract. The zero-divisor graph $\Gamma(R)$ of a commutative ring R is the graph whose vertices consist of the nonzero zero-divisors of R such that distinct vertices x and y are adjacent if and only if $xy = 0$. In this paper, a characterization is provided for zero-divisor graphs of Boolean rings. Also, commutative rings R such that $\Gamma(R)$ is isomorphic to the zero-divisor graph of a direct product of integral domains are classified, as well as those whose zero-divisor graphs are central vertex complete.

1 Introduction

Let R be a commutative ring with $1 \neq 0$, and define the *zero-divisors* of R to be the elements in the set $Z(R) = \{r \in R \mid rs = 0 \text{ for some } 0 \neq s \in R\}$. Given any vertex v of any simple graph Γ (that is, any undirected graph Γ with no loops or multiple edges), the *neighborhood* of v is the set $N(v)$ of all vertices that are adjacent to v. The *zero-divisor graph* of R is the simple graph $\Gamma(R)$ whose vertices are the nonzero zero-divisors of R such that $r \in N(s)$ if and only if $r \neq s$ and $rs = 0$. The notion of a zero-divisor graph was introduced in [3], where every element in R was considered to be a vertex. The present definition is due to D. F. Anderson and P. S. Livingston [2].

While many subjects in the area have been explored, one topic of interest in zero-divisor graph theory involves the investigation of properties satisfied by neighborhoods. In particular, one attempts to classify rings whose zero-divisor graphs have neighborhoods that satisfy certain criteria. For example, a *complement* of a vertex v is defined in [1] as any vertex w such that v is adjacent to w, and no vertex of the graph is adjacent to both v and w. A graph is called *complemented* if every vertex has a complement. A characterization of commutative rings whose zero-divisor graphs are complemented is given in [1, Corollary 3.10, Theorem 3.14] (see Theorem 4.3). Rings having zero-divisor graphs such that all vertices have unique complements are classified in [7, Theorem 2.5]. In [12], any nonempty simple graph is called *uniquely determined* if all distinct vertices have distinct neighborhoods; that is, $N(v) = N(w)$ if and only if $v = w$. A characterization of rings whose zero-divisor graphs are uniquely determined is provided in [12, Theorem 2.5]. In this paper, we continue the investigations of [1, 7, 12].

Let Γ be a simple graph with vertex-set $\mathcal{V}(\Gamma)$. Define the *neighborhood* of any $A \subseteq \mathcal{V}(\Gamma)$ by $N(\varnothing) = \mathcal{V}(\Gamma)$, and $N(A) = \bigcap\{N(a) \mid a \in A\}$ if $A \neq \varnothing$. If $A = \{a_1, \dots, a_n\}$, then $N(A)$ will be denoted by $N(a_1, \dots, a_n)$. Recall that a ring R is a *Boolean* ring if $r^2 = r$ for all $r \in R$. In [12, Theorem 2.5], it was shown that the zero-divisor graph of any commutative ring R is uniquely determined if and only if either

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R is a Boolean ring, or the *total quotient ring of* R (that is, the ring $T(R) = R_{R\setminus Z(R)}$) is local and $x^2 = 0$ for all $x \in Z(R)$. In [7, Theorem 2.5], it was shown that any commutative ring R is a Boolean ring if and only if either R is isomorphic to one of the rings in the set $\{\mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2\}$, or R has at least three nonzero zero-divisors and every vertex of $\Gamma(R)$ has a unique complement. The idea of a graph being uniquely determined is generalized by considering graphs with the property that $N(A) = N(x)$ for some $A \subseteq \mathcal{V}(\Gamma)$ if and only if $A = \{x\}$. In Section 2, it is shown that $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is the only Boolean ring that realizes a zero-divisor graph satisfying this stronger condition (Theorem 2.4). As a corollary, another characterization of zero-divisor graphs of Boolean rings is provided (Corollary 2.5).

It is well known that the zero-divisor graph of any direct product of integral domains is isomorphic to that of a direct product of fields, namely, the zero-divisor graph of its total quotient ring [1, Theorem 2.2]. Lest one attempt to make generalizations based on this scenario, note that the ring $R = \{r \in \prod_{\mathbb{N}} \mathbb{R} \mid |\{r(i)\}_{i\in\mathbb{N}}| < \infty\}$ is a total quotient ring, *i.e.*, $R = T(R)$, such that $\Gamma(R) \simeq \Gamma(\prod_{\mathbb{N}} \mathbb{R})$, but R is not isomorphic to any direct product of integral domains. On the other hand, the *maximal ring of quotients* $Q(R)$ (discussed in Section 3) of R is a direct product of fields; in fact, $Q(R) = \prod_{\mathbb{N}} \mathbb{R}$ [7, Example 3.5].

