A LEMMA ON MAXIMAL SETS AND THE THEOREM OF DENJOY-VITALI

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1. Introduction and summary

By an R-related family $\mathscr{F}$ we mean a non-empty family $\mathscr{F}$ of elements such that to each element $F \in \mathscr{F}$ is associated a set $R(F)$ of elements of $\mathscr{F}$, called the R-class of F, which contains F. An element $G \in R(F)$ is said to be R-related to F. By an R-section S of $\mathscr{F}$ we mean a set of elements of $\mathscr{F}$ such that for any elements F_1, F_2 of S either $F_1 \in R(F_2)$ or $F_2 \in R(F_1)$. If $R(F) = \{F\}$ for each $F \in \mathscr{F}$ then the only R-sections are the sets $\{F\}$. The interesting applications of the lemma proved below are to those cases when there exist R-sections which do not contain a finite number of elements.

Two R-sections of $\mathscr{F}$, S_1, S_2 are said to be R-related, and we write $(S_1)R(S_2)$, if either $F_1 \in R(F_2)$ or $F_2 \in R(F_1)$ for each $F_1 \in S_1$ and each $F_2 \in S_2$. Evidently $(S)R(S)$ for each R-section S and $(S_1)R(S_2)$ implies $(S_2)R(S_1)$. That is the relation R is reflexive and symmetric between R-sections.

An R-section S of $\mathscr{F}$ is said to be maximal if there is no other R-section of $\mathscr{F}$ which contains S as a proper subset. The fundamental result of this paper, lemma (2.1) asserts that an R-related family always has a maximal R-section. The lemma is equivalent to the axiom of choice since we prove it by means of Zorn's lemma and then show that it implies the axiom of choice.

We then use the lemma to establish an extension of Vitali's covering theorem to abstract spaces. The form of this extension is similar to that of Denjoy [1] and Trjitzinsky [3].

Another application of the lemma occurs in Finch [2].

In section 3 the elements of $\mathscr{F}$ will be subsets of an abstract space $\mathscr{X}$ of points x and the R-class of any element $F \in \mathscr{F}$ will consist of F and all the elements of $\mathscr{F}$ which are disjoint to F regarded as subsets of $\mathscr{X}$.

2. The lemma

We assert

LEMMA (2.1). *An R-related family possesses a maximal R-section.*

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PROOF. Denote by $\mathscr{R}$ the set of all R-sections of $\mathscr{S}$. This set may be partially ordered by inclusion. A chain in $\mathscr{R}$ is a subset of $\mathscr{R}$ in which the partial ordering by inclusion is a total ordering, that is, a set $\mathscr{C} \subset \mathscr{R}$ such that if S_1, S_2 are two R-sections belonging to $\mathscr{C}$ then either $S_1 \subset S_2$ or $S_2 \subset S_1$. If S_1, S_2 are two elements of a chain $\mathscr{C}$ in $\mathscr{R}$ such that $S_1 \subset S_2$ then S_1, S_2 are R-related R-sections of $\mathscr{S}$. It follows that the union of all the elements of $\mathscr{S}$ contained in the chain is an R-section. Thus this union is an upper bound of the chain in $\mathscr{R}$. It follows from Zorn's lemma that there exists a maximal chain in $\mathscr{R}$. The union of all the elements of $\mathscr{S}$ contained in the chain is easily seen to be a maximal R-section of $\mathscr{S}$. This completes the proof of the lemma.

We show now that lemma (2.1) implies the axiom of choice, namely, "If X_a is a non-empty set for each element a of an index set A, then there is a function F^* on A such that $F^*(a) \in X_a$ for each a in A".

For let $\mathscr{S}$ be the family of all functions F such that the domain of F is a subset of A and $F(a) \in X_a$ for each a in the domain of F. To each element $F \in \mathscr{S}$ we associate an R-class $R(F)$ as follows. For each $F \in \mathscr{S}$, $R(F)$ contains F and all the elements of $\mathscr{S}$ whose domains are disjoint to that of F. By lemma (2.1) this family possesses a maximal R-section. A maximal R-section of $\mathscr{S}$ is itself an element of $\mathscr{S}$, that is, a function $F^* \in \mathscr{S}$. The domain of F^* is the set A for if there is an $a \in A$ which is not in the domain of F^* then since X_a is non-empty there is an $x \in X_a$ and $F = \{a, x\} \in \mathscr{S}$. Hence F^* is contained in the R-section $F^* \cup F$ contradicting the maximality of F^*. This completes the derivation of the axiom of choice from lemma (2.1).