Let $\mathcal{F}$ denote the class of graphs that are realizable as zero-divisor graphs of direct products of integral domains. The members of $\mathcal{F}$ are completely characterized in [9]. In Section 3, it is shown that the zero-divisor graph of any commutative ring R is isomorphic to a member of $\mathcal{F}$ if and only if either $\Gamma(R)$ is a *star graph* (*i.e.*, any graph with at least two vertices such that there exists a vertex that is adjacent to every other vertex, and these are the only adjacency relations), or $Q(R)$ is isomorphic to a direct product of fields and $\Gamma(R) \simeq \Gamma(Q(R))$ (Theorem 3.4). In contrast to total quotient rings, the zero-divisor graph of any *rationally complete* commutative ring R (that is, $R = Q(R)$) is isomorphic to a member of $\mathcal{F}$ if and only if either $\Gamma(R)$ is a star graph or R is isomorphic to a direct product of fields.

A graph Γ is *central vertex complete*, or c.v.-complete, if for every $\varnothing \neq A \subseteq \mathcal{V}(\Gamma)$ such that $N(A) \neq \varnothing$, there exists a $v \in \mathcal{V}(\Gamma)$ such that $N(v) = N(A)$. This condition was studied in [7,8] as an invariant of zero-divisor graphs of rationally complete commutative rings without nonzero nilpotents (Corollary 4.2). For example, it is known that any Boolean ring R is rationally complete if and only if $\Gamma(R)$ is c.v.-complete [8, Theorem 3.4]. In Section 4, commutative rings whose zero-divisor graphs are c.v.-complete are classified (Theorem 4.5 and Remark 4.6). Moreover, it is shown that connected simple c.v.-complete graphs having at least two vertices are complemented (Theorem 4.1). As a corollary, it is shown that the zero-divisor graph of any finite commutative ring having at least two vertices is complemented if and only if it is c.v.-complete (Corollary 4.7).

2 The Zero-Divisor Graph of a Boolean Ring

Recall that [7, Theorem 2.5] classifies zero-divisor graphs of Boolean rings in terms of (graph-theoretic) complements. In this section, zero-divisor graphs of Boolean rings are characterized by strengthening a graph-theoretic condition investigated in [12]. In particular, we shall investigate zero-divisor graphs $\Gamma(R)$ such that $A = \{x\}$

whenever $A \subseteq \mathcal{V}(\Gamma(R))$, $x \in \mathcal{V}(\Gamma(R))$, and $N(A) = N(x)$.

Given any $A \subseteq R$, let $\operatorname{ann}(A) = \{r \in R \mid ra = 0 \text{ for all } a \in A\}$. If $A = \{a_1, \dots, a_n\}$, then write $\operatorname{ann}(A) = \operatorname{ann}(a_1, \dots, a_n)$. The sufficiency portion of [12, Theorem 2.5] is generalized in the following lemma. The converse of Lemma 2.1 is handled in Proposition 2.2.

Lemma 2.1 *Let R be a commutative ring and suppose that $0 \neq x \in R$ such that $x^2 = 0$. Let $A \subseteq \mathcal{V}(\Gamma(R))$. Then $N(A) = N(x)$ if and only if $A = \{x\}$.*

Proof The sufficiency portion is clear. To prove the converse, suppose that $N(A) = N(x)$ for some $A \subseteq \mathcal{V}(\Gamma(R))$. Since zero-divisor graphs are connected [2, Theorem 2.3], the equality $N(x) = \varnothing$ implies that $\mathcal{V}(\Gamma(R)) = \{x\}$. Then the result is clear if $N(x) = \varnothing$.

Assume that $N(x) \neq \varnothing$. To the contrary, suppose that $N(A) = N(x)$ for some $A \subseteq \mathcal{V}(\Gamma(R))$ with $A \neq \{x\}$. Clearly $A \neq \varnothing$ ($\Gamma(R)$ is simple). Also, $ax \neq 0$ for all $a \in A \setminus \{x\}$. Thus $\operatorname{ann}(x) \cap \operatorname{ann}(A) = N(x) \cup \{0\}$. In particular, $N(x) \cup \{0\}$ is an ideal. Let $y \in N(x)$. Then $x + y \in \operatorname{ann}(x) \setminus \{x\} = N(x) \cup \{0\}$, and therefore $x = x + y - y \in N(x) \cup \{0\}$. Since $x \neq 0$, it follows that $x \in N(x)$. This is a contradiction, and therefore $A = \{x\}$. ■

Proposition 2.2 *Let R be a commutative ring and $x \in \mathcal{V}(\Gamma(R))$. Given any $A \subseteq \mathcal{V}(\Gamma(R))$, suppose that $N(A) = N(x)$ if and only if $A = \{x\}$. Then $x^2 \in \{0, x\}$.*

Proof Suppose that $x^2 \notin \{0, x\}$. If $x^3 \neq 0$, then $x^2 \notin N(x)$, and thus $N(x) \subseteq N(x, x^2)$. The reverse inclusion is clear, contradicting the assumptions on x. Therefore, assume that $x^3 = 0$. Note that the equality $x = -x$ holds since $N(x) = N(-x)$. Then the assumption $x^2 \neq x$ implies that $x^2 + x \neq 0$. Also, $x(x^2 + x) = x^2 \neq 0$. Thus $N(x) \subseteq N(x, x^2 + x)$. The reverse inclusion is clear, contradicting the assumptions on x. This exhausts all possibilities, and hence $x^2 \in \{0, x\}$. ■

Observe that Proposition 2.2 fails if the assumption on x is weakened to the defining condition for being uniquely determined. For example, let $R = \mathbb{Z}_4 \oplus \mathbb{Z}_2$. Then $N(v) = N((2,1))$ for some $v \in \mathcal{V}(\Gamma(R))$ if and only if $v = (2,1)$, but $(2,1)^2 \notin \{(0,0), (2,1)\}$. On the other hand, if the weaker condition is imposed on *all* elements of $\mathcal{V}(\Gamma(R))$, then the following lemma is a consequence of [12, Theorem 2.5].

Lemma 2.3 *Let R be a commutative ring such that $Z(R) \neq \{0\}$. If $\Gamma(R)$ is uniquely determined, then either R is a Boolean ring or $x^2 = 0$ for all $x \in Z(R)$.*

The next theorem captures the effect of strengthening the "uniquely determined" hypothesis in the previous lemma.

Theorem 2.4 *Let R be a commutative ring such that $Z(R) \neq \{0\}$. Then the following are equivalent:*

(i) *Given any $x \in \mathcal{V}(\Gamma(R))$ and $A \subseteq \mathcal{V}(\Gamma(R))$, the equality $N(A) = N(x)$ holds if and only if $A = \{x\}$.*
(ii) *Either $R \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $x^2 = 0$ for all $x \in Z(R)$.*

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Proof Clearly (i) holds if $R \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Therefore, (ii) implies (i) by Lemma 2.1. It remains to show that (i) implies (ii).

Suppose that (i) holds. Then $\Gamma(R)$ is uniquely determined, and therefore Lemma 2.3 shows that either R is a Boolean ring or $x^2 = 0$ for all $x \in Z(R)$. Assume that R is a Boolean ring such that $R \ncong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since $Z(R) \neq \{0\}$, it follows that $|\mathcal{V}(\Gamma(R))| > 2$. Hence there exists $x, y \in \mathcal{V}(\Gamma(R))$ such that $x \notin \{y, 1 + y\}$ $(= \{1 + (1 + y), 1 + y\})$. Moreover, if $x = xy$, then $x \neq x(1 + y)$. Therefore, it can be assumed that $x \notin \{1 + t, xt\}$ for some $t \in \mathcal{V}(\Gamma(R))$. Suppose that $xt \neq 0$. Then $x(xt) = xt \neq 0$, and hence $N(x) = N(x, xt)$, a contradiction. Suppose that $xt = 0$. Then $x(1 + t) = x$, and thus $N(1 + t) = N(x, 1 + t)$. Again, this is a contradiction. Therefore, if (i) holds and R is a Boolean ring, then $R \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The result now follows by Lemma 2.3. ∎

Corollary 2.5 *The following are equivalent for a commutative ring R:*

(i) *R is a Boolean ring.*
(ii) *Either $R \cong B$ for some $B \in \{\mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2\}$, or $|\mathcal{V}(\Gamma(R))| > 2$ and every element of $\mathcal{V}(\Gamma(R))$ has a unique complement.*
(iii) *Either $R \cong B$ for some $B \in \{\mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2\}$, or $\Gamma(R)$ is uniquely determined and $N(A) = N(x)$ for some $x \in \mathcal{V}(\Gamma(R))$ and $A \subseteq \mathcal{V}(\Gamma(R))$ with $A \neq \{x\}$.*

Proof The equivalence of (i) and (ii) is established in [7, Theorem 2.5]. It remains to verify the equivalence of (i) and (iii).

Suppose that R is a Boolean ring and R is not isomorphic to any ring in the set $\{\mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2\}$. Clearly $\Gamma(R)$ is uniquely determined, *e.g.*, by (ii), every vertex has a unique complement. Since $R \ncong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and R has no nonzero nilpotents, Theorem 2.4 implies that $N(A) = N(x)$ for some $x \in \mathcal{V}(\Gamma(R))$ and $A \subseteq \mathcal{V}(\Gamma(R))$ with $A \neq \{x\}$.