3. Application to the theorem of Denjoy-Vitali

Let $\mathscr{X}$ be an abstract space of points x. Let $\mathscr{M}$ be a σ-field of subsets of $\mathscr{X}$, that is, a family of subsets of $\mathscr{X}$ such that (i) $\mathscr{X} \in \mathscr{M}$, (ii) if $M \in \mathscr{M}$ then $\mathscr{X} - M \in \mathscr{M}$, and (iii) if $\{M_j\}$ is a sequence of elements of $\mathscr{M}$ then $U_j M_j \in \mathscr{M}$. We suppose that a σ-finite measure $\mu(\cdot)$ is defined on $\mathscr{M}$, that is, $\mu(M)$ is a real valued function defined for each $M \in \mathscr{M}$ such that (i) $\mu(M) \geqq 0$ (ii) $\mu(\Sigma_j M_j) = \Sigma_j \mu(M_j)$ for each sequence $\{M_j\}$ of disjoint elements of $\mathscr{M}$ and (iii) there exists a sequence of disjoint elements of $\mathscr{M}$ such that $\Sigma_j M_j = \mathscr{X}$ and $\mu(M_j) < \infty$ for each j.

As is well-known one can define a Carathéodory outermeasure $\mu^*(\cdot)$ on the family $C(\mathscr{X})$ of all subsets of $\mathscr{X}$ by writing

$$\mu^*(X) = \inf \{\mu(M) : X \subset M, M \in \mathscr{M}\}, \qquad X \in C(\mathscr{X}).$$

The outer measure μ^* is such that (i) $\mu^*(X) = \mu(X)$ if $X \in \mathscr{M}$, (ii) $\mu^*(X_1) \leqq \mu^*(X_2)$ if $X_1 \subset X_2$ and (iii) $\mu^*(U_j X_j) \leqq \Sigma_j \mu^*(X_j)$ for any

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sequence $\{X_j\}$ of elements of $C(\mathscr{X})$. An element $M^* \in C(\mathscr{X})$ is said to be μ^*-measurable if

$$\mu^*(X) = \mu^*(XM^*)+\mu^*(X(\mathscr{X}-M^*))$$

for all $X \in C(\mathscr{X})$.

Denote by $\mathscr{M}^*$ the family of all μ^*-measurable elements of $C(\mathscr{X})$. Then $\mathscr{M}^* \supset \mathscr{M}$, is a σ-field of subsets of $\mathscr{X}$ and μ^* is a σ-finite measure on $\mathscr{M}^*$. In order to avoid ambiguity we shall identify $\mathscr{M}^*$ with $\mathscr{M}$. Elements of $\mathscr{M}$ will be said to be μ-measurable and we reserve the use of μ^* for the outer measure of sets which are not necessarily μ-measurable. We shall assume also that μ is a complete measure, that is if $M \in \mathscr{M}$ and $\mu(M) = 0$ then each $M' \subset M$ is μ-measurable and $\mu(M') = 0$.

Throughout the remainder of this paper we shall assume that the measure is such that $\mu^*(\{x\}) = 0$ for any point $x \in \mathscr{X}$.

If $\mathscr{F}$ is a family of subsets of $\mathscr{X}$ and x is a point of $\mathscr{X}$ we say that x is covered finely by $\mathscr{F}$ if there is a sequence $\{F_n\}$, $n \geqq 1$ of elements of $\mathscr{F}$ such that for each n,

$$\mu^*(F_n) > 0, \quad x \in F_n \quad \text{and} \quad \lim_{n\to\infty} \mu^*(F_n) = 0.$$

We denote by $\mathscr{F}^*$ the set of points of $\mathscr{X}$ which are covered finely by $\mathscr{F}$.

We prove the following theorem.