Conversely, suppose that (iii) holds. If $R \cong B$ for some $B \in \{\mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2\}$, then R is a Boolean ring. Suppose that $\Gamma(R)$ is uniquely determined and $N(A) = N(x)$ for some $x \in \mathcal{V}(\Gamma(R))$ and $A \subseteq \mathcal{V}(\Gamma(R))$ with $A \neq \{x\}$. Then $x^2 \neq 0$ by Lemma 2.1, and therefore R is a Boolean ring by Lemma 2.3. ∎

3 The Complete Ring of Quotients

In [9, Theorem 2.2], graphs that are realizable as zero-divisor graphs of direct products of integral domains are characterized. In this section, we describe commutative rings R such that $\Gamma(R)$ is isomorphic to the zero-divisor graph of a direct product of integral domains. Recall that the zero-divisor graph of any commutative ring R is isomorphic to that of its total quotient ring $T(R)$ [1, Theorem 2.2]. It may happen that $\Gamma(R)$ is isomorphic to the zero-divisor graph of a direct product of fields even if $T(R)$ is not isomorphic to any direct product of fields [7, Example 3.5]. However, the ring-theoretic structure is less ambiguous for a particular generalization of $T(R)$, which we now describe.

A subset D of a ring R is called *dense* if $r \in R$ with $rD = \{0\}$ implies $r = 0$. Let D_1 and D_2 be dense ideals of R and let $f_i \in \mathrm{Hom}_R(D_i, R)$ $(i = 1, 2)$. Note that $f_1 + f_2$ is an R-module homomorphism on the dense ideal $D_1 \cap D_2$, and $f_1 \circ f_2$ is an R-module homomorphism on the dense ideal $f_2^{-1}(D_1) = \{r \in R \mid f_2(r) \in D_1\}$. Then the *complete ring of quotients* $Q(R) = F/\sim$ of R is a commutative ring, where

$$F = \{f \in \mathrm{Hom}_R(D, R) \mid D \subseteq R \text{ is a dense ideal}\}$$

and $\sim$ is the congruence relation defined by $f_1 \sim f_2$ if and only if there exists a dense ideal $D \subseteq R$ such that $f_1(d) = f_2(d)$ for all $d \in D$ [11, Proposition 2.3.1]. Given any ring T, it is straightforward to check that any ring-isomorphism from R onto T will induce a congruence-preserving bijection from F onto the set

$$\{f \in \mathrm{Hom}_T(D, T) \mid D \subseteq T \text{ is a dense ideal}\}.$$

It follows that $Q(R) \cong Q(T)$ whenever $R \cong T$.

The mapping $h\colon R \to Q(R)$ that assigns any $t \in R$ to the congruence class containing the element $(r \mapsto tr) \in \mathrm{Hom}_R(R, R)$ is easily seen to be a ring monomorphism [11, Proposition 2.3.1]. Any ring S containing R is called a *ring of quotients of* R if there exists a monomorphism $H\colon S \to Q(R)$ such that $H|_R = h$. Equivalently, the ideal $s^{-1}R = \{r \in R \mid sr \in R\}$ of R is dense in S for all $0 \neq s \in S$ [11, Proposition 2.3.6]. For example, $T(R)$ is a ring of quotients of R since $dR \subseteq (r/d)^{-1}R$ for every $r \in R$ and $d \in R \setminus Z(R)$. Clearly maximal (with respect to inclusion) rings of quotients exist and are isomorphic to $Q(R)$. Therefore, any maximal ring of quotients of R will be denoted by $Q(R)$.

A ring R is called *rationally complete* if $R = Q(R)$. For example, every finite commutative ring is rationally complete, *e.g.*, by [6, Theorem 80] finite rings do not properly contain any dense ideals. If R is any commutative ring, then $Q(R)$ is (up to isomorphism) the unique rationally complete ring of quotients of R [11, Proposition 2.3.7]. Moreover, if $R \subseteq S \subseteq Q(R)$, then $Q(R)$ is a ring of quotients of S [5, 1.4]. It follows that $Q(R) \cong Q(S)$ whenever $R \subseteq S$ is a ring of quotients. For in-depth discussions on rings of quotients, see [5, 10, 11].

Recall that a commutative ring T is a *von Neumann regular* ring if for all $r \in T$, there exists an $s \in T$ such that $r = r^2 s$, *e.g.*, Boolean rings and direct products of fields. Also, as in [1], a graph is called *uniquely complemented* if it is complemented and $N(u) = N(v)$ whenever there exists a vertex w such that w is a complement of both u and v. The next lemma follows from [1, Theorem 3.5].

Lemma 3.1 *Let R be a commutative ring. Then $\Gamma(R)$ is uniquely complemented if and only if either $T(R)$ is a von Neumann regular ring or $\Gamma(R)$ is a star graph.*

Given any von Neumann regular ring T, let $B(T)$ denote the Boolean algebra of idempotents of T, and let $[r]_T = \{s \in T \mid \mathrm{ann}(s) = \mathrm{ann}(r)\}$. By [1, Theorem 4.1], any two von Neumann regular rings S and T have isomorphic zero-divisor graphs if and only if there exists a Boolean algebra isomorphism $\gamma\colon B(S) \to B(T)$ such that $|[e]_S| = |[\gamma(e)]_T|$ for all $1 \neq e \in B(S)$. The following lemma shows that if the zero-divisor graph of a commutative von Neumann regular ring T is isomorphic to that of a direct product of fields, then $B(T)$ is *atomic* (for more on Boolean algebras and the Boolean algebra of idempotents, see [11, 14]).

Lemma 3.2 *Let T be a commutative von Neumann regular ring such that $\Gamma(T)$ is isomorphic to the zero-divisor graph of a direct product of fields. If $b \in B(T) \setminus \{0\}$, then there exists an $a \in B(T) \setminus \{0\}$ such that $ab = a$ and $ae \in \{0, a\}$ for all $e \in B(T)$.*

Proof Suppose that $F = \prod_{i \in I} F_i$ is a direct product of fields such that $\Gamma(F) \simeq \Gamma(T)$. By [1, Theorem 4.1], there exists an isomorphism of Boolean algebras $\gamma\colon B(F) \to B(T)$ such that $|[e]_F| = |[\gamma(e)]_T|$ for all $1 \neq e \in B(F)$. Let $b \in B(T) \setminus \{0\}$. Since $\gamma^{-1}(b)$ is a nonzero element of F, there exists a $j \in I$ such that $\gamma^{-1}(b)(j) \neq 0$. Let $t \in B(F)$ be the element such that $t(j) = 1$, and $t(i) = 0$ for all $i \in I \setminus \{j\}$. Set $a = \gamma(t)$. Then $\gamma^{-1}(ab) = \gamma^{-1}(a)\gamma^{-1}(b) = t\gamma^{-1}(b)$ is nonzero, and therefore $ab \neq 0$. It is clear that $t\gamma^{-1}(e) \in \{0, t\}$ for all $e \in B(T)$. Thus

$$ae = \gamma(t)\gamma(\gamma^{-1}(e)) = \gamma(t\gamma^{-1}(e)) \in \{\gamma(0), \gamma(t)\} = \{0, a\}$$

for all $e \in B(T)$. Since $ab \neq 0$, it follows that $ab = a$. ∎

Suppose that T is a commutative von Neumann regular ring. If $0 \neq a \in B(T)$ such that $ae \in \{0, a\}$ for all $e \in B(T)$, then aT is a field. Indeed, suppose that $r \in T$ such that $ar \neq 0$. Choose an $s \in T$ such that $r = r^2 s$. Then $0 \neq rs \in B(T)$ with $a(rs) \neq 0$. Thus $(ar)(as) = a(rs) = a$, showing that as is the multiplicative inverse (in aT) of ar.

Lemma 3.3 *Let T be a commutative von Neumann regular ring such that $\Gamma(T)$ is isomorphic the zero-divisor graph of a direct product of fields. Then*

$$\mathcal{A} = \{a \in B(T) \setminus \{0\} \mid ae \in \{0, a\} \text{ for all } e \in B(T)\}$$

is a dense subset of T and $\Gamma(T) \simeq \Gamma(\prod_{a \in \mathcal{A}} aT)$.

Proof Let $0 \neq r \in T$. There exists an $s \in T$ such that $r = r^2 s$. Clearly $rs \in B(T) \setminus \{0\}$. If $rs \in \mathcal{A}$, then the observation $r(rs) = r \neq 0$ shows that $r\mathcal{A} \neq \{0\}$. Suppose that $rs \notin \mathcal{A}$. By Lemma 3.2 there exists an $a \in \mathcal{A}$ such that $a(rs) = a \neq \{0\}$. In particular, $ra \neq 0$. This shows that $r\mathcal{A} \neq \{0\}$ for all $0 \neq r \in T$. Thus $\mathcal{A}$ is dense in T.

Let F and γ be as in Lemma 3.2. If $t_j \in F$ is the element such that $t_j(j) = 1$ and $t_j(i) = 0$ for all $i \neq j$, then the mapping $\alpha\colon I \to \mathcal{A}$ defined by $\alpha(j) = \gamma(t_j)$ is a bijection (since γ is an isomorphism of Boolean algebras). Given any $a \in \mathcal{A}$, it is straightforward to check that $[a]_T = aT \setminus \{0\}$. Hence

$$|F_i| = |[t_i]_F| + 1 = |[\gamma(t_i)]_T| + 1 = |\alpha(i)T|$$

for all $i \in I$. Thus $\Gamma(F) \simeq \Gamma(\prod_{a \in \mathcal{A}} aT)$ [1, Theorem 2.1]. Therefore, $\Gamma(T) \simeq \Gamma(\prod_{a \in \mathcal{A}} aT)$. ∎

It is known that any zero-divisor graph $\Gamma(R)$ is a finite star graph (that is, a star graph with finitely many vertices) if and only if either $R \cong A$, where $A \in \{\mathbb{Z}_9, \mathbb{Z}_3[X]/(X^2), \mathbb{Z}_8, \mathbb{Z}_2[X]/(X^3), \mathbb{Z}_4[X]/(2X, X^2 - 2)\}$, or $R \cong \mathbb{Z}_2 \times F$, where F is a finite field [4, Corollary 1.11]. Moreover, $\Gamma(R)$ is an infinite star graph if and only if either $R \cong \mathbb{Z}_2 \times F$ for some infinite integral domain F, or there exists a $0 \neq x \in R$

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such that $Z(R) = \operatorname{ann}(x)$, $\operatorname{nil}(R) = \{0, x\}$, and $R/\operatorname{nil}(R)$ is an infinite integral domain [4, Theorem 1.12].