THEOREM (3.1). *Let $\mathscr{A}$, $\mathscr{B}$ be two families of subsets of $\mathscr{X}$ such that*
(i) *$\mu^*(A) > 0$ for each $A \in \mathscr{A}$;*
(ii) *each element of $\mathscr{B}$ is μ-measurable, $\mu(B) > 0$ for each $B \in \mathscr{B}$ and there is a one-one mapping $b(\cdot)$ of $\mathscr{A}$ onto $\mathscr{B}$ such that $B = b(A) \subset A$ for each $A \in \mathscr{A}$;*
(iii) *if $x \in \mathscr{B}^*$ then any sequence $\{B_n\}$, $n \geqq 1$ of elements of $\mathscr{B}$ such that $x \in B_n$, $n \geqq 1$, and $\lim_{n\to\infty} \mu(B_n) = 0$ has a subsequence $\{B_{n'}\}$ such that if $A_{n'} = b^{-1}(B_{n'})$ then $\lim_{n\to\infty} \mu^*(A_{n'}) = 0$;*
(iv) *$\mu^*\{\cup B : B \in \mathscr{B}\} < \infty$;*
(v) *for each $B \in \mathscr{B}$ the union of all points of $\mathscr{A}^* - B \cdot \mathscr{A}^*$ which are covered finely by the elements of $\mathscr{A}$ which intersect B is a set of μ-measure zero.*

Then, to each $\alpha > 0$ there exists a sequence $\{B_j^\alpha\}$, $j \geqq 1$ of mutually disjoint elements of $\mathscr{B}$ such that if $B^{(\alpha)} = \sum_{j=1}^\infty B_j^\alpha$ then

$$\mu(\mathscr{A}^* - B^{(\alpha)} \cdot \mathscr{A}^*) = 0. \tag{3.1}$$

Further the sets $\mathscr{A}^*$ and $\mathscr{B}^*$ are μ-measurable, and

$$\mu(\mathscr{A}^*) = \mu(\mathscr{B}^*) \tag{3.2}$$

$$\mu(\mathscr{B}^*) \leqq \mu(B^{(\alpha)}) < \mu(\mathscr{B}^*)+\alpha. \tag{3.3}$$

Remarks. The conditions of theorem (3.1) are similar to those imposed by Denjoy [1]. In fact the only difference between theorem (3.1) and the 'théorème général' of Denjoy is that condition (iii) above replaces Denjoy's condition 2°, namely

2° There exist two positive numbers α, β, $(1 < \alpha < \beta)$ such that for any $B \in \mathscr{B}$ the union $\mathscr{A}(B)$ of all elements A' of $\mathscr{A}$ which intersect B and satisfy $\mu(B') < \alpha\mu(B)$ where $B' = b(A')$, has an outer μ-measure less than $\beta\mu(B)$.

This condition implies (iii) above for if $x \in \mathscr{B}^*$ and $\{B_n\}$, $n \geqq 1$, is a sequence of elements of $\mathscr{B}$ containing x and such that $\lim_{n\to\infty} \mu(B_n) = 0$ there is no loss of generality in supposing that $\mu(B_{n+1}) < \alpha\mu(B_n)$, $n \geqq 1$. If $A_n = b^{-1}(B_n)$ then $\mu^*(A_m) < \beta\mu(B_n)$ for all $m > n$ and so $\lim_{n\to\infty} \mu^*(A_n) = 0$.

Proof of theorem (3.1). Let $\{\alpha_n\}$, $n \geqq 1$ be a monotonic decreasing sequence of positive real number such that $\lim \alpha_n = 0$ and $\mu(B) < \alpha_1$ for each $B \in \mathscr{B}$. Note that $\alpha_1 < \infty$ because of condition (iv) of the theorem. Denote by $\mathscr{B}^n$ the family of all elements of $\mathscr{B}$ such that $\mu(B) < \alpha_n$, $n \geqq 1$. For each $B \in \mathscr{B}^n$ define the R-class of B to contain B and all the elements of $\mathscr{B}^n$ which are disjoint to B. For each $n \geqq 1$ the family $\mathscr{B}^n$ has a maximal R-section by lemma (2.1). Since $\{\cup B : B \in \mathscr{B}^n\}$ has finite outer μ-measure and since each element of $\mathscr{B}^n$ has positive μ-measure a maximal R-section of $\mathscr{B}^n$ consists of at most a countable number of mutually disjoint elements B^n_j, $j \geqq 1$ of $\mathscr{B}^n$. Then $B^{(n)} = \sum_{j=1}^{\infty} B^n_j$ is μ-measurable and $\mu(B^{(n)}) < \infty$.

Let E^n_j be the set of points of $\mathscr{A}^* - B^n_j \cdot \mathscr{A}^*$ which are covered finely by elements of $\mathscr{A}$ which intersect B^n_j. By condition (v) of the theorem $\mu(E^n_j) = 0$. We shall prove that

$$(3.4) \qquad \mathscr{A}^* - B^{(n)} \cdot \mathscr{A}^* = E^{(n)}$$

where $E^{(n)} = \bigcup_{j=1}^{\infty} E^n_j$ and hence that the left-hand side of (3.4) has μ-measure zero. This will prove (3.1).