The following theorem determines when $\Gamma(R)$ is isomorphic to the zero-divisor graph of a direct product of integral domains.

Theorem 3.4 *Let R be a commutative ring. Then $\Gamma(R)$ is isomorphic to the zero-divisor graph of a direct product of integral domains if and only if either*

(i) $\Gamma(R)$ *is a star graph, or*

(ii) *the ring $Q(R)$ is isomorphic to a direct product of fields and $\Gamma(R) \simeq \Gamma(Q(R))$.*

If (ii) *holds, then $Q(R) \cong \prod_{a\in\mathcal{A}} aT(R)$, where*

$$\mathcal{A} = \{a \in B(T(R)) \setminus \{0\} \mid ae \in \{0, a\} \text{ for all } e \in B(T(R))\}.$$

Proof Suppose that $\Gamma(R)$ is a star graph. If $\mathcal{V}(\Gamma(R))$ is infinite, then $\Gamma(R)$ is isomorphic to $\Gamma(\mathbb{Z}_2 \oplus F)$, where F is any integral domain with the appropriate cardinality. By checking the list given prior to the statement of this theorem, it follows that $\Gamma(R)$ is isomorphic to $\Gamma(\mathbb{Z}_2 \oplus F)$ for some integral domain F. The sufficiency portion of the theorem is now clear.

Suppose that $\Gamma(R)$ is not a star graph, but is isomorphic to the zero-divisor graph of a direct product of integral domains F. By [1, Theorem 4.2], $\Gamma(R) \simeq \Gamma(T(F))$, the zero-divisor graph of a direct product of fields. Any direct product of fields is a von Neumann regular ring. Then $\Gamma(R)$ is isomorphic to the zero-divisor graph of a von Neumann regular ring, and is therefore uniquely complemented by Lemma 3.1. Since $\Gamma(R)$ is not a star graph, Lemma 3.1 shows that $T(R)$ is a von Neumann regular ring.

Define $\varphi\colon T(R) \to \prod_{a\in\mathcal{A}} aT(R)$ by $\varphi(r)(a) = ar$ for all $a \in \mathcal{A}$. Then φ is a homomorphism of rings (a is idempotent). Also, φ is injective since $\mathcal{A}$ is dense by Lemma 3.3. Thus $T(R) \cong \varphi(T(R))$. Let $f \in \prod_{a\in\mathcal{A}} aT(R)$. Any product of distinct elements in $\mathcal{A}$ is 0, and thus $f\varphi(a) = \varphi(f(a)) \in \varphi(T(R))$ for all $a \in \mathcal{A}$. Therefore, $\varphi(\mathcal{A}) \subseteq f^{-1}\varphi(T(R))$ for all $f \in \prod_{a\in\mathcal{A}} aT(R)$. Also, $f\varphi(a) \neq 0$ whenever $f(a) \neq 0$, showing that $\varphi(\mathcal{A})$ is dense in $\prod_{a\in\mathcal{A}} aT(R)$. This verifies that $\prod_{a\in\mathcal{A}} aT(R)$ is a ring of quotients of $\varphi(T(R))$.

Note that $Q(K) = K$ for any field K since every dense set in K contains a unit (if $f \in Q(K)$ and $0 \neq u \in f^{-1}K$, then $f = (fu)u^{-1} \in K$). In particular, every direct product of fields is rationally complete [11, Proposition 2.3.8]. The comments prior to Lemma 3.3 imply that $\prod_{a\in\mathcal{A}} aT(R)$ is a direct product of fields. Therefore, the observations at the beginning of this section show that

$$Q(R) \cong Q(T(R)) \cong Q(\varphi(T(R))) \cong Q\big(\prod_{a\in\mathcal{A}} aT(R)\big) = \prod_{a\in\mathcal{A}} aT(R).$$

By [1, Theorem 4.2] and Lemma 3.3, it follows that $\Gamma(R) \simeq \Gamma(T(R)) \simeq \Gamma(Q(R))$. ■

Corollary 3.5 *Suppose that R is a rationally complete commutative ring such that $\Gamma(R)$ is not a star graph. Then R is isomorphic to a direct product of fields if and only if its zero-divisor graph is isomorphic to the zero-divisor graph of a direct product of fields.*

Proof The necessity portion is trivial. The converse holds by Theorem 3.4 since $R = Q(R)$. ■

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4 Complemented Zero-Divisor Graphs and Central Vertex Completeness

Recall that a graph Γ is c.v.-complete if for every $\varnothing \neq A \subseteq \mathcal{V}(\Gamma)$ such that $N(A) \neq \varnothing$, there exists a $v \in \mathcal{V}(\Gamma)$ such that $N(v) = N(A)$. Commutative rings with complemented zero-divisor graphs are described in [1] (see Theorem 4.3 below). This characterization, together with the following graph-theoretic lemma, simplifies the task of classifying rings with c.v.-complete zero-divisor graphs.

Theorem 4.1 *Let Γ be a connected simple graph such that $|\mathcal{V}(\Gamma)| > 1$. If Γ is c.v.-complete, then Γ is complemented.*

Proof Suppose that $v \in \mathcal{V}(\Gamma)$ does not have a complement. Let $A = N(v)$. Then $A \neq \varnothing$ since Γ is connected with $|\mathcal{V}(\Gamma)| > 1$, and $N(A) \neq \varnothing$ since clearly $v \in N(A)$. Then there exists a $w \in \mathcal{V}(\Gamma)$ such that $N(w) = N(A)$. Since $v \in N(A) = N(w)$ and v does not have a complement, there exists a $u \in N(v, w)$. Hence $u \in N(w) = N(A)$. But $u \in N(v)$ implies that $u \in A$, contradicting that Γ is simple. Thus Γ is complemented. ■

The following corollary is stated in [8, Theorem 3.3], where the hypothesis of the "if" statement includes the condition $\Gamma(R)$ *is complemented*. By Theorem 4.1, this assumption is superfluous.

Corollary 4.2 *Let R be a commutative ring such that $\operatorname{nil}(R) = \{0\}$, $|R| < \aleph_\omega$, and $2 \notin Z(R)$. Then $\Gamma(R) \simeq \Gamma(Q(R))$ if and only if $\Gamma(R)$ is c.v.-complete.*

Note that the converse of Theorem 4.1 can fail. For example, if R is any Boolean ring that is not rationally complete, then $\Gamma(R)$ is complemented, but is not c.v.-complete (see Theorem 4.3 and Lemma 4.4). However, it will be shown that the converse is true for finite rings having at least two nonzero zero-divisors. First, we state the characterization from [1] of rings with complemented zero-divisor graphs.

Theorem 4.3 ([1, Corollary 3.10, Theorem 3.14]) *Let R be a commutative ring. Then $\Gamma(R)$ is complemented if and only if at least one of the following conditions is satisfied:*

(i) *$T(R)$ is a von Neumann regular ring,*
(ii) *$\Gamma(R)$ is a star graph,*
(iii) *$R \cong D \oplus B$, where D is an integral domain and B is either $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$.*

It is clear that star graphs are c.v.-complete. Suppose that $R \cong D \oplus B$, where D is an integral domain and B is either $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. Then $|Z(B)| = 2$; say $Z(B) = \{0, x\}$. Let $\varnothing \neq A \subseteq \mathcal{V}(\Gamma(D \oplus B))$ with $N(A) \neq \varnothing$. Define $s = (s_1, s_2)$ to be the element of $D \oplus B$ defined by

$$s_1 = \begin{cases} 0 & \text{if } a_1 = 0 \text{ for all } (a_1, a_2) \in A, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$s_2 = \begin{cases} 0 & \text{if } a_2 = 0 \text{ for all } (a_1, a_2) \in A, \\ 1 & \text{if } a_2 \notin Z(B) \text{ for some } (a_1, a_2) \in A, \\ x & \text{otherwise.} \end{cases}$$

Then it is straightforward to check that $N(s) = N(A)$, and it follows that R is c.v.-complete. On the other hand, it has already been observed that there exist von Neumann regular rings whose zero-divisor graphs are not c.v.-complete.

Given any $r \in \mathcal{V}(\Gamma(R))$, let $[r] = \{s \in \mathcal{V}(\Gamma(R)) \mid N(s) = N(r)\}$. Define $\Gamma^*(R)$ to be the graph with $\mathcal{V}(\Gamma^*(R)) = \{[r] \mid r \in \mathcal{V}(\Gamma(R))\}$, such that $[r]$ is adjacent to $[s]$ in $\Gamma^*(R)$ if and only if r and s are adjacent in $\Gamma(R)$. Also, recall that the Boolean algebra of idempotents $B(R)$ of any ring R becomes a Boolean ring with multiplication defined the same as in R and addition defined by the mapping $(r, s) \mapsto r + s - 2rs$. Moreover, any two Boolean algebras of idempotents are isomorphic if and only if they are isomorphic as rings [11, Proposition 1.1.3].

The following lemma summarizes some past results to provide conditions equivalent to c.v.-completeness for zero-divisor graphs of von Neumann regular rings.

Lemma 4.4 *The following conditions are equivalent for a commutative von Neumann regular ring R:*

(i) $\Gamma(R)$ *is c.v.-complete.*
(ii) $B(R)$ *is a complete Boolean algebra.*
(iii) $B(R) = B(Q(R))$.
(iv) $\Gamma^*(R) \simeq \Gamma^*(Q(R))$.