If $x \in \mathscr{A}^* - B^{(n)} \cdot \mathscr{A}^* - E^{(n)}$ then x is covered finely by elements of $\mathscr{A}$ which do not intersect $B^{(n)}$ and so there is an $A^n_0 \in \mathscr{A}$ such that $x \in A^n_0$, $\mu^*(A^n_0) < \alpha_n$ and $A^n_0 B^n_j = 0$, $j \geqq 1$. Let $B^n_0 = b(A^n_0)$, since $B^n_0 \subset A^n_0$, $\mu(B^n_0) < \alpha_n$, that is $B^n_0 \in \mathscr{B}^n$. Further $B^n_0 B^n_j = 0$, $j \geqq 1$ and this contradicts the maximality of the R-section $\{B^n_j\}$, $j \geqq 1$. This proves (3.4) and hence (3.1).

Write $\bar{B}^n = \{\bigcup B : B \in \mathscr{B}^n\}$, then $\mathscr{B}^* = \bigcap_{n=1}^{\infty} \bar{B}^n$. For if $x \in \mathscr{B}^*$ then for each $n \geqq 1$ there is an element of $\mathscr{B}^n$ which contains x and hence $x \in \bar{B}^n$ for each $n \geqq 1$. Conversely if $x \in \bar{B}^n$ for each $n \geqq 1$ there is an element of $\mathscr{B}^n$ which contains x for each $n \geqq 1$, that is, $x \in \mathscr{B}^*$.

Write $\mathscr{B}^{(\cdot)} = \bigcap_{n=1}^{\infty} B^{(n)}$ then since $B^{(n)} \subset \bar{B}^n$ we have $\mathscr{B}^{(\cdot)} \subset \mathscr{B}^*$. Since (iii) implies that $\mathscr{B}^* \subset \mathscr{A}^*$ we obtain

$$\mathscr{B}^{(\cdot)} \subset \mathscr{B}^* \subset \mathscr{A}^*. \tag{3.5}$$

Note that $\mathscr{B}^{(\cdot)}$ is μ-measurable since each $B^{(n)}$ is μ-measurable.

It is easily verified from (3.4) that

$$\mathscr{A}^* - \mathscr{B}^{(\cdot)} = E$$

where $E = \bigcup_{n=1}^{\infty} E^{(n)}$ has μ-measure zero since each $E^{(n)}$ has μ-measure zero. Since $\mathscr{B}^{(\cdot)}$ is μ-measurable it follows that $\mathscr{A}^*$ is μ-measurable and that $\mu(\mathscr{A}^*) = \mu(\mathscr{B}^{(\cdot)})$. Because of (3.5) we have

$$\mathscr{B}^* = \mathscr{A}^* - E'$$

where $E' \subset E$ has μ-measure zero since μ is complete. Thus $\mathscr{B}^*$ is measurable and

$$\mu(\mathscr{B}^*) = \mu(\mathscr{A}^*).$$

To prove (3.3) we note firstly that

$$\mu(B^{(n)}) \geqq \mu(\mathscr{B}^{(\cdot)}) = \mu(\mathscr{B}^*)$$

and hence

$$\liminf_{n\to\infty} \mu(B^{(n)}) \geqq \mu(\mathscr{B}^*). \tag{3.6}$$

On the other hand $B^{(n)} \subset \bar{B}^n$ and $\bar{B}^{n+1} \subset \bar{B}^n$, thus

$$\limsup B^{(n)} = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} B^{(n)}$$
$$\subset \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bar{B}^n = \bigcap_{m=1}^{\infty} \bar{B}^m = \mathscr{B}^*.$$

Thus

$$\limsup_{n\to\infty} \mu(B^{(n)}) \leqq \mu(\limsup B^{(n)}) \leqq \mu(\mathscr{B}^*). \tag{3.7}$$

From (3.6) and (3.7) we deduce that $\lim_{n\to\infty} \mu(B^{(n)}) = \mu(\mathscr{B}^*)$, this establishes (3.3) and completes the proof of the theorem.

Remark. If the condition (v) of the theorem is replaced by

(vi) *for each $B \in \mathscr{B}$ there are no points of $\mathscr{A}^*$ which do not belong to B and which are covered finely by elements of $\mathscr{A}$ which intersect B,*

a simple modification of the argument shows that (3.4) is replaced by $\mathscr{A}^* \subset B^{(n)}$ for each $n \geqq 1$ and so $\mathscr{A}^* \subset \mathscr{B}^{(\cdot)}$. Because of (3.5) we obtain $\mathscr{A}^* = \mathscr{B}^* = \mathscr{B}^{(\cdot)}$ and so we can state

THEOREM (3.2) *Under conditions (i), (ii), (iii), (iv) and (vi) we have $\mathscr{A}^* = \mathscr{B}^*$ and to each $\alpha > 0$ there exists a sequence $\{B_j^\alpha\}$, $j \geqq 1$ of mutually disjoint elements of $\mathscr{B}$ such that if $B^{(\alpha)} = \sum_{j=1}^{\infty} B_j^\alpha$*

$$\mathscr{A}^* \subset B^{(\alpha)} \tag{3.8}$$

and

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(3.9) $$\mu(\mathscr{A}^*) \leqq \mu(B^{(\alpha)}) < \mu(\mathscr{A}^*)+\alpha.$$

Note that theorem (3.2) is applicable to the hypotheses of Vitali's theorem in the form B of Denjoy [1], section 5, since the fact that the elements of $\mathscr{B}$ are closed sets and the elements of $\mathscr{A}$ are open sets implies that (vi) is true.

Following Denjoy we call a regular family of sets in the family that results from the identification of the families $\mathscr{A}$ and $\mathscr{B}$ in the hypothesis of theorem (3.1). Explicitly, a family $\mathscr{A}$ of subsets of $\mathscr{X}$ is a regular family if

(i) each element of $\mathscr{A}$ is μ-measurable and $\mu(A) > 0$ for each $A \in \mathscr{A}$

(ii) $\mu^*\{\cup A : A \in \mathscr{A}\} < \infty$

(iii) for each $A \in \mathscr{A}$ the union of all points of $\mathscr{A}^* - A \cdot \mathscr{A}^*$ which are covered finely by the elements of $\mathscr{A}$ which intersect A is a set of μ-measure zero.

Note that condition (iii) of theorem (3.1) is trivially true when $\mathscr{A} \equiv \mathscr{B}$ whereas Denjoy's condition 2°, which (iii) replaces, must still be retained when $\mathscr{A} \equiv \mathscr{B}$.

If, in the definition of a regular family we replace (iii) by

(iii)′ for each $A \in \mathscr{A}$ there are no points $\mathscr{A}^*$ which do not belong to A and which are covered finely by elements of $\mathscr{A}$ which intersect A,

we shall say that the family $\mathscr{A}$ is completely regular.

Every subfamily of a regular family is regular and every subfamily of a completely regular family is completely regular. From theorems (3.1) and (3.2) we obtain

THEOREM (3.3). *If $\mathscr{A}$ is a regular family of subsets of $\mathscr{X}$ then to each $\alpha > 0$ there is a sequence $\{A_j^\alpha\}$, $j \geqq 1$ of mutually disjoint elements of $\mathscr{A}$ such that if $A^{(\alpha)} = \sum_{j=1}^\infty A_j^\alpha$*

(3.10) $$\mu(\mathscr{A}^* - A^{(\alpha)} \cdot \mathscr{A}^*) = 0$$

$\mathscr{A}^*$ is μ-measurable and

(3.11) $$\mu(\mathscr{A}^*) \leqq \mu(A^{(\alpha)}) < \mu(\mathscr{A}^*)+\alpha$$

and

THEOREM (3.4). *If $\mathscr{A}$ is a completely regular family of subsets of $\mathscr{X}$ then to each $\alpha > 0$ there is a sequence $\{A_j^\alpha\}$, $j \geqq 1$ of mutually disjoint elements of $\mathscr{A}$ such that if $A^{(\alpha)} = \sum_{j=1}^\infty A_j^\alpha$*

(3.12) $$\mathscr{A}^* \subset A^{(\alpha)}.$$

$\mathscr{A}^*$ is μ-measurable and

(3.11) $$\mu(\mathscr{A}^*) \leqq \mu(\mathscr{A}^\alpha) < \mu(\mathscr{A}^*)+\alpha.$$

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All the results of Denjoy [1] which are derived from the conclusions of Theorems (3.1) and (3.3) remain valid and a family which is regular in the sense of Denjoy is also regular in the sense of this paper.

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