Proof The equivalence of (i) and (ii) is established in [8, Lemma 3.1]. The equivalence of (ii) and (iii) can be found in [5, Theorem 11.9]. It remains to justify the equivalence of (iii) and (iv).

Note that $Q(R)$ is a von Neumann regular ring [11, Proposition 2.4.1] and $\Gamma^*(R) \simeq \Gamma(B(R))$ for any commutative von Neumann regular ring R [1, Proposition 4.5]. Thus (iii) implies (iv). Since any two Boolean rings R and S are isomorphic if and only if $\Gamma(R) \simeq \Gamma(S)$ [7, Theorem 4.1], it follows that $B(R)$ and $B(Q(R))$ are isomorphic whenever (iv) holds. Thus (iv) implies $B(R) = B(Q(R))$ by [5, Theorem 11.9]. ■

The next theorem determines when any zero-divisor graph is c.v.-complete, and Remark 4.6 translates the result into purely ring-theoretic terms.

Theorem 4.5 *Let R be a commutative ring with nonzero zero-divisors. Then $\Gamma(R)$ is c.v.-complete if and only if at least one of the following conditions is satisfied:*

(i) $\mathrm{nil}(R) = \{0\}$ *and* $\Gamma^*(R) \simeq \Gamma^*(Q(R))$,
(ii) $\Gamma(R)$ *is a star graph,*
(iii) $R \cong D \oplus B$, *where either $D = \{0\}$ or D is an integral domain, and B is either $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$.*

Proof Suppose that $\Gamma(R)$ is c.v.-complete and that (ii) and (iii) fail. Since (iii) fails, $|\mathcal{V}(\Gamma(R))| > 1$ [13, Section 5]. Then $T(R)$ is a von Neumann regular ring by Theorem 4.1 and Theorem 4.3. Thus $\mathrm{nil}(R) = \{0\}$. Also, $\Gamma(T(R))$ is c.v.-complete since $\Gamma(R) \simeq \Gamma(T(R))$ [1, Theorem 2.2]. Therefore, $\Gamma^*(T(R)) \simeq \Gamma^*(Q(T(R)))$ by Lemma 4.4. But $\Gamma(R) \simeq \Gamma(T(R))$ and $Q(T(R)) = Q(R)$. Hence $\Gamma^*(R) \simeq \Gamma^*(Q(R))$.

Conversely, $\Gamma(R)$ is c.v.-complete whenever (ii) or (iii) is satisfied (see the discussion prior to Lemma 4.4, and note that $\Gamma(\mathbb{Z}_4)$ and $\Gamma(\mathbb{Z}_2[X]/(X^2))$ are trivially c.v.-complete). Suppose that (i) holds. Then $Q(R)$ is a von Neumann regular ring since

$\mathrm{nil}(R) = \{0\}$ [11, Proposition 2.4.1]. Hence $\Gamma(Q(R))$ is c.v.-complete by Lemma 4.4. Since $\Gamma^*(R) \simeq \Gamma^*(Q(R))$, it clearly follows that $\Gamma(R)$ is c.v.-complete. ∎

Remark 4.6 Let R be a commutative ring with nonzero zero-divisors, and suppose that the statement in Theorem 4.5(i) holds. Then $\Gamma(R)$ is c.v.-complete with $|\mathcal{V}(\Gamma(R))| > 1$, and is therefore complemented by Theorem 4.1. Since $\mathrm{nil}(R) = \{0\}$, it is clear that the statement in Theorem 4.5(iii) fails, and if the statement in Theorem 4.5(ii) holds, then the list prior to Theorem 3.4 shows that $T(R)$ must be a direct product of fields. By Theorem 4.3, it follows that the statement in Theorem 4.5(i) implies $T(R)$ is a von Neumann regular ring. Thus $B(T(R)) = B(Q(R))$ by Lemma 4.4 (since $\Gamma(T(R)) \simeq \Gamma(R)$ is c.v.-complete and $Q(T(R)) = Q(R)$). Conversely, another application of Lemma 4.4 will prove that $\Gamma^*(R) \simeq \Gamma^*(T(R)) \simeq \Gamma^*(Q(R))$ whenever $T(R)$ is a von Neumann regular ring and $B(T(R)) = B(Q(R))$. Therefore, the statement in Theorem 4.5(i) holds if and only if $T(R)$ is a von Neumann regular ring and $B(T(R)) = B(Q(R))$. Since rings whose zero-divisor graphs are star graphs have been classified, Theorem 4.5 provides a purely ring-theoretic characterization of any R such that $\Gamma(R)$ is c.v.-complete.

Corollary 4.7 *If R is a finite ring and $|\mathcal{V}(\Gamma(R))| > 1$, then $\Gamma(R)$ is complemented if and only if it is c.v.-complete.*

Proof If $\Gamma(R)$ is c.v.-complete, then it is complemented by Theorem 4.1. The converse holds by Theorem 4.3 and Theorem 4.5 since $\mathrm{nil}(R) = \{0\}$ and $R = Q(R)$ for every finite von Neumann regular ring R. ∎

